Advanced Textbooks in Control and Signal Processing

Basil Kouvaritakis
Mark Cannon

Model Predictive Control

Classical, Robust and Stochastic

Springer

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To Niko Kouvaritakis, who introduced us to the sustainable development problem and initiated our interest in stochastic predictive control.

Series Editors' Foreword

The *Advanced Textbooks in Control and Signal Processing* series is designed as a vehicle for the systematic textbook presentation of both fundamental and innovative topics in the control and signal processing disciplines. It is hoped that prospective authors will welcome the opportunity to publish a more rounded and structured presentation of some of the newer emerging control and signal processing technologies in this textbook series. However, it is useful to note that there will always be a place in the series for contemporary presentations of foundational material in these important engineering areas.

In 1995, our monograph series *Advances in Industrial Control* published *Model Predictive Control in the Process Industries* by Eduardo F. Camacho and Carlos Bordons (ISBN 978-3-540-19924-3, 1995). The subject of model predictive control in all its different varieties is a popular control technique and the original monograph benefited from that popularity and consequently moved to the *Advanced Textbooks in Control and Signal Processing* series. In 2004, it was republished in a thoroughly updated second edition now simply entitled *Model Predictive Control* (ISBN 978-1-85233-694-3, 2004). A decade on, the new edition is a successful and well-received textbook within the textbook series.

As demonstrated by the continuing demand for Prof. Camacho and Prof. Bordon's textbook, the technique of model predictive control or "MPC" has been startlingly successful in both the academic and industrial control communities. If the reader considers the various concepts and principles that are combined in MPC, the reasons for this success are not so difficult to identify.

"M $\sim$ Model" From an early beginning with transfer-function models using the Laplace transform through the 1960s' revolution of state-space system descriptions leading on to the science of system identification, the use of a system or process model in control design is now very well accepted.

"P $\sim$ Predictive" The art of looking forward from a current situation and planning ahead to achieve an objective is simply a natural human activity. Thus, once a system model is available it can be used to predict ahead from a currently measured position to anticipate the future and avoid constraints and other restrictions.

"C ~ Control" This is the computation of the control action to be taken. The enabling idea here is automated computation achieved using optimization. A balance between output error and control effort used is captured in a cost function that is usually quadratic for mathematical tractability.

There may be further reasons for its success connected with nonlinearities, the future process output values to be attained, and any control signal restrictions; these combine to require constrained optimization. In applying the forward-looking control signal, the one-step-at-a-time receding-horizon principle is implemented.

Academic researchers have investigated so many theoretical aspects of MPC that it is a staple ingredient of innumerable journal and conference papers, and monographs. However, looking at the industrial world, the two control techniques that appear to find extensive real application seem to be PID control for simple applications and MPC for the more complicated situations. In common with the ubiquitous PID controller, MPC has intuitive depth that makes it easily understood and used by industrial control engineers. For these reasons alone, the study of PID control and MPC cannot be omitted from today's modern control course.

This is not to imply that all the theoretical or computational problems in MPC have been solved or are even straightforward. But it is the adding in of more process properties that leads to a need for a careful analysis of the MPC technique. This is the approach of this *Advanced Textbooks in Control and Signal Processing* entry entitled *Predictive Control: Classical, Robust and Stochastic* by Basil Kouvaritakis and Mark Cannon. The authors' work on predictive control at Oxford has been carried out over a long period and they have been very influential in stimulating interest in new algorithms for both linear and nonlinear systems.

Divided into three parts, the text considers linear system models subject to process-output and control constraints. The three parts are as follows: "Classical" refers to deterministic formulations, "Robust" incorporates uncertainty to the system description and "Stochastic" considers system uncertainty that has probabilistic properties. In the presence of constraints, the authors seek out conditions for closed-loop system stability, control feasibility, convergence and algorithmic computable control solutions. The series editors welcome this significant contribution to the MPC textbook literature that is also a valuable entry to the *Advanced Textbooks in Control and Signal Processing* series.

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Preface

One of the motivations behind this book was to collect together the many results of the Oxford University predictive control group. For this reason we have, rather unashamedly, included a number of ideas that were developed at Oxford and in this sense some of the discussions in the book are included as background material that some readers may wish to skip on an initial reading. Elsewhere, however, the preference for our own methodology is quite deliberate on account of the distinctive nature of some of the Oxford results. Thus, for example, in Stochastic MPC our attention is focussed on algorithms with guaranteed control theoretic properties, including that of recurrent feasibility. On account of this, contrary to common practice, we often eschew the normal distribution, which despite its mathematical convenience neither lends itself to the proof of stability and feasibility, nor does it allow accurate representations of model and measurement uncertainties, as these rarely assume arbitrarily large values. On the other hand, we have clearly attempted to incorporate all the major developments in the field, some of which are rather recent and as yet may not be widely known. We apologise to colleagues whose work did not get a mention in our account of the development of MPC; mostly this is due to fact that we had to be selective of our material so as to give a fuller description over a narrower range of concepts and techniques.

Over the past few decades the state of the art in MPC has come closer to the optimum trade-off between computation and performance. But it is still nowhere near close enough for many control problems. In this respect the field is wide open for researchers to come up with fresh ideas that will help bridge the gap between ideal performance and what is achievable in practice.

October 2014

Basil Kouvaritakis
Mark Cannon

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Part III Stochastic MPC

Chapter 1
Introduction

The benefits of feedback control have been known to mankind for more than 2,000 years and examples of its use can be found in ancient Greece, notably the float regulator of the water clock invented by Ktesibios in about 270 BC [1]. The formal development of the field as a mathematical tool for the analysis of the behaviour of dynamical systems is much more recent, beginning around 150 years ago when Maxwell published his work on governors [2]. Since then the field has seen spectacular developments, promoted by the work of mathematicians, engineers and physicists. Laplace, Lyapunov, Kolmogorov, Wiener, Nyquist, Bode, Bellman are just a few of the major contributors to the edifice of what is known today as control theory.

A development of particular interest, both from a theoretical point of view but also one that has enjoyed considerable success in terms of practical applications, is that of optimal control. Largely based on the work of Pontryagin [3] and Bellman [4], optimal control theory is an extension of the calculus of variations [5, 6] addressing the problem of optimizing a cost index that measures system performance through the choice of system parameters that are designated as control inputs. The appeal of this work from a control engineering perspective is obvious because it provides a systematic approach to the design of strategies that achieve optimal performance. Crucially, the optimal control solution has the particularly simple form of linear state feedback for the case of linear systems and quadratic cost functions, and the feedback gains can be computed by solving an equation known as the steady-state Riccati equation [7]. This applies to both continuous time systems described by sets of differential equations and to discrete time systems formulated in terms of difference equation models. The latter description is, of course, of special importance to modern applications, which are almost entirely implemented using digital microprocessors.

The benefits of optimal control are, however, difficult to achieve in the case of systems with nonlinear models and systems that are subject to constraints on input variables or model states. For both these cases, in general, it is not possible to derive analytic expressions for the optimal control solution. Given the continuing

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improvements in the processing power of inexpensive microprocessors, one might hope that optimal solutions could be computed numerically. However, the associated optimization problem is difficult to solve for all but the simplest cases, and hence it is impracticable for the majority of realistic control problems. In the pursuit of optimality one is therefore forced to consider approximate solutions, and this is perhaps the single most important reason behind the phenomenal success of model predictive control (MPC). MPC is arguably the most widely accepted modern control strategy because it offers, through its receding horizon implementation, an eminently sensible compromise between optimality and speed of computation.

The philosophy of MPC can be described simply as follows. Predict future behaviour using a system model, given measurements or estimates of the current state of the system and a hypothetical future input trajectory or feedback control policy. In this framework future inputs are characterized by a finite number of degrees of freedom, which are used to optimize a predicted cost. Only the first control input of the optimal control sequence is implemented, and, to introduce feedback into this strategy, the process is repeated at the next time instant using newly available information on the system state. This repetition is instrumental in reducing the gap between the predicted and the actual system response (in closed-loop operation). It also provides a certain degree of inherent robustness to the uncertainty that can arise from imperfect knowledge or unknown variations in the model parameters (referred to as multiplicative uncertainty), as well as to model uncertainty in the form of disturbances appearing additively in the system dynamics (referred to as additive uncertainty).

