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Supplement to *Simulation and Visualization of 3D-Spherical Distributions*

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A Supplement, Spherical harmonics

Orthonormal spherical harmonics with complex values $Y_\ell^m(\vartheta, \varphi)$, $\ell = 0, 1, 2, \ldots$, $m = -\ell, -\ell+1, \ldots -1, 0, 1, \ldots, \ell-1, \ell$ of **degree** ℓ and **order** m (rank ℓ and projection m)

$$Y_\ell^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\vartheta) e^{im\varphi}, \ \varphi \in [0, 2\pi], \ \vartheta \in [0, \pi], \tag{1}$$

where P_ℓ^m denotes *associated normalized* **Legendre function** *of the first kind.* The spherical harmonics are eigenfunctions of the square of the orbital angular momentum operator.

$$Y_\ell^0(\vartheta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\vartheta), \tag{2}$$

$$Y_0^0(\vartheta, \varphi) = \sqrt{\frac{1}{4\pi}},$$

more over

$$Y_\ell^m\left(\underline{\widetilde{N}}\right) = \delta_{m,0}\sqrt{\frac{2\ell+1}{4\pi}}. \tag{3}$$

Y_ℓ^m is fully normalized

$$\int_0^{2\pi} \int_0^{\pi} \left|Y_\ell^m(\vartheta, \varphi)\right|^2 \sin\vartheta d\vartheta d\varphi = 1.$$

Some detailed account of spherical harmonics Y_ℓ^m can be found in [**?**] and [**?**].
some authors do not apply $1/\sqrt{4\pi}$ in the definition of Y_ℓ^m, also for a sphere with radius R *spherical harmonics are normalized additionally* $Y_\ell^m(\vartheta,\varphi)/R$. It also follows

$$\begin{aligned} Y_\ell^{m*}(\vartheta,\varphi) &= Y_\ell^m(\vartheta,-\varphi) \\ &= (-1)^m Y_\ell^{-m}(\vartheta,\varphi), \\ Y_\ell^{-m}(\vartheta,\varphi) &= (-1)^m e^{-i2m\varphi} Y_\ell^m(\vartheta,\varphi). \end{aligned}$$

Inversion $\widetilde{\underline{x}} \to -\widetilde{\underline{x}}$, $(\vartheta,\varphi) \to (\pi-\vartheta, \pi+\varphi)$

$$Y_\ell^m(-\widetilde{\underline{x}}) = (-1)^\ell Y_\ell^m(\widetilde{\underline{x}}). \tag{4}$$

Addition formula (see [**?**], 8.814,[**?**], 11.4(8)),

$$\sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\widetilde{\underline{x}}_1) Y_\ell^m(\widetilde{\underline{x}}_2) = \frac{2\ell+1}{4\pi} P_\ell(\cos\vartheta), \tag{5}$$

where $\cos\vartheta = \widetilde{\underline{x}}_1 \cdot \widetilde{\underline{x}}_2$.

$$\sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\widetilde{\underline{x}}) Y_\ell^m(\widetilde{\underline{x}}) = \frac{2\ell+1}{4\pi}, \tag{6}$$

B Supplement, MATLAB Scripts for Figures

In this section we list all the MATLAB Scripts of Figures contained in the paper

MATLAB Script of Figure 2

```
pV1 = [0.1, 0.1, 0.43, 0.82]; pV2 = [0.5, 0.1, 0.43, 0.82];
Psi=0; Mu= [0,0,1]; resolution=100; gamm=6.364; bet1=4.5; bet2=1.5;
figure( 'Position',[1 1 756 343])
subplot('Position',pV1); gx5 = Density\_FB5(gamm,bet1,Mu,Psi,resolution);
subplot('Position',pV2); gx5 = Density_FB5(gamm, bet2,Mu,Psi,resolution);
```

MATLAB Script of Figure 3

```
Mu=  [0,0,1]; Psi=0; resolution=100;
pV1 = [0.05,0.1,0.28,0.82];pV2 = [0.38,0.1,0.28,0.82];pV3 = [0.69,0.1,0.28,0.82];
figure( 'Position',[300 300 1300 343])
subplot('Position',pV1);  gx6 =  Density_FB6(0,3.2,-1.1,Mu,Psi,resolution);
subplot('Position',pV2);  gx6 =  Density_FB6(0,3.2,3.2,Mu,Psi,resolution);
subplot('Position',pV3); gx6 =  Density_FB6(0,3.2,4.1,Mu,Psi,resolution);
```

MATLAB Script of Figure 4

```
n=2\^12; nSide= 2$^$3; nPix= nSide2nPix(nSide);
Y = Random_Uni_Inv(n);
hY = Hist2DSphere(Y,nPix);
pV1 = [0.1,0.1,0.3,0.82]; pV2 = [0.5,0.1,0.43,0.82];
figure( 'Position',[1 1 756 343])
subplot('Position',pV1); Plot_DataRandomS2(Y);
subplot('Position',pV2); Plot_Hist2DSphere(hY);
```