Many early MPC strategies took account of predicted system behaviour over a finite horizon only and therefore lacked guarantees of nominal stability (i.e. closed-loop stability in the absence of any uncertainty). This difficulty was initially overcome by imposing additional conditions, known as equality terminal constraints, on the predicted model states. Such conditions were chosen in order to ensure that the desired steady state was reached at the end of a finite prediction horizon. The effect of these constraints was to render a finite horizon equivalent to an infinite horizon, thereby ensuring various stability and convergence properties.

Imposing the requirement that predicted behaviour reaches steady state over a finite future horizon is in general an overly-stringent requirement, and furthermore it presents computational challenges in the case of systems described by nonlinear models. Instead it was proposed that a stabilizing feedback law could be used to define the predicted control inputs at future times beyond the initial, finite prediction horizon. This feedback law is known as a terminal control law, and is often taken to be the optimal control law for the actual system dynamics in the absence of constraints (if that is available), or otherwise can be chosen as the optimal control law for the unconstrained, linearized dynamics about the desired steady state. To ensure that this control law meets the system constraints, thereby ensuring the future feasibility of the receding horizon strategy, additional constraints known as terminal constraints are imposed. These typically require that the system state at the end of the initial finite prediction horizon should belong to a subset of state space with the property that once entered, the state of the constrained system will never leave it.

We refer to the MPC algorithms that are derived from the collection of ideas discussed above as Classical MPC. These cause the controlled system, in closed-loop operation, to be stable, to meet constraints and to converge (asymptotically) to the desired steady state. However, it is often of paramount importance that a controller should have an acceptable degree of robustness to model uncertainty. This constitutes a much more challenging control problem. We refer to the case in which the uncertainty has known bounds but no further information is assumed as Robust MPC (or RMPC), and to the case in which model uncertainty is assumed to be random with known probability distribution, and where some or all of the constraints are probabilistic in nature, as Stochastic MPC (or SMPC).

Thus let the state model of a system be $x^+ = Ax + Bu + Dw$, where x and x^+ denote, respectively, the current model state and the successor state (i.e. the state at the next time instant), u is the vector of control inputs and w represents an unknown vector of external disturbance inputs. For such a model it may be the case that the numerical values of the elements of the matrices A, B, D are not known (or indeed knowable) precisely; this corresponds to the case of multiplicative uncertainty. The model parameters may however be known to lie in particular intervals, whether they are constant or vary with time, in which case the uncertainty is bounded by a known set of values. Typically these uncertainty sets will be polytopic sets defined by known vertices or by a number of linear inequalities. To give an example of this, consider the payloads of a robotic arm that may differ from time to time, depending on the task performed. This naturally leads to uncertainty which can be modelled (albeit conservatively) as multiplicative bounded uncertainty in a linear model. Similarly, the additive disturbance representing, for example, torques arising from static friction in the robot arm system discussed above, though unknown, will lie in a bounded set of additive uncertainty.

Such a problem would almost certainly be subject to constraints implied by maxima on possible applied motor torques, or consideration of safety and/or singularities which impose limits on the angular positions of the various links of the robotic arm. A general linear representation of such constraints, whether they apply only to the control input, or to the state, or are mixed input and state constraints, is $Fx + Bu \leq 1$. The concern then for RMPC would be to guarantee stability, constraint satisfaction and convergence of the state vector to a given steady-state condition or set of states, for all possible realizations of uncertainty.

In a number of applications, however, uncertainty is subject to some statistical regularity and can be modelled as random but with known probability distribution. Thus consider the problem of controlling the pitch of the blades of a wind turbine with the aim of maximizing electrical power generation while at the same time limiting the fatigue damage to the turbine tower due to fore-and-aft movement caused by fluctuations in the aerodynamic forces experienced by the blades. Although the wind speed is subject to random variations, it can be modelled in terms of given probability distributions. Furthermore, fatigue damage occurs when tower movement exceeds given limits frequently. Therefore the implied constraint is not that extreme tower movement is not allowed but rather that it happens with a probability which is below a given threshold. This situation cannot be described by a hard constraint such as

$Fx + Gu \leq 1$, but can more conveniently be modelled by probabilistic constraints of the form $\Pr\{Fx + Gu \leq 1\} \leq p$ where p represents a given probability. It is the object of Stochastic MPC to ensure that such constraints, together with any additional hard constraints that may be present, are met in closed-loop operation, and to simultaneously stabilize the system, for example by causing the state to converge to a given steady-state set.

Classical, Robust and Stochastic MPC are the main topics of this book; the three distinct parts of the book discuss each of these in turn. Our tendency has been in each part of the book to start with background material that helps define the basic concepts and then progressively present more sophisticated algorithms. By and large these are capable of affording advantages over the earlier presented results in terms of ease of computation, or breadth of applicability in terms of the size of their allowable set of initial conditions or degree of optimality of performance. This in a way is a reflection of the development of the field as a whole, which continuously aspires for optimality tempered with ease of implementation.

This book only explicitly addresses systems with linear dynamics, but the reader should be aware that most of the results presented have obvious extensions to the nonlinear case, provided certain assumptions about convexity are satisfied. There are clear implications of course in terms of ease of online computation, but the hope is that with the increasing speed and storage capabilities of computing hardware for control, this aspect will become less significant. Below, we give a brief description of the salient features to be presented in each of the three parts of the book.

1.1 Classical MPC

The first part comprises a single chapter that describes developments concerning the nominal case only, but simultaneously lays the foundations for the remainder of the book by introducing key concepts (e.g. set invariance, recurrent feasibility, Lyapunov stability, etc.). It begins with the problem definition and an unconventional derivation of the optimal unconstrained control law; this is done deliberately so as to avoid repeating the classical presentation of ideas of calculus of variations and Pontryagin's maximum principle. It then moves on to describe the dual mode prediction setting. Some readers may object to the use of the term "dual mode" here since it is sometimes reserved to describe a control strategy that switches between two different control laws (e.g. when the system state transitions into a particular region of state space around the desired steady state). Our use of the term "dual mode" refers instead to the split of the predicted control sequence into the control inputs at the first N predicted time-steps (which are not predetermined) and those that apply to the remainder of the prediction horizon (which are fixed by a terminal feedback law). The terms mode 1 and mode 2 constitute in our opinion a useful and intuitive shorthand for the two modes of predicted operation.

Discussion then turns to set invariance, which in the first instance is introduced in connection with a terminal control law. Subsequently this is developed into the concept of controlled invariance, which is a convenient concept for describing the feasibility properties of the system constraints combined with terminal constraints. Instrumental in this and in the attendant proof of stability is the idea of what we call, for lack of a better descriptive terminology, the "tail". This constitutes a method of extending a future trajectory computed at any given time instant to a subsequent time instant, and provides a convenient tool for establishing both recurrent feasibility and a monotonic non-increasing property of the cost. The former relates to the property that feasibility at current time implies feasibility at the next and subsequent instants, whereas the latter can be used to establish closed-loop stability.

A formulation of interest concerns an alternative representation of the prediction dynamics that makes use of a lifted autonomous state space model. Here the state is augmented to include the degrees of freedom in the control sequence of mode 1; this provides a framework that expedites many of the arguments presented in the subsequent parts of the book, and it also leads to a particularly efficient implementation of nominal MPC. Moreover, it enables the design of prediction dynamics that can be optimized, for example with the aim of maximizing the size of the set of allowable initial conditions.

The chapter also considers aspects of computation and presents early MPC algorithms, one of which enables the introduction of a Youla parameter into the MPC problem.

1.2 Robust MPC

The presence of bounded uncertainty leads to a generally more challenging control problem which is addressed by RMPC and is discussed in the second part of the book. We begin by considering the case of additive disturbances, examining first a class of MPC strategies that perform prediction optimization over open-loop input trajectories. These are distinct from strategies employing optimization over control policies, which in essence are closed-loop prediction strategies in that they take account of future realizations of uncertainty which, though not known to the controller at current time, will be available when the control law is computed at a future time.

We begin our account by describing a state decomposition into nominal and uncertain components. This, in conjunction with an augmented predicted state model, enables the treatment of robust invariance and recursive feasibility and also suggests convenient ways to define and compute maximal and minimal robust invariant sets. Then, using the dual mode prediction paradigm and a nominal predicted cost, it is possible to develop an RMPC strategy with guaranteed stability and convergence properties.

Next we consider a game theoretic approach, a control strategy that is revisited later when the case of mixed additive and multiplicative uncertainty is examined. This approach uses a min-max optimization in which the cost is defined so as to set up a dynamic game between the uncertainty, over which RMPC performs a maximization, and the controller, which selects the control input by minimizing the maximum cost. Over and above the usual control theoretic properties of recursive feasibility and convergence to some steady-state set, this approach also provides a quantification of the disturbance rejection properties of RMPC.

The exposition then moves on to the tube RMPC methodology that appears to have dominated the relevant literature over the past 15 years or so. According to this, constraints are enforced by inclusion conditions that ensure the uncertain future state and input trajectories lie in sequences of sets, known as tubes, which are contained entirely within sets in which constraints of the control problem are satisfied. The sets defining the tubes were originally taken to be low-complexity polytopic sets (i.e. affine transformations of hypercubes), but were later replaced by more general sets which are either fixed or scalable. Such tubes can be used to guarantee recursive feasibility through the use of suitably defined constraints. The topic of open-loop strategies for the additive uncertainty case is brought to a close through a review of some early Tube RMPC strategies that deployed tubes with low-complexity polytopic cross sections as well as a review of an RMPC strategy that achieved recurrent feasibility with respect to the entire class of additive uncertainty through an artificial tightening of constraints.

Open-loop strategies are computationally convenient but can be conservative since they ignore information about future uncertainty that, though not available at current time, will be available to the controller. For a given prediction horizon and terminal control law, the best possible performance and the largest set of admissible initial conditions is obtained by optimizing a multistage min-max control problem over all feedback policies. This problem, and its solution through the use of dynamic programming (DP), is considered next in the book. The drawback of this approach and its implementation within an MPC framework is that computation grows exponentially with the prediction horizon and system dimension.

The optimal solution can be shown to be an affine function of the system state which is dictated by the set of active constraints. Thus, in theory, one could potentially compute this function offline at the regions of the state space defined by different sets of active constraints. The number of such regions however typically grows exponentially with the size of the system and therefore this approach is not practicable for anything other than low order systems and short prediction horizons. To avoid this problem it is possible to use an approach based on an online interpolation between the current state and a state at which the optimal control law is known. The active constraint set is updated during this interpolation and an equality constrained optimization problem is solved at each active set change. Although this active set dynamic programming approach can lead to significant reductions in online computation, it still requires the offline computation of controllability sets which may be computationally demanding. In such cases it may be preferable to perform the optimization over a restricted class of control policies, since this may provide a good approximation of the optimal solution at a fraction of the computational cost.

One such policy employs a feedback parameterization with an affine dependence on past disturbance inputs, resulting in a convex optimization problem in a number of variables that grows quadratically with the prediction horizon. Through the use of a separable prediction scheme with a triangular structure it is possible to provide disturbance feedback that is piecewise affine (rather than simply affine) in the disturbance. The resulting RMPC algorithm, Parameterized Tube MPC (PTMPC), provides a greater degree of optimality for a similar computational load, with the number of optimization variables again depending quadratically on the prediction horizon. Computation can be reduced by using a scheme with a striped triangular structure (for which the dependence of the number of optimization variables on horizon length is linear) rather than a triangular prediction scheme. With this approach it is possible to outperform PTMPC since the effects of the striped prediction structure are allowed to extend into mode 2, thus effectively replacing the fixed terminal law by one that is optimized online.

We next consider RMPC in the presence of multiplicative uncertainty. Early work on this topic used a parameterization of predicted control inputs in terms of a linear state feedback law in which the feedback gain is taken to be an optimization variable computed online at each time instant. The approach uses quadratic constraints, expressed as linear matrix inequalities, to ensure that the predicted state is contained in ellipsoids within which constraints are satisfied. These ellipsoidal sets also guarantee a monotonic non-increasing property for the predicted cost for a polytopic class of uncertainty. The prediction structure of this approach was subsequently enriched by introducing additional optimization variables in the form of a perturbation sequence applied to the predicted linear feedback law. However, this increases the required online computation considerably, making it overly demanding for high order systems or systems with many uncertain parameters. A very significant reduction in online computation can be achieved by using a lifted autonomous model for the prediction dynamics. This latter approach also enables the dynamics defining the predicted state and input trajectories to be optimized through an offline optimization, thus maximizing the volume of an ellipsoidal region of attraction.

As in the case of additive uncertainty, multiplicative uncertainty can be handled conveniently through the introduction of tubes defining the evolution of the predicted state and control trajectories. This is considered next, first with tubes consisting of low-complexity polytopic sets, and then with general polytopic tube cross sections through appropriate use of Farkas' lemma. A combination of these ideas with the lifted optimized dynamics is discussed for the derivation of a min-max RMPC algorithm for the case of mixed additive and multiplicative uncertainty.

1.3 Stochastic MPC

The final part of this book is dedicated to SMPC which deals with the case when uncertainty is random, with known probability distribution, rather than simply being known to lie in a given set. This opens up the possibility of incorporating statistical information in the definition of optimal performance, and of allowing some or all of the constraints to be of a probabilistic nature. Early versions of SMPC were concerned with the unconstrained case and considered the minimization of the one-step-ahead variance or the expected value of a quadratic cost over a given prediction horizon. However, use of only expected values removes much of the stochastic nature of the problem, so these early algorithms, although historically important, cannot really be classified as SMPC.

We begin our treatment of SMPC by defining the stochastic system models together with the probabilistic constraints and discussing the basic assumption of mean-square stability. Use is made of the dual mode prediction paradigm and the unconstrained optimal control law is developed for a particular form of predicted cost. This is then followed by the treatment of a mean-variance predicted cost SMPC. The discussion is concluded by a review of some earlier stochastic predictive control work, namely Minimum Variance Control and MPC through the use of moving average models in conjunction with chance constraints. Also included is a description of earlier work based on a fully stochastic formulation in which both the constraints and the cost are defined using the probabilistic information about the uncertainty. This approach was developed in the context of a sustainable development problem which is also discussed. Such a formulation appears to be eminently appropriate for a problem with such a strong stochastic element: it involves a prediction horizon, which by definition has to be the inter-generational gap, and over which it is unrealistic to model the uncertainties of world economy in a deterministic manner.

The following chapter presents useful tools for the construction of SMPC algorithms such as recursive feasibility, supermartingale convergence analysis, probabilistic invariance and Markov chain models. Using these ingredients, SMPC algorithms are proposed using an expectation cost and a mean variance cost as well as algorithms based on probabilistically invariant ellipsoids and algorithms based on tubes with polytopic cross sections constructed on the basis of Markov chain models.

Our discussion of SMPC concludes by considering algorithms that use tubes with polytopic and elliptical cross sections constructed on the basis of information on the probability distributions of additive and multiplicative uncertainty. One feature of these algorithms is that they achieve recursive feasibility by treating, at any prediction time, the uncertainty up to the previous prediction time robustly. This is because, at any given time, all earlier realizations of uncertainty will have already occurred and will have taken any allowable value in the uncertainty class. We emphasize that our preference is for uncertainty with finite support, despite the mathematical convenience of distributions such as the Gaussian distribution. In general uncertainty distributions with infinite support do not accord well with realistic applications, where model uncertainty never assumes arbitrarily large values. Moreover, assumptions of unbounded model uncertainty preclude the possibility of establishing control theoretic properties such as stability and feasibility. Consideration is also given to SMPC which addresses constraints on the average number of constraint violations. The case of multiplicative uncertainty poses interesting problems in respect of the online calculation of probability distributions of predicted states. We discuss how this difficulty can be overcome through the use of techniques based on numerical integration and random sampling.