MATLAB Script of Figure 5

```
n=2^12; nSide=2^3; nPix= nSide2nPix(nSide); resolution=100;
MuB=[1,-1,1];    betB =4.5 ; PsiB=  pi/2;
Y = Random_FB4_Beta_GyT( betB,MuB,PsiB,n);
hY=Hist2DSphere(Y,nPix);
pV1 = [0.05,0.1,0.3, 0.82]; pV2 = [0.4,0.1,0.22,0.82]; pV3 = [0.67,0.1,0.3,0.82];
figure( 'Position',[300 300 1300 343])
subplot('Position',pV1); dx = Density_FB4_Beta(betB,MuB,PsiB);
subplot('Position',pV2);  Plot_DataRandomS2(Y);
subplot('Position',pV3); Plot_Hist2DSphere(hY);
```

MATLAB Script of Figure 6

```
n=2^12; nSide=2^3; nPix= nSide2nPix(nSide); resolution=100;
kappaK = 5 ; betK =2 ; MuK=[-0.9 -1 .2]; PsiK= 0;
Y = Random_FB5_GyT(kappaK,betK,MuK,PsiK,n);
hY=Hist2DSphere(Y,nPix);
pV1 = [0.05,0.1,0.3, 0.82]; pV2 = [0.4,0.1,0.22,0.82]; pV3 = [0.67,0.1,0.3,0.82];
figure( 'Position',[300 300 1300 343])
subplot('Position',pV1);  gx5 =  Density_FB5(kappaK ,betK,MuK,PsiK,resolution);
subplot('Position',pV2);  Plot_DataRandomS2(Y);
subplot('Position',pV3); Plot_Hist2DSphere(hY);
```

MATLAB Script of Figure 7

```
n=2^12; nSide=2^3; nPix= nSide2nPix(nSide); resolution=100;
kappaK = 5 ; betK =5 ; MuK=[-.5 0 .5]; PsiK= pi/4;
Y = Random_FB5_GyT(kappaK,betK,MuK,PsiK,n);
hY=Hist2DSphere(Y,nPix);
pV1 = [0.05,0.1,0.3, 0.82]; pV2 = [0.4,0.1,0.22,0.82]; pV3 = [0.67,0.1,0.3,0.82];
figure( 'Position',[300 300 1300 343])
subplot('Position',pV1);  gx5 =  Density_FB5(kappaK ,betK,MuK,PsiK,resolution);
subplot('Position',pV2);  Plot_DataRandomS2(Y);
subplot('Position',pV3); Plot_Hist2DSphere(hY);
```

MATLAB Script of Figure 8

```
n=2^12; nSide=2^3; nPix= nSide2nPix(nSide); resolution=100;
Psi6= 0; Mu6=  [1,-1,1]; kappa6 = 1.5; bet6 = .61 ; gamm6 = -0.31;
pV1 = [0.05,0.1,0.3,0.82]; pV2 = [0.4,0.1,0.22,0.82]; pV3 = [0.67,0.1,0.3,0.82];
Y = Random_FB6( kappa6,bet6,0,Mu6,Psi6,n);
hY=Hist2DSphere(Y,nPix);
figure( 'Position',[300 300 1300 343])
subplot('Position',pV1);
gx6 =  Density_FB6(kappa6,bet6,gamm6,Mu6,Psi6,resolution);
subplot('Position',pV2);  Plot_DataRandomS2(Y);
subplot('Position',pV3); Plot_Hist2DSphere(hY);
```

MATLAB Script of Figure 9

```
pV1 = [0.05,0.1,0.28,0.82];pV2 = [0.38,0.1,0.28,0.82];pV3 = [0.69,0.1,0.28,0.82];
figure( 'Position',[300 300 1300 343]);  Real=true;
subplot('Position',pV1);
px = Density_SphHarm(4, 1, Real);
subplot('Position',pV2);
px = Density_SphHarm(4, 2, Real);
subplot('Position',pV3);
px = Density_SphHarm(4, 3, Real);
```

MATLAB Script of Figure 10

```
n=2^12; nSide=2^3; nPix= nSide2nPix(nSide);
L=3; m=2; Mu=[1 -1 1];Psi=pi/2;  Real=false;
Y = Random_Y3_2Compl_square( Mu,n);
hY=Hist2DSphere(Y,nPix);
pV1 = [0.05,0.1,0.3,0.82]; pV2 = [0.4,0.1,0.22,0.82]; pV3 = [0.67,0.1, 0.3,0.82];
figure( 'Position',[300 300 1300 343])
subplot('Position',pV1); px = Density_SphHarm(L, m,Real);
subplot('Position',pV2);  Plot_DataRandomS2(Y);
subplot('Position',pV3); Plot_Hist2DSphere(hY);
```