1.4 Concluding Remarks and Comments on the Intended Readership

In summary, predictive control has experienced a phenomenal amount of development, and this has been matched by wide acceptability in practice. Classical MPC is now a mature research area in which further major developments are, in our opinion, unlikely. The same however is certainly not true of RMPC, where the race is still on for the development of approaches that provide an improved balance between optimality and practicability of implementation. It is to be expected that fresh ideas about the structure of predictions will emerge that will narrow the gap between what can realistically be implemented and the ideal optimal solution. SMPC has itself seen some significant advances but is still more in a state of flux, even in respect of what its aims ought to be. This area will undoubtedly see in the future several major stages of development. It is our hope that this book will seed some of the ideas that will make this possible.

The levels of difficulty and technical detail of the book differ from chapter to chapter. The intention is that Chap. 2 should be accessible to all undergraduates specializing in control, whereas Chaps. 2, 3 and 5 should be of interest to graduate students of control. With that in mind we have provided exercises (with solutions) to these chapters in the hope that the students will be able to test their understanding by solving the given problems. This can be done either as a paper-and-pencil exercise, or with the aid of mathematical software such as MATLAB. It is anticipated that Chaps. 4 and 6–8 would be read mostly by research students, researchers and academics. The technical level here is more involved and testing one's understanding could only be attempted using rather sophisticated suites of (MATLAB) programs and for this reason we have refrained from providing exercises for these chapters.

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Part I
Classical MPC

Chapter 2
MPC with No Model Uncertainty

2.1 Problem Description

This section provides a review of some of the key concepts and techniques in classical MPC. Here the term "classical MPC" refers to a class of control problems involving linear time invariant (LTI) systems whose dynamics are described by a discrete time model that is not subject to any uncertainty, either in the form of unknown additive disturbances or imprecise knowledge of the system parameters. In the first instance the assumption will be made that the system dynamics can be described in terms of the LTI state-space model

$$x_{k+1} = Ax_k + Bu_k \tag{2.1a}$$

$$y_k = Cx_k \tag{2.1b}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$ are, respectively, the system state, the control input and the system output, and k is the discrete time index. If the system to be controlled is described by a model with continuous time dynamics (such as an ordinary differential equation), then the implicit assumption is made here that the controller can be implemented as a sampled data system and that (2.1a) defines the discrete time dynamics relating the samples of the system state to those of its control inputs.

Assumption 2.1 Unless otherwise stated, the state x_k of the system (2.1a) is assumed to be measured and made available to the controller at each sampling instant $k = 0, 1, \ldots$

The controlled system is also assumed to be subject to linear constraints. In general these may involve both states and inputs and are expressed as a set of linear inequalities

$$Fx + Gu \leq \mathbf{1} \tag{2.2}$$

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where $F \in \mathbb{R}^{n_C \times n_x}$, $G \in \mathbb{R}^{n_C \times n_u}$ and the inequality applies elementwise. We denote by $\mathbf{1}$ a vector with elements equal to unity, the dimension of which is context dependent, i.e. $\mathbf{1} = [1 \cdots 1]^T \in \mathbb{R}^{n_C}$ in (2.2). Setting F or G to zero results in constraints on inputs or states alone. A feasible pair (x_k, u_k) or feasible sequence $\{(x_0, u_0), (x_1, u_1), \ldots\}$ for (2.2) is any pair or sequence satisfying (2.2). The constraints in (2.2) are symmetric if $(-x_k, -u_k)$ is feasible whenever (x_k, u_k) is feasible, and non-symmetric otherwise. Although the form of (2.2) does not encompass constraints involving states or inputs at more than one sampling instant (such as, for example rate constraints or more general dynamic constraints), these can be handled through a suitable and obvious extension of the results to be presented.

The classical regulation problem is concerned with the design of a controller that drives the system state to some desired reference point using an acceptable amount of control effort. For the case that the state is to be steered to the origin, the controller performance is quantified conveniently for this type of problem by a quadratic cost index of the form

$$J\left(x_0, \{u_0, u_1, u_2 \ldots\}\right) \doteq \sum_{k=0}^{\infty} \left(\|x_k\|_Q^2 + \|u_k\|_R^2 \right). \tag{2.3}$$

Here $\|v\|_S^2$ denotes the quadratic form $v^T S v$ for any $v \in \mathbb{R}^{n_v}$ and $S = S^T \in \mathbb{R}^{n_v \times n_v}$, and Q, R are weighting matrices that specify the emphasis placed on particular states and inputs in the cost. We assume that R is a symmetric positive-definite matrix (i.e. the eigenvalues of R are real and strictly positive, denoted $R \succ 0$) and that Q is symmetric and positive semidefinite (all eigenvalues of Q are real and non-negative, denoted $Q \succeq 0$). This allows, for example, the choice $Q = C^T Q_y C$ for some positive-definite matrix Q_y, which corresponds to the case that the output vector, y, rather than the state, x, is to be steered to the origin. At time k, the optimal value of the cost (2.3) with respect to minimization over admissible control sequences $\{u_k, u_{k+1}, u_{k+2}, \ldots\}$ is denoted

$$J^*(x_k) \doteq \min_{u_k, u_{k+1}, u_{k+2}, \ldots} J\big(x_k, \{u_k, u_{k+1}, u_{k+2} \ldots\}\big).$$

This problem formulation leads to an optimal control problem whereby the controller is required to minimize at time k the performance cost (2.3) subject to the constraints (2.2). To ensure that the optimal value of the cost is well defined, we assume that the state of the model (2.1) is stabilizable and observable.

Assumption 2.2 In the system model (2.1) and cost (2.3), the pair (A, B) is stabilizable, the pair (A, Q) is observable, and R is positive-definite.

Given the linear nature of the controlled system, the problem of setpoint tracking (in which the output y is to be steered to a given constant setpoint) can be converted into the regulation problem considered here by redefining the state of (2.1a) in terms of the deviation from a desired steady-state value. The more general case of tracking a time-varying setpoint (e.g. a ramp or sinusoidal signal) can also be tackled within the framework outlined here provided the setpoint can itself be generated by applying a constant reference signal to a system with known LTI dynamics.

2.2 The Unconstrained Optimum

The problem of minimizing the quadratic cost of (2.3) in the unconstrained case (i.e. when $F = 0$ and $G = 0$ in (2.2)) is addressed by Linear Quadratic (LQ) optimal control, which forms an extension of the calculus of variations. The solution is usually obtained either using Pontryagin's Maximum Principle [1] or Dynamic Programming and the recursive Bellman equation [2]. Rather than replicating these solution methods, here we first characterize the optimal linear state feedback law that minimizes the cost of (2.3), and later show (in Sect. 2.7) through a lifting formulation that this control law is indeed optimal over all input sequences.

We first obtain an expression for the cost under linear feedback, $u = Kx$, for an arbitrary stabilizing gain matrix $K \in \mathbb{R}^{n_u \times n_x}$, using the closed-loop system dynamics

$$x_{k+1} = (A + BK)x_k$$

to write $x_k = (A + BK)^k x_0$ and $u_k = K(A + BK)^k x_0$, for all k. Therefore $J(x_0) = J(x_0, \{Kx_0, Kx_1, \ldots\})$ is a quadratic function of x_0,

$$J(x_0) = x_0^T W x_0, \tag{2.4a}$$

$$W = \sum_{k=0}^{\infty} (A + BK)^{k^T} (Q + K^T RK)(A + BK)^k. \tag{2.4b}$$

If $A + BK$ is strictly stable (i.e. each eigenvalue of $A + BK$ is strictly less than unity in absolute value), then it can easily be shown that the elements of the matrix W defined in (2.4b) are necessarily finite. Furthermore, if R is positive-definite and (A, Q) is observable, then $J(x_0)$ is a positive-definite function of x_0 (since then $J(x_0) \geq 0$, for all x_0, and $J(x_0) = 0$ only if $x_0 = 0$), which implies that W is a positive-definite matrix.

The unique matrix W satisfying (2.4) can be obtained by solving a set of linear equations rather than by evaluating the infinite sum in (2.4b). This is demonstrated by the following result, which also shows that $(A + BK)$ is necessarily stable if W in (2.4) exists.

Lemma 2.1 (Lyapunov matrix equation) *Under Assumption 2.2, the matrix W in (2.4) is the unique positive definite solution of the Lyapunov matrix equation*

$$W = (A + BK)^T W(A + BK) + Q + K^T RK \tag{2.5}$$

if and only if $A + BK$ is strictly stable.

Proof Let W_n denote the sum of the first n terms in (2.4b), so that

$$W_n \doteq \sum_{k=0}^{n-1} (A + BK)^{k^T} (Q + K^T RK)(A + BK)^k.$$

Then $W_1 = Q + K^T RK$ and $W_{n+1} = (A + BK)^T W_n (A + BK) + Q + K^T RK$ for all $n > 0$. Assuming that $A + BK$ is strictly stable and taking the limit as $n \to \infty$, we obtain (2.5) with $W = \lim_{n\to\infty} W_n$. The uniqueness of W satisfying (2.5) is implied by the uniqueness of W_{n+1} in this recursion for each $n > 0$, and $W \succ 0$ follows from the positive-definiteness of $J(x_0)$.

If we relax the assumption that $A + BK$ is strictly stable, then the existence of $W \succ 0$ satisfying (2.5) implies that there exists a Lyapunov function demonstrating that the system $x_{k+1} = (A + BK)x_k$ is asymptotically stable, since (A, Q) is observable and $R \succ 0$ by Assumption 2.2. Hence $A + BK$ must be strictly stable if (2.5) has a solution $W \succ 0$. □

The optimal unconstrained linear feedback control law is defined by the stabilizing feedback gain K that minimizes the cost in (2.3) for all initial conditions $x_0 \in \mathbb{R}^{n_x}$. The conditions for an optimal solution to this problem can be obtained by considering the effect of perturbing the value of K on the solution, W, of the Lyapunov equation (2.5). Let $W + \delta W$ denote the sum in (2.4b) when K is replaced by $K + \delta K$. Then $W + \delta W$ and $K + \delta K$ satisfy the Lyapunov equation

$$\begin{aligned} W + \delta W = \big[A + B(K + \delta K)\big]^T (W + \delta W)\big[A + B(K + \delta K)\big] \\ + Q + (K + \delta K)^T R(K + \delta K) \end{aligned}$$

which, together with (2.5), implies that δW satisfies

$$\begin{aligned} \delta W = {} & \delta K^T \big[B^T W(A + BK) + RK\big] + \big[(A + BK)^T WB + K^T R\big]\delta K \\ & + (A + BK)^T \delta W(A + BK) + \delta K^T (B^T WB + R)\delta K \\ & + \delta K^T B^T \delta W(A + BK) + (A + BK)^T \delta W\ B\ \delta K + \delta K^T B^T \delta W\ B\ \delta K. \end{aligned} \tag{2.6}$$

For given $\delta K_1 \in \mathbb{R}^{n_u \times n_x}$, consider a perturbation of the form

$$\delta K = \epsilon\, \delta K_1,$$

and consider the effect on δW of varying the scaling parameter $\epsilon \in \mathbb{R}$. Clearly K is optimal if and only if $x_0^T (W + \delta W)x_0 \geq x_0^T W x_0$, for all $x_0 \in \mathbb{R}^{n_x}$, for all

$\delta K_1 \in \mathbb{R}^{n_u \times n_x}$ and for all sufficiently small ϵ. It follows that K is optimal if and only if the solution of (2.6) has the form

$$\delta W = \epsilon^2\,\delta W_2 + \epsilon^3\,\delta W_3 + \cdots$$

for all $\epsilon \in \mathbb{R}$, where δW_2 is a positive semidefinite matrix. Considering terms in (2.6) of order ϵ and order ϵ^2, we thus obtain the following necessary and sufficient conditions for optimality:

$$B^T W(A + BK) + RK = 0, \tag{2.7a}$$

$$\delta W_2 \succeq 0, \tag{2.7b}$$

$$\delta W_2 = (A + BK)^T \delta W_2 (A + BK) + \delta K_1^T (B^T W B + R)\delta K_1. \tag{2.7c}$$

Solving (2.7a) for K gives $K = -(B^T W B + R)^{-1} B^T W A$ as the optimal feedback gain, whereas Lemma 2.1 and (2.7c) imply that

$$\delta W_2 = \sum_{k=0}^{\infty} (A + BK)^{k\,T} \delta K_1^T (B^T W B + R)\delta K_1 (A + BK)^k$$

and therefore (2.7b) is necessarily satisfied since $A + BK$ is strictly stable and $B^T W B + R$ is positive-definite.

These arguments are summarized by the following result.

Theorem 2.1 (Discrete time algebraic Riccati equation) *The feedback gain matrix K for which the control law*

$$u = Kx$$

minimizes the cost of (2.3) for any initial condition x_0 under the dynamics of (2.1a) is given by

$$K = -(B^T W B + R)^{-1} B^T W A, \tag{2.8}$$

where $W \succ 0$ is the unique solution of

$$W = A^T W A + Q - A^T W B (B^T W B + R)^{-1} B^T W A. \tag{2.9}$$

Under Assumption 2.2, $A + BK$ is strictly stable whenever there exists $W \succ 0$ satisfying (2.9).

Proof The optimality of (2.8) is a consequence of the necessity and sufficiency of the optimality conditions in (2.7a), (2.7b) and (2.7c). Equation (2.9) (which is known as the discrete time algebraic Riccati equation) is obtained by substituting K in (2.8) into (2.5). From Lemma 2.1, we can conclude that, under Assumption 2.2, the solution of (2.9) for W is unique and positive-definite if and only if $A + BK$ is strictly stable. □

2.3 The Dual-Mode Prediction Paradigm

The control law that minimizes the cost (2.3) is not in general a linear feedback law when constraints (2.2) are present. Moreover, it may not be computationally tractable to determine the optimal controller as an explicit state feedback law. Predictive control strategies overcome this difficulty by minimizing, subject to constraints, a predicted cost that is computed for a particular initial state, namely the current plant state. This constrained minimization of the predicted cost is solved online at each time step in order to derive a feedback control law. The predicted cost corresponding to (2.3) can be expressed

$$J(x_k, \{u_{0|k}, u_{1|k}, \ldots\}) = \sum_{i=0}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) \tag{2.10}$$

where $x_{i|k}$ and $u_{i|k}$ denote the predicted values of the model state and input, respectively, at time $k + i$ based on the information that is available at time k, and where $x_{0|k} = x_k$ is assumed.

The prediction horizon employed in (2.10) is infinite. Hence if every element of the infinite sequence of predicted inputs $\{u_{0|k}, u_{1|k}, \ldots\}$ were considered to be a free variable, then the constrained minimization of this cost would be an infinite-dimensional optimization problem, which is in principle intractable. However predictive control strategies provide effective approximations to the optimal control law that can be computed efficiently and in real time. This is possible because of a parameterization of predictions known as the dual-mode prediction paradigm, which enables the MPC optimization to be specified as a finite-dimensional problem.

The dual-mode prediction paradigm divides the prediction horizon into two intervals. Mode 1 refers to the predicted control inputs over the first N prediction time steps for some finite horizon N (chosen by the designer), while mode 2 denotes the control law over the subsequent infinite interval. The mode 2 predicted inputs are specified by a fixed feedback law, which is usually taken to be the optimum for the problem of minimizing the cost in the absence of constraints [3–6]. Therefore the predicted cost (2.10) can be written as

$$J(x_k, \{u_{0|k}, u_{1|k}, \ldots\}) = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_W^2 \tag{2.11}$$

where, by Theorem 2.1, W is the solution of the Riccati equation (2.9). The term $\|x_{N|k}\|_W^2$ is referred to as a terminal penalty term and accounts for the cost-to-go after N prediction time steps under the mode 2 feedback law.