MATLAB Script of Figure 11

```
n=2^12; nSide=2^3; nPix= nSide2nPix(nSide);
L=3; m=2; Mu=[1 -1 1];Psi=pi/2;  Real=true;
Y = Random_Y3_2Real_square( Mu,Psi,n);
hY=Hist2DSphere(Y,nPix);
pV1 = [0.05,0.1,0.3,0.82]; pV2 = [0.4,0.1,0.22,0.82]; pV3 = [0.67,0.1, 0.3,0.82];
figure( 'Position',[300 300 1300 343])
subplot('Position',pV1); px = Density_SphHarm(L, m,Real);
subplot('Position',pV2);  Plot_DataRandomS2(Y);
subplot('Position',pV3); Plot_Hist2DSphere(hY);
```

MATLAB Script of Figure 12

```
n=2^13; nSide=2^3; nPix= nSide2nPix(nSide);  Mu=[0 0 1]; Psi=  pi/2;
pV1 = [0.05,0.1, 0.3,0.82]; pV2 = [0.4,0.1,0.22,0.82]; pV3 = [0.67,0.1,0.3,0.82];
Y = Random_U_Distr(Mu, n);
hY=Hist2DSphere(Y,nPix);
figure( 'Position',[300 300 1300 343])
subplot('Position',pV1); gx = Density_U(Mu,Psi);
subplot('Position',pV2); Plot_DataRandomS2(Y);
subplot('Position',pV3);  Plot_Hist2DSphere(hY);
```

C Supplement, Algorithms

In this section we list all the algorithms of our paper scripted in a MATLAB package '3D-Directional Statistics, Simulation and Visualization'. We introduce the indicator function $\mathbf{1}_A = 1$ if A is true otherwise it is 0.

Algorithm 1 Rotation

Require: $\underline{\widetilde{\mu}} \in \mathbb{S}_2$ and $\psi \in [0, 2\pi]$ are given
Ensure: Rotated frame of reference
1: Calculate $\underline{\widetilde{N}} \times \underline{\widetilde{\mu}}$ the #1 axis of rotation
2: Rotate $\underline{\widetilde{N}}$ to $\underline{\widetilde{\mu}}$
3: Use $\underline{\widetilde{\mu}} = \underline{\widetilde{N}}$ for the #2 axis of rotation and rotate the sphere by the angle ψ.
{ If only $\underline{\widetilde{\mu}}$ is given, this algorithm simplifies to steps 1. and 2 or set $\psi = 0$.}

Algorithm 2 Rejection

Require: $f_X(x)$ (evaluable), $h(y)$ (samplable), and proportionality constant c are given
Ensure: Variate X is distributed according to density $f_X(x)$
1: Generate two random variates Y and U, from $h(y)$ and $Unif(0, 1)$ respectively
2: **if** $U \leq g(Y) = f_X(Y)/(c \cdot h(Y))$ **then**
3:     **return** $X = Y$ as a variate generated from f_X
4: **else**
5:     reject Y and **go to** 1
6: **end if**

Algorithm 3 Random_vMF_Inv

Require: $\kappa \in \mathbb{R}$ is given
Ensure: Variate $\underline{\widetilde{Y}}$ is distributed according to the von Mises-Fisher distribution, $vMF(\kappa)$
1: Generate independent U_1, U_2 from $Unif(0, 1)$
2: Calculate $X = \cos\Theta$, by

$$X = \frac{\log\left(2U_1 \sinh\kappa + e^{-\kappa}\right)}{\kappa}$$

3: Calculate longitude $\Phi = 2\pi U_2$
4: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1 - X^2}\cos(\Phi), \sqrt{1 - X^2}\sin(\Phi), X\right)^\top$.

Algorithm 4 Random_vMF_Wood

Require: κ and $d \geq 2$ are given

Ensure: Variate $\underline{\widetilde{X}}$ is distributed according to the von Mises-Fisher distribution, $vMF(\kappa)$on $\mathbb{S}_{d-1}$

1: Set

$$b = \frac{-2\kappa + \sqrt{4\kappa^2 + (d-1)^2}}{d-1},$$
$$x_0 = \frac{1-b}{1+b},$$
$$c = \kappa x_0 + (d-1)\log\left(1 - x_0^2\right)$$

2: Generate variate B from $Beta(\alpha, \beta)$ with shape parameters given by

$$\alpha = (d-1)/2, \quad \beta = (d-1)/2,$$

3: Calculate

$$X = \frac{1-(1+b)B}{1-(1-b)B},$$

and generate U from $Unif(0,1)$

4: **if** $\kappa X + (d-1)\log(1 - x_0 X) - c < \log U$ **then**

5: **go to** step 2.

6: **else**

7: Generate a $d-1$−dimensional spherical uniform vector W,

8: **return** $\underline{\widetilde{X}} = \left(\sqrt{1-X^2}W^\top, X\right)^\top$.

9: **end if**

Algorithm 5 Random_Watson_Fish

Require: $\gamma \in \mathbb{R}$ is given
Ensure: Variate $\underline{\widetilde{Y}}$ is distributed according to the Dimroth-Watson distribution, $DW(\gamma)$
1: **if** $\gamma > 0$, (Bipolar) **then**
2: Set $c = 1/\left(e^{\gamma} - 1\right)$
3: Generate independent U_1, U_2 and U from $Unif(0,1)$
4: Set $X = \left(\log\left(U_1/c + 1\right)\right)/\gamma$
5: **if** $U_2 \leq \exp\left(\gamma X^2 - \gamma X\right)$ **then**
6: $X = \left(\mathbf{1}_{U<1/2} - \mathbf{1}_{U \geq 1/2}\right) X$, {Since X is positive and the density is symmetric to the equator}
7: $\Phi = 2\pi U_2$
8: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1 - X^2}\cos(\Phi), \sqrt{1 - X^2}\sin(\Phi), X\right)$
9: **else**
10: **go to** step 3.
11: **end if**
12: **else** $\{\gamma < 0$, (Girdle)$\}$
13: Set $c_1 = \sqrt{|\gamma|}, \quad c_2 = \arctan c_1$.
14: Generate independent U_1, U_2 and U from $Unif(0,1)$
15: Set $X = (1/c_1)\tan(U_1 c_2)$
16: **if** $U_2 \leq \left(1 - \gamma X^2\right)\exp\left(\gamma X^2\right)$ **then**
17: $X = \left(\mathbf{1}_{U<1/2} - \mathbf{1}_{U \geq 1/2}\right) X$; {Since S is positive and the density is symmetric to the equator}
18: $\Phi = 2\pi U_0$,
19: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1 - X^2}\cos(\Phi), \sqrt{1 - X^2}\sin(\Phi), X\right)^{\top}$.
20: **else**
21: **go to** step 13.
22: **end if**
23: **end if**

Algorithm 6 Random_Watson_LW

Require: $\gamma \in \mathbb{R}$ is given

Ensure: Variate $\widetilde{\underline{Y}}$ is distributed according to the Dimroth-Watson distribution, $DW(\gamma)$

1: **if** $\gamma > 0$, (Bipolar) **then**

2: Set

$$\rho = \frac{4\gamma}{2\gamma + 3 + \sqrt{(2\gamma+3)^2 - 16\gamma}}, \; r = \left(\frac{3\rho}{2\gamma}\right)^3 e^{-3+2\gamma/\rho},$$

3: Generate independent U_1 and U_2 from $Unif(0,1)$

4: Set

$$S = \frac{U_1^2}{1 - \rho\left(1 - U_0^2\right)}, \; V = \frac{rU_2^2}{(1-\rho S)^3}, W = \gamma S,$$

5: **if** $V \leq e^{2W}$ **then**

6: Put $\Theta = \arccos\sqrt{S}$

7: Generate U_3 from $Unif(0,1)$

8: Calculate

$$X = \cos \mathbf{1}_{U_3<1/2}\left(\pi - \Theta\right), \; \Phi = 4\pi\mathbf{1}_{U_3<1/2}U_3 + 2\pi\left(2U_3 - 1\right),$$

9: **else**

10: **go to** step 3.

11: **end if**

12: **return** $\widetilde{\underline{Y}} = \left(\sqrt{1-X^2}\cos\left(\Phi\right), \sqrt{1-X^2}\sin\left(\Phi\right), X\right)^\top.$

13: **else** $\{\gamma < 0$, (Girdle) $\}$

14: Set $b = e^{2\gamma} - 1$,

{ Begin, generate variates X_k from trucated normal distribution}

15: Generate independent U_1, U_2 from $Unif(0,1)$

16: Set $V = \ln\left(1 + U_1 b\right)/\gamma$, $\xi = 2\pi U_2$ and $c = \cos\xi$

17: Set $S_1 = Vc^2$, and $S_2 = V - S_1$

18: **if** either $S_1 > 1$ or $S_2 > 1$ **then**

19: **go to** step 15.

20: **else**

21: Set $X_1 = \sqrt{V}c$, $X_2 = \sin\xi$

22: **end if**

{End of Generation X_k }

23: Generate independent U_4, U_5 from $Unif(0,1)$

24: Calculate $\Phi_1 = 2\pi U_4$, $\Phi_2 = 2\pi U_5$

25: **return**

$$\widetilde{\underline{Y}}_1 = \left(\sqrt{1-X_1^2}\cos\left(\Phi\right), \sqrt{1-X_1^2}\sin\left(\Phi\right), X_1\right)^\top,$$

$$\widetilde{\underline{Y}}_2 = \left(\sqrt{1-X_2^2}\cos\left(\Phi\right), \sqrt{1-X_2^2}\sin\left(\Phi\right), X_2\right)^\top,$$

two independent variates from DW distribution.