To simplify notation we express the predicted cost as an explicit function of the initial state of the prediction model and the degrees of freedom in predictions. Hence for the dual-mode prediction paradigm in which the control inputs over the prediction horizon of mode 1 are optimization variables, we write (2.11) as

$$J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_W^2. \tag{2.12}$$

where $\mathbf{u}_k = \{u_{0|k}, u_{1|k}, \ldots, u_{N-1|k}\}$.

The receding horizon implementation of MPC stipulates that at each time instant k the optimal mode 1 control sequence $\mathbf{u}_k^* = \{u_{0|k}^*, \ldots, u_{N-1|k}^*\}$ is computed, and only the first element of this sequence is implemented, namely $u_k = u_{0|k}^*$. Thus at each time step the most up-to-date measurement information (embodied in the state x_k) is employed. This creates a feedback mechanism that provides some compensation for any uncertainty present in the model of (2.1a). It also reduces the gap between the optimal value of the predicted cost $J(x_k, \mathbf{u}_k)$ in (2.12) and the optimal cost for the infinite-dimensional problem of minimizing (2.10) over the infinite sequence of future inputs $\{u_{0|k}, u_{1|k}, \ldots\}$.

The rationale behind the dual-mode prediction paradigm is as follows. Let $\{u_{0|k}^0, u_{1|k}^0, \ldots\}$ denote the optimal control sequence for the problem of minimizing the cost (2.10) over the infinite sequence $\{u_{0|k}, u_{1|k}, \ldots\}$ subject to the constraints $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$, for all $i \ge 0$, for an initial condition $x_{0|k} = x_k$ such that this problem is feasible. If the weights Q and R satisfy Assumption 2.2, then this notional optimal control sequence drives the predicted state of the model (2.1a) asymptotically to the origin, i.e. $x_{i|k} \to 0$ as $i \to \infty$. Since $(x, u) = (0, 0)$ is strictly feasible for the constraints $Fx + Gu \le \mathbf{1}$, there exists a neighbourhood, $\mathcal{S}$, of $x = 0$ with the property that these constraints are satisfied at all times along trajectories of the model (2.1a) under the unconstrained optimal feedback law, $u = Kx$, starting from any initial condition in $\mathcal{S}$. Hence there necessarily exists a horizon N_∞ (which depends on x_k) such that $x_{i|k} \in \mathcal{S}$, for all $i \ge N_\infty$. Since the optimal trajectory for $i \ge N_\infty$ is necessarily optimal for the problem with initial condition $x_{N_\infty|k}$ (by Bellman's Principle of Optimality [7]), the constrained optimal sequence must therefore coincide with the unconstrained optimal feedback law, i.e. $u_{i|k}^0 = Kx_{i|k}$, for all $i \ge N_\infty$. It follows that if the mode 1 horizon is chosen to be sufficiently long, namely if $N \ge N_\infty$, then the mode 1 control sequence, $\mathbf{u}_k^*$, that minimizes the cost of (2.12) subject to the constraints $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$ for $i = 0, 1, \ldots, N-1$ must be equal to the first N elements of the infinite sequence that minimizes the cost (2.10), namely $u_{i|k}^* = u_{i|k}^0$ for $i = 0, \ldots, N-1$.

For completeness we next give a statement of this result; for a detailed proof and further discussion we refer the interested reader to [4, 5].

Theorem 2.2 *There exists a finite horizon N_∞, which depends on x_k, with the property that, whenever $N \geq N_\infty$: (i). the sequence $\mathbf{u}_k^*$ that achieves the minimum of $J(x_k, \mathbf{u}_k)$ in (2.12) subject to $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$ for $i = 0, 1, \ldots, N-1$ is equal to the first N terms of the infinite sequence $\{u^0_{0|k}, u^0_{1|k}, \ldots\}$ that minimizes $J(x_k, \{u_{0|k}, u_{1|k}, \ldots\})$ in (2.10) subject to $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$, for all $i \geq 0$; and (ii). $J(x_k, \mathbf{u}_k^*) = J(x_k, \{u^0_{0|k}, u^0_{1|k}, \ldots\})$.*

It is generally convenient to consider the LQ optimal feedback law $u = Kx$ as underlying both mode 1 and mode 2, and to introduce perturbations $c_{i|k} \in \mathbb{R}^{n_u}$, $i = 0, 1, \ldots, N-1$ over the horizon of mode 1 in order to meet constraints. Then the predicted sequence of control inputs is given by

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \qquad i = 0, 1, \ldots, N-1 \tag{2.13a}$$

$$u_{i|k} = Kx_{i|k}, \qquad i = N, N+1, \ldots \tag{2.13b}$$

with $x_{0|k} = x_k$. This prediction scheme is sometimes referred to as the closed-loop paradigm because the term Kx provides feedback in the horizons of both modes 1 and 2.

We argue in Sect. 3.1 (in the context of robustness to model uncertainty) that (2.13) should be classified as an open-loop prediction scheme because K is fixed rather than computed on the basis of measured information (namely x_k). Nevertheless, the feedback term Kx forms a pre-stabilizing feedback loop around the dynamics of (2.1a), which assume the form

$$x_{i+1|k} = \Phi x_{i|k} + Bc_{i|k}, \qquad i = 0, 1, \ldots, N-1 \tag{2.14a}$$

$$x_{i+1|k} = \Phi x_{i|k}, \qquad i = N, N+1, \ldots \tag{2.14b}$$

where $\Phi = A + BK$, with $x_{0|k} = x_k$. The strict stability property of Φ prevents numerical ill-conditioning that could arise in the prediction equations and the associated MPC optimization problem in the case of open-loop unstable models [8].

For the closed-loop paradigm formulation in (2.13), the predicted state trajectory can be generated by simulating (2.14a) forwards over the mode 1 prediction horizon, giving

$$\mathbf{x}_k = M_x x_k + M_c \mathbf{c}_k, \tag{2.14c}$$

where

$$\mathbf{x}_k \doteq \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix}, \quad \mathbf{c}_k \doteq \begin{bmatrix} c_{0|k} \\ \vdots \\ c_{N-1|k} \end{bmatrix}$$

$$M_x = \begin{bmatrix} \Phi \\ \Phi^2 \\ \vdots \\ \Phi^N \end{bmatrix}, \quad M_c = \begin{bmatrix} B & 0 & \cdots & 0 \\ \Phi B & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^{N-1} B & \Phi^{N-2} B & \cdots & B \end{bmatrix}.$$

On the basis of these prediction equations and the fact that the predicted cost over mode 2 is given by $\|x_{N|k}\|_W^2$ (where W is the solution of the Lyapunov equation (2.5)), the predicted cost of (2.11) can be written as a quadratic function of the degrees of freedom, namely the vector of predicted perturbations $\mathbf{c}_k$. The details of this computation are straightforward and will not be given here. Instead we derive an equivalent but more convenient form for the predicted cost in Sect. 2.7. For simplicity (but with a slight abuse of notation) in the following development, we denote the cost of (2.11) evaluated along the predicted trajectories of (2.13a) and (2.14a) as $J(x_k, \mathbf{c}_k)$, thus making explicit the dependence of the cost on the optimization variables $\mathbf{c}_k$.

2.4 Invariant Sets

The determination of the minimum prediction horizon N which ensures that the predicted state and input trajectories in mode 2 meet constraints (2.2) is not a trivial matter. Instead lower bounds for this horizon were proposed in [4, 5]. However such bounds could be conservative, leading to the use of unnecessarily long prediction horizons. This in turn could make the online optimization of the predicted cost computationally intractable as a result of large numbers of free variables and large numbers of constraints in the minimization of predicted cost. In such cases it becomes necessary to use a shorter horizon N while retaining the guarantee that predictions over mode 2 satisfy constraints on states and inputs. This can be done by imposing a terminal constraint which requires that the state at the end of the mode 1 horizon should lie in a set which is positively invariant under the dynamics defined by (2.13b) and (2.14b) and under the constraints (2.2).