26: **end if**

Algorithm 7 Random_FB4

Require: κ, γ are given

Ensure: Variate $\underline{\widetilde{Y}}$ is distributed according to the Fisher-Bingham$_4$ ($\mathbf{GFB}_4$) distribution

1: **if** $\gamma < 0$ **then**
2: **if** $0 \leq \kappa \leq -2\gamma$ **then**
3: Generate Z from normal $\mathcal{N}\left(-\kappa/2\gamma, -1/2\gamma\right)$
4: **if** $Z \in [-1, 1]$ **then**
5: Generate U_1 from $\mathit{Unif}(0,1)$
6: Set $\Phi = 2\pi U_1$,
7: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1-Z^2}\cos(\Phi), \sqrt{1-Z^2}\sin(\Phi), Z\right)^\top$
8: **else**
9: Reject Z and **go to** 3
10: **end if**
11: **else**
12: Generate U from $\mathit{Unif}(0,1)$
13: Generate X from $\mathit{vMF}(\kappa + 2\gamma)$
14: **if** $U \leq e^{\gamma} \exp\left(\gamma\left(X^2 - 2X\right)\right)$ **then**
15: Generate U_2 from $\mathit{Unif}(0,1)$
16: Set $\Phi = 2\pi U_2$,
17: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1-X^2}\cos(\Phi), \sqrt{1-X^2}\sin(\Phi), X\right)^\top$.
18: **else**
19: reject X and **go to** 12
20: **end if**
21: **end if**
22: **else if** $\gamma > 0$ **then**
23: Set

$$c_\kappa = \frac{\kappa}{2\pi\left(e^{\gamma} - e^{-\gamma x}\right)}, \quad p_1 = \frac{c_{\kappa+\gamma}}{c_{\kappa+\gamma} + c_{\kappa-\gamma}},$$

24: Generate independent U and U_1 from $\mathit{Unif}(0,1)$
25: Generate independent X_1 and X_2 from vMF with parameters $\kappa - \gamma$, and $\kappa + \gamma$ respectively
26: Calculate the mixture X,

$$X = \mathbf{1}_{U_1 \leq p_1} X_1 + \mathbf{1}_{U_1 > p_1} X_2,$$

27: **if**

$$U \leq \left(1 + e^{-2\gamma}\right) \frac{e^{\gamma X^2}}{e^{\gamma X} + e^{-\gamma X}}.$$

then
28: Generate U_3 with uniform distribution
29: set $\Phi = 2\pi U_3$,
30: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1-X^2}\cos(\Phi), \sqrt{1-X^2}\sin(\Phi), X\right)^\top$.
31: **else**
32: reject X and **go to** 24
33: **end if**
34: **else**
35: $\gamma = 0$, generate $\mathit{vMF}(\kappa)$
36: **end if**

Algorithm 8 Random_FB4_Beta

Require: $\beta \neq 0$ is given

Ensure: Variate $\underline{\widetilde{Y}}$ is distributed according to the $\mathbf{GFB}_{4,\beta}$ distribution

1: Calculate

$$c_- = \int_{-1}^{1} e^{-\beta x^2} dx\,; \quad c_+ = \int_{-1}^{1} e^{\beta x^2} dx$$
$$c = e^{\beta} c_- + e^{-\beta} c_+\,; \quad p_1 = e^{\beta} c_- / c$$

2: Generate independent U_1 and U_2 from $Unif\,(0,1)$

3: Generate independent V_1 and V_2 from DW with parameters β and $-\beta$ respectively

4: Set $V = (\mathbf{1}_{U_1<p_1})\, V_1 + (1 - \mathbf{1}_{U_1<p_1})\, V_2$

5: **if** $U_2 \leq I_0\left(\beta\left(1 - V^2\right)\right) / \cosh\left(\beta\left(1 - V^2\right)\right)$ **then**

6: Set $X = \cos\Theta$, by $X = V$,

7: **else**

8: **go to** step 2

9: **end if**

10: Use Algorithm 9 with $\beta\left(1 - X^2\right)$ to get Φ

11: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1 - X^2}\cos(\Phi)\,, \sqrt{1 - X^2}\sin(\Phi)\,, X\right)^{\top}$

Algorithm 9 vMF_Circ_Phi

Require: $\beta \neq 0$ and x

Ensure: Variate Φ distributed vMF on $\mathbb{S}_1$ extended to $[0, 2\pi]$

1: Generate random variate Φ_1 from $vMF\big(\beta\left(1 - x^2\right)\big)$, on $\mathbb{S}_1$

2: Generate random variate U uniformly distributed on integers $(1, 2, 3, 4)$

3: **return** $\Phi = (\mathbf{1}_{U=1} + \mathbf{1}_{U=3})\,\Phi + (\mathbf{1}_{U=2} + \mathbf{1}_{U=3})\,\pi - (\mathbf{1}_{U=2} + \mathbf{1}_{U=4})\,\Phi + 2\pi\mathbf{1}_{U=4}$

Algorithm 10 Random_FB5_Kent

Require: $\beta \geq 0$, $\kappa \geq 0$ are given such that we have the case where, $2\beta \leq \kappa$ (equal-area projection), for $\kappa < 0$, we use $|\kappa|$ and transform back in a final step

Ensure: Variate $\underline{\widetilde{Y}}$ is distributed according to the Kent distribution (Model $\mathbf{GFB}_{5,K}$)

1: Set

$$a = 4\kappa - 8\beta\,; \quad b = 4\kappa + 8\beta\,; \quad \gamma = 8\beta$$
$$\lambda_1 = \sqrt{a + 2\sqrt{\gamma}}\,; \quad \lambda_2 = \sqrt{b}\,; \quad c_2 = b/8\kappa$$

2: Generate independent U_1 and U_2 from $\mathit{Unif}(0,1)$

3: Generate independent R_1 and R_2 from exponential distribution with parameters λ_1 and λ_2 respectively

4: **if** $U_1 \leq \exp\left(-\left(aR_1^2 + \lambda_1 R_1^4\right)/2 + \lambda_1 R_1 - 1\right)$ **then**

5: Accept R_1

6: **else**

7: **go to** step 3

8: **end if**

9: **if** $U_2 \leq \exp\left(-\left(bR_2^2 - \gamma R_2^4\right)/2 + \lambda_2 R_2 - c_2\right)$ **then**

10: Accept R_2

11: **else**

12: **go to** step 3

13: **end if**

14: **if** $R_1^2 + R_2^2 < 1$ **then**

15: Accept (R_1, R_2)

16: **else**

17: **go to** step 3

18: **end if**

19: Using trigonometric identities

$$\cos\vartheta = 1 - 2\left(R_1^2 + R_2^2\right), \quad \sin\varphi = \frac{R_2}{\sqrt{R_1^2 + R_2^2}}, \quad \cos\varphi = \frac{R_1}{\sqrt{R_1^2 + R_2^2}},$$

calculate

$$X = 1 - 2\left(R_1^2 + R_2^2\right)\;; \quad S_\varphi = \frac{R_2}{\sqrt{R_1^2 + R_2^2}}; \quad C_\varphi = \frac{R_1}{\sqrt{R_1^2 + R_2^2}},$$

20: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1 - X^2}C_\varphi, \sqrt{1 - X^2}S_\varphi, X\right)^\top$

Algorithm 11 Random_FB5_GyT

Require: $\beta \geq 0$, $\kappa \geq 0$ are given, for $\kappa < 0$, we use $|\kappa|$ and transform back in a final step

Ensure: Variate $\underline{\widetilde{Y}}$ is distributed according to the Kent distribution (Model $\mathbf{GFB}_{5,K}$)

1: **if** $2\beta \leq \kappa$ **then**
2: Set: $\alpha_1 = \kappa - 2\beta\,;\quad \alpha_2 = \kappa + 2\beta\,;\quad \sigma_1^2 = 1/\left(4\alpha_1 + \sqrt{8\beta}\right)\,;\quad \sigma_2^2 = 1/4\kappa$
3: Generate U_1 from $\mathit{Unif}(0,1)$ and Z_1 from normal $\mathcal{N}(0,\sigma_1^2)$
4: **if** $U_1 \leq \exp\left(-\left(2\alpha_1 Z_1^2 + 4\beta Z_1^4\right) - \frac{1}{2} + \left(2\alpha_1 + \sqrt{8\beta}\right) Z_1^2\right)$ **then**
5: Accept Z_1
6: **else**
7: **go to** step 2
8: **end if**
9: Generate U_2 from $\mathit{Unif}(0,1)$ and Z_2 from normal $\mathcal{N}(0,\sigma_2^2)$
10: **if** $U_2 \leq \exp\left(-\left(2\alpha_2 Z_2^2 + 4\beta Z_2^4\right) + (2\alpha_2 - 4\beta) Z_2^2\right)$ **then**
11: Accept Z_2
12: **else**
13: **go to** step 9
14: **end if**
15: **if** $Z_1^2 + Z_2^2 < 1$ **then**
16: **go to** step 37
17: **else**
18: **go to** step 2
19: **end if**
20: **else**
21: Set $\alpha_1 = \kappa - 2\beta;\quad \alpha_2 = \kappa + 2\beta;\quad \eta = 1 - \kappa/(2\beta);\quad y_0 = \sqrt{\eta/2};\quad p_{y_0} = (\beta/2)\eta^2;$
$\sigma_1^2 = 1/2\beta\eta\,;\quad \sigma_2^2 = 1/4\kappa.$
22: Generate U_1 from $\mathit{Unif}(0,1)$ and Z_1 from normal, $\mathcal{N}(y_0,\,\sigma_1^2)$
23: **if** $U_1 \leq \exp\left(-\left(2\alpha_1 Z_1^2 + 4\beta Z_1^4\right) + \beta\eta\,(Z_1 - y_0)^2 - 2p_{y_0}\right)$ **then**
24: Accept Z_1
25: **else**
26: **go to** step 22
27: **end if**
28: Generate U from $\mathit{Unif}(0,1)$
29: Set $Z_1 = Z_1\left(\mathbf{1}_{U<1/2} - \mathbf{1}_{U\geq 1/2}\right)$
30: Generate U_2 from $\mathit{Unif}(0,1)$ and Z_2 from normal, $\mathcal{N}(p_{y_0},\,\sigma_2^2)$
31: **if** $U_4 \leq \exp\left(-\left(\alpha_2 R_2^2 + 4\beta R_2^4\right) + (2\alpha_2 - 4\beta) Z_2^2\right)$ **then**
32: Accept Z_2
33: **else**
34: **go to** step 27
35: **end if**
36: **if** $Z_1^2 + Z_2^2 < 1$ **then**
37: Using trigonometric identities, calculate
$X = 1 - 2\left(Z_1^2 + Z_2^2\right)\,;\quad S_\varphi = Z_2/\sqrt{Z_1^2 + Z_2^2};\quad C_\varphi = Z_1/\sqrt{Z_1^2 + Z_2^2},$
38: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1-X^2}C_\varphi, \sqrt{1-X^2}S_\varphi, X\right)^\top.$
39: **else**
40: **go to** step 22
41: **end if**
42: **end if**