Definition 2.1 (*Positively invariant set*) A set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is positively invariant under the dynamics defined by (2.13b) and (2.14b) and the constraints (2.2) if and only if $(F + GK)x \leq \mathbf{1}$ and $\Phi x \in \mathcal{X}$, for all $x \in \mathcal{X}$.

The use of invariant sets within the dual prediction mode paradigm is illustrated in Fig. 2.1 for a second-order system. The predicted state at the end of mode 1 is constrained to lie in an invariant set $\mathcal{X}_T$ via the constraint $x_{N|k} \in \mathcal{X}_T$. Thereafter, in

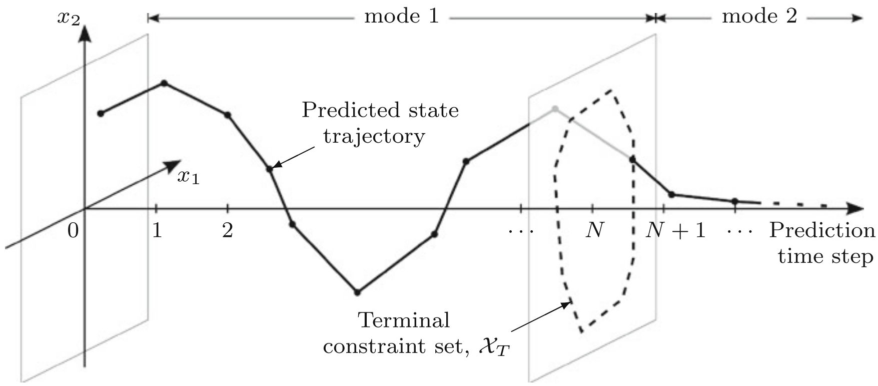

Fig. 2.1 The dual-mode prediction paradigm with terminal constraint. The control inputs in mode 1 are chosen so as to satisfy the system constraints as well as the constraint that the N step ahead predicted state should be inside the invariant set $\mathcal{X}_T$. Over the infinite mode 2 prediction horizon the predicted state trajectory is dictated by the prescribed feedback control law $u = Kx$

mode 2, the evolution of the state trajectory is that prescribed by the state feedback control law $u_k = Kx_k$.

In order to increase the applicability of the MPC algorithm, and in particular to increase the size of the set of initial conditions $x_{0|k}$ for which the terminal condition $x_{N|k} \in \mathcal{X}_T$ can be met, it is important to choose the maximal positively invariant set as the terminal constraint set. This set is defined as follows.

Definition 2.2 (*Maximal positively invariant set*) The maximal positively invariant (MPI) set under the dynamics of (2.13b) and (2.14b) and the constraints (2.2) is the union of all sets that are positively invariant under these dynamics and constraints.

It was shown in [9] that, for the case of linear dynamics and linear constraints considered here, the MPI set is defined by a finite number of linear inequalities. This result is summarized next.

Theorem 2.3 ([9]) *The MPI set for the dynamics defined by (2.13b) and (2.14b) and the constraints (2.2) can be expressed*

$$\mathcal{X}^{\text{MPI}} \doteq \{x : (F + GK)\Phi^i x \leq \mathbf{1},\ i = 0, \ldots, \nu\} \tag{2.15}$$

where ν is the smallest positive integer such that $(F + GK)\Phi^{\nu+1}x \leq \mathbf{1}$, for all x satisfying $(F+GK)\Phi^i x \leq \mathbf{1}$, $i = 0, \ldots, \nu$. If Φ is strictly stable and $(\Phi, F+GK)$ is observable, then ν is necessarily finite.

Proof Let $\mathcal{X}^{(n)} = \{x : (F + GK)\Phi^i x \leq \mathbf{1},\ i = 0, \ldots, n\}$ for $n \geq 0$, then it can be shown that (2.15) holds for some finite ν using Definition 2.2 to show that the MPI set $\mathcal{X}^{\text{MPI}}$ is equal to $\mathcal{X}^{(\nu)}$ for finite ν.

In particular, if $x_{0|k} \notin \mathcal{X}^{(n)}$ for given n, then the constraint (2.2) must be violated under the dynamics of (2.13b) and (2.14b). By Definition 2.2 therefore, any $x \notin \mathcal{X}^{(n)}$ cannot lie in $\mathcal{X}^{\text{MPI}}$, so $\mathcal{X}^{(n)}$ must contain $\mathcal{X}^{\text{MPI}}$, for all $n \geq 0$.

Furthermore, if $(F + GK)\Phi^{\nu+1}x \leq \mathbf{1}$, for all $x \in \mathcal{X}^{(\nu)}$, then $\Phi x \in \mathcal{X}^{(\nu)}$ must hold whenever $x \in \mathcal{X}^{(\nu)}$ (since $x \in \mathcal{X}^{(\nu)}$ and $(F + GK)\Phi^{\nu+1}x \leq \mathbf{1}$ imply $(F + GK)\Phi^{i}(\Phi x) \leq \mathbf{1}$ for $i = 0, \ldots \nu$). But from the definition of $\mathcal{X}^{(\nu)}$ we have $(F + GK)x \leq \mathbf{1}$ for all $x \in \mathcal{X}^{(\nu)}$, and therefore $\mathcal{X}^{(\nu)}$ is positively invariant under (2.13b), (2.14b) and (2.2). From Definition 2.2 it can be concluded that $\mathcal{X}^{(\nu)}$ is a subset of, and therefore equal to $\mathcal{X}^{\text{MPI}}$.

Finally, for $\nu \geq n_x$, the set $\mathcal{X}^{(\nu)}$ is necessarily bounded if $(\Phi, F + GK)$ is observable, and, since Φ is strictly stable, the set $\{x : (F + GK)\Phi^{(\nu+1)}x \leq \mathbf{1}\}$ must contain $\mathcal{X}^{(\nu)}$ for finite ν; therefore $\mathcal{X}^{\text{MPI}}$ must be defined by (2.15) for some finite ν. □

The value of ν satisfying the conditions of Theorem 2.3 can be computed by solving at most νn_C linear programs (LPs), namely

$$\underset{x}{\text{maximize}}\ (F + GK)_j\Phi^{n+1}x \ \text{ subject to }\ (F + GK)\Phi^{i}x \leq \mathbf{1},\ i = 0, \ldots, n$$

for $j = 1, \ldots, n_C, n = 1, \ldots, \nu$, where $(F+GK)_j$ denotes the jth row of $F+GK$. The value of ν clearly does not depend on the system state, and this procedure can therefore be performed offline. In general $\nu \geq n_x$, and (2.15) defines the MPI set as a polytope. Therefore if $\mathcal{X}_T$ is equal to the MPI set, the terminal constraint $x_{N|k} \in \mathcal{X}_T$ can be invoked via linear inequalities on the degrees of freedom in mode 1 predictions. It will be convenient to represent the terminal set $\mathcal{X}_T$ in matrix form

$$\mathcal{X}_T = \{x : V_T x \leq \mathbf{1}\},$$

so that with $\mathcal{X}_T$ chosen as the MPI set (2.15), V_T is given by

$$V_T = \begin{bmatrix} F + GK \\ (F + GK)\Phi \\ \vdots \\ (F + GK)\Phi^{\nu} \end{bmatrix}.$$

Example 2.1 Figure 2.2 gives an illustration of the MPI set for a second-order system with state-space matrices

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix} \tag{2.16a}$$

and constraints $-\mathbf{1} \leq x/8 \leq \mathbf{1}$, $-1 \leq u \leq 1$, which correspond to the following constraint matrices in (2.2),