Algorithm 12 Random_FB6

Require: $\kappa \geq 0$, $\beta \neq 0$, and $\gamma \in \mathbb{R}$ are given

Ensure: Variate $\underline{\widetilde{Y}}$ is distributed according to the Fisher-Bingham$_6$ distribution (Model $\mathbf{GFB}_6$)

1: Calculate

$$c_- = \int_{-1}^{1} e^{\kappa x + (\gamma - \beta) x^2} dx; \quad c_+ = \int_{-1}^{1} e^{\kappa x + (\gamma + \beta) x^2} dx$$

$$c_6 = e^{\beta} c_- + e^{-\beta} c_+, \quad p_1 = \frac{e^{\beta} c_-}{c_6}$$

2: Generate independent U_1 and U_2 from $Unif(0,1)$

3: Generate independent V_1 and V_2 from $\mathbf{GFB}_4$ distribution on the sphere with parameters $(\kappa, \gamma - \beta)$ and $(\kappa, \gamma + \beta)$ respectively

4: Set $V = (\mathbf{1}_{U_1 < p_1}) V_1 + (1 - \mathbf{1}_{U_1 < p_1}) V_2$

5: **if** $U_2 \leq I_0\left(\beta\left(1 - V^2\right)\right) / \cosh\left(\beta\left(1 - V^2\right)\right)$ **then**

6: $\quad X = V$

7: **else**

8: $\quad$ **go to** step 2

9: **end if**

10: Use Algorithm 9 with $\beta\left(1 - V^2\right)$ to get Φ

11: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1 - X^2}\cos(\Phi), \sqrt{1 - X^2}\sin(\Phi), X\right)^{\top}$.

Algorithm 13 Random_Y3_2Compl_square

Require: $n \in \mathbb{N}$ is given (sample size)

Ensure: Variate $\underline{\widetilde{Y}}$ distributed according to the distribution characterized by $\left|Y_3^2\left(\underline{\widetilde{x}}\right)\right|^2$

{ Use a Beta envelope for generating X with the density $2f_3^2\left(u\right)$ where $u \in (0,1)$ }

1: Generate U_1 from $Unif(0,1)$

2: Generate variate B from $Beta(\alpha,\beta)$ distribution with shape parameters given by

$$\alpha = 3.08, \quad \beta = 2.5249,$$

3: **if**

$$U_1 > \frac{2g_3^2\left(B\right)}{b\left(B,\alpha,\beta\right)}.$$

then

4: **go to** step 1

5: **else**

6: Accept $X = B$ with acceptance ratio $c^{-1} = 1/1.074 = 0.9311$.

7: **end if**

{ End of genereation of X }

8: Generate U from $Unif(0,1)$

9: Set $X = X\left(\mathbf{1}_{U<1/2} - \mathbf{1}_{U\geq 1/2}\right)$

10: Generate independent Z_1 and Z_2 from standard normal, $\mathcal{N}(0,1)$

11: Calculate variate W, from the uniform distribution on circle, given by

$$W = \left(Z_1/\sqrt{Z_1^2+Z_2^2}, Z_2/\sqrt{Z_1^2+Z_2^2}\right)^\top,$$

12: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1-X^2}W, X\right)^\top$.

Algorithm 14 Random_Y3_2Real_squared

Require: $n \in \mathbb{N}$ is given (sample size)

Ensure: Variate $\underline{\widetilde{Y}}$ distributed according to the distribution characterized by

$$f(\underline{\widetilde{x}}) \cong P_3^2 (\cos\vartheta)^2 \cos^2 2\varphi$$

1: Generate X by as in Algorithm 13,
{ Simulate Φ by density $\cos^2 K\varphi$. on $[0, 2\pi]$ }

2: Generate variate B from $Beta(\alpha, \beta)$ with shape parameters given by

$$\alpha = 3/2\,, \quad \beta = 1/2$$

with density $b(x, \alpha, \beta)$

3: Calculate $\Phi = \arccos\sqrt{B} \in [0, \pi/2]$,

4: Generate variate U from $Unif(0, 1)$

5: Generate variate K from *Discrete Uniform* on the set $\{1, 2, 3, 4\}$

6: Set

$$\Phi = \Phi\left(\mathbf{1}_{K=1} + \mathbf{1}_{K=2} - \mathbf{1}_{K=3} - \mathbf{1}_{K=4}\right) + \pi\left(\mathbf{1}_{K=2} - 2\mathbf{1}_{K=3} + \mathbf{1}_{K=4}\right) + \left(\mathbf{1}_{U<1/2}\right)2\pi,$$
$$\Phi = \Phi/2,$$

7: **return** $\underline{\widetilde{Y}} = \left(\sqrt{1-X^2}\cos(\Phi)\,, \sqrt{1-X^2}\sin(\Phi)\,, X\right)^\top.$

References

[1] Atkinson, A. C. and Pearce, M. C. (1976). The computer generation of beta, gamma and normal random variables. *Journal of the Royal Statistical Society. Series A (General)*, pages 431–461.

[2] Bingham, C. (1974). An antipodally symmetric distribution on the sphere. *The Annals of Statistics*, pages 1201–1225.

[3] Dimroth, E. (1962). Untersuchungen zum mechanismus von blastesis und syntexis in phylliten und hornfelsen des südwestlichen fichtelgebirges i. die statistische auswertung einfacher grteldiagramme. *Tschermaks mineralogische und petrographische Mitteilungen*, 8:248–274.

[4] Fisher, R. A. (1953). Dispersion on a sphere. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 217(1130):295–305.

[5] Gorski, K. M., Hivon, E., Banday, A. J., Wandelt, B. D., Hansen, F. K .and Reinecke, M., and Bartelmann, M. (2005). HEALPix a framework for high-resolution discretization and fast analysis of data distributed on the sphere. *The Astrophysical Journal*, 622(2):759.

[6] Kent, J. T. (1982). The Fisher-Bingham distribution on the sphere. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 71–80.

[7] Kent, J. T., Ganeiber, A. M., and Mardia, K. V. (2018). A new unified approach for the simulation of a wide class of directional distributions. *Journal of Computational and Graphical Statistics*, 27(2):291–301.

[8] Kent, J. T. and Hamelryck, T. (2005). Using the Fisher-Bingham distribution in stochastic models for protein structure. *Quantitative Biology, Shape Analysis, and Wavelets*, 24:57–60.

[9] Ley, C. and Verdebout, T. (2017). *Modern directional statistics*. Chapman and Hall/CRC.

[10] Mardia, K. V. and Jupp, P. E. (2009). *Directional Statistics*, volume 494. John Wiley & Sons.

[11] Neumann, J. v. (1951). Various techniques used in connection with random digits. *U.S. Nat. Bur. Stand. Appl. Math. Ser.,*, 12:36–38.

[12] Rao Jammalamadaka, S. and Terdik, H. G. (to appear in JMVA May, 2019). Harmonic analysis and distribution-free inference for spherical distributions. arxiv:1710.00253 [stat.me], UCSB and UD.

[13] Rubinstein, R. Y. (1981). *Simulation and the Monte Carlo Method*. John Wiley & Sons, Inc.

[14] Ulrich, G. (1984). Computer generation of distributions on the m-sphere. *Applied Statistics*, pages 158–163.

[15] Watson, G. S. (1965). Equatorial distributions on a sphere. *Biometrika*, 52(1/2):193–201.

[16] Watson, G. S. (1983). *Statistics on spheres*. Number 6 in University of Arkansas Lecture Notes in the Mathematical Sciences. John Wiley & Sons, Inc., New York.

[17] Wood, A. T. A. (1987). The simulation of spherical distributions in the Fisher-Bingham family. *Communications in Statistics-Simulation and Computation*, 16(3):885–898.

[18] Wood, A. T. A. (1988). Some notes on the Fisher–Bingham family on the sphere. *Communications in Statistics-Theory and Methods*, 17(11):3881–3897.

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