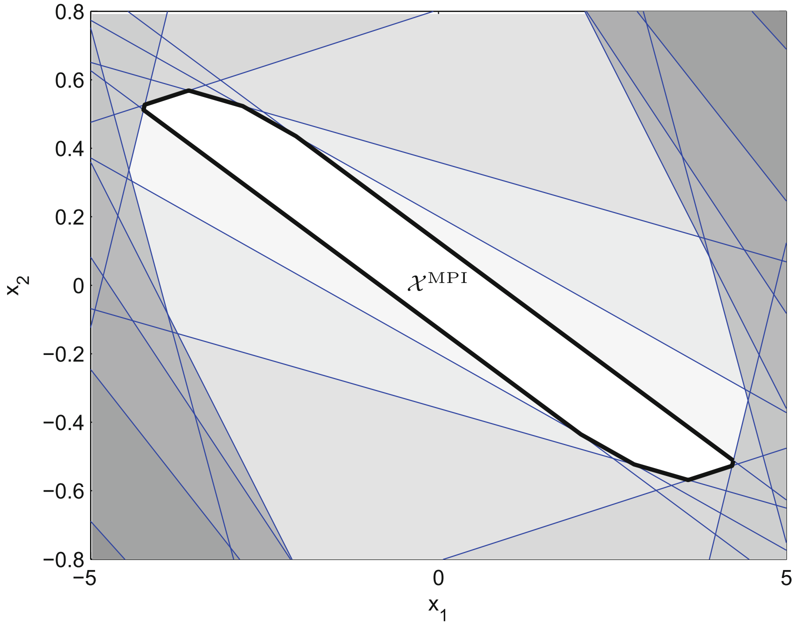

Fig. 2.2 The maximal positively invariant (MPI) set, $\mathcal{X}^{\rm MPI}$, for the system of (2.16a), (2.16b). Each of the inequalities defining $\mathcal{X}^{\rm MPI}$ is represented by a straight line on the diagram

$$F = \begin{bmatrix} 0 & 1/8 \\ 1/8 & 0 \\ 0 & -1/8 \\ -1/8 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \tag{2.16b}$$

The mode 2 feedback law is taken to be the optimal unconstrained linear feedback law $u = Kx$, with cost weights $Q = C^T C$ and $R = 1$, for which $K = -\begin{bmatrix}1.19 & 7.88\end{bmatrix}$. The MPI set is given by (2.15) with $\nu = 5$. After removing redundant constraints, this set is defined by 10 inequalities corresponding to the 10 straight lines that intersect the boundary of the MPI set, marked $\mathcal{X}^{\rm MPI}$ in Fig. 2.2. ◊

2.5 Controlled Invariant Sets and Recursive Feasibility

Collecting the ideas discussed in the previous sections we can state the following MPC algorithm:

Algorithm 2.1 (*MPC*) At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization

$$\underset{\mathbf{c}_k}{\text{minimize}} \quad J(x_k, \mathbf{c}_k) \tag{2.17a}$$

$$\text{subject to} \quad (F + GK)x_{i|k} + Gc_{i|k} \leq \mathbf{1}, i = 0, \ldots, N-1 \tag{2.17b}$$

$$V_T x_{N|k} \leq \mathbf{1} \tag{2.17c}$$

where $J(x_k, \mathbf{c}_k)$ is the cost of (2.11) evaluated for the predicted trajectories of (2.13a) and (2.14a).

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ for problem (2.17). ◁

The terminal condition (2.17c) is sometimes referred to as a stability constraint because it provides a means of guaranteeing the closed-loop stability of the MPC law. It does this by ensuring that the mode 2 predicted trajectories (2.13b) and (2.14b) satisfy the constraint $(F + GK)x_{i|k} \leq \mathbf{1}$, thus ensuring that the predicted cost over mode 2 is indeed given by $\|x_{N|k}\|^2_W$, and also by guaranteeing that Algorithm 2.1 is feasible at all time instants if it is feasible at initial time. The latter property of recursive feasibility is a fundamental requirement for closed-loop stability since it guarantees that the optimization problem (2.17) is solvable and hence that the control law of Algorithm 2.1 is defined at every time instant if (2.17) is initially feasible.

Recall that the feasibility of predicted trajectories in mode 2 is ensured by constraining the terminal state to lie in a set which is positively invariant. The feasibility of Algorithm 2.1 can be similarly ensured by requiring that the state x_k lies in an invariant set. However, since there are degrees of freedom in the predicted trajectories of (2.13a) and (2.14a), the relevant form of invariance is controlled positive invariance.

Definition 2.3 (*Controlled positively invariant set*) A set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is controlled positively invariant (CPI) for the dynamics of (2.1a) and constraints (2.2) if, for all $x \in \mathcal{X}$, there exists $u \in \mathbb{R}^{n_u}$ such that $Fx + Gu \leq \mathbf{1}$ and $Ax + Bu \in \mathcal{X}$. Furthermore $\mathcal{X}$ is the maximal controlled positively invariant (MCPI) set if it is CPI and contains all other CPI sets.

To show that Algorithm 2.1 is recursively feasible, we demonstrate next that its feasible set is a CPI set. Algorithm 2.1 is feasible whenever x_k belongs to the feasible set $\mathcal{F}_N$ defined by

$$\mathcal{F}_N \doteq \big\{x_k : \exists\, \mathbf{c}_k \text{ such that } (F + GK)x_{i|k} + Gc_{i|k} \leq \mathbf{1},\ i = 0, \ldots, N-1 \\ \text{and } V_T x_{N|k} \leq \mathbf{1}\big\}. \tag{2.18}$$

Clearly this is the same as the set of states of (2.1a) that can be driven to the terminal set $\mathcal{X}_T = \{x : V_T x \le \mathbf{1}\}$ in N steps subject to the constraints (2.2), and it therefore has the following equivalent definition:

$$\mathcal{F}_N = \big\{x_0 : \exists\, \{u_0, \ldots, u_{N-1}\} \text{ such that } Fx_i + Gu_i \le \mathbf{1},\ i = 0, \ldots, N-1, \\ \text{and } x_N \in \mathcal{X}_T\big\}. \tag{2.19}$$

Theorem 2.4 *If $\mathcal{X}_T$ in (2.19) is positively invariant for (2.13b), (2.14b) and (2.2), then $\mathcal{F}_N \subseteq \mathcal{F}_{N+1}$, for all $N > 0$, and $\mathcal{F}_N$ is a CPI set for the dynamics of (2.1a) and constraints (2.2).*

Proof If $x_0 \in \mathcal{F}_N$, then by definition there exists a sequence $\{u_0, \ldots, u_{N-1}\}$ such that $Fx_i + Gu_i \le \mathbf{1}$, $i = 0, \ldots, N-1$ and $x_N \in \mathcal{X}_T$. Also, since $\mathcal{X}_T$ is positively invariant, the choice $u_N = Kx_N$ would ensure $Fx_N + Gu_N \le \mathbf{1}$ and $x_{N+1} \in \mathcal{X}_T$, and this in turn implies $x_0 \in \mathcal{F}_{N+1}$ whenever $x_0 \in \mathcal{F}_N$. Furthermore if $x_0 \in \mathcal{F}_N$, then by definition u_0 exists such that $Fx_0 + Gu_0 \le \mathbf{1}$ and $x_1 \in \mathcal{F}_{N-1}$, and since $\mathcal{F}_{N-1} \subset \mathcal{F}_N$, it follows that $\mathcal{F}_N$ is CPI. □

Although the proof of Theorem 2.4 considers the sequence of control inputs $\{u_0, \ldots, u_{N-1}\}$, the same arguments apply to the optimization variables $\mathbf{c}_k$ in (2.17), since for each feasible u_k, $k = 0, \ldots, N-1$, there exists a feasible c_k such that $u_k = Kx_k + c_k$. Therefore, the fact that $\mathcal{F}_N$ is a CPI set for (2.1a) and (2.2) also implies that $\mathcal{F}_N$ is CPI for the dynamics (2.14a) and constraints (2.17b). Hence for any $x_k \in \mathcal{F}_N$ there must exist c_k such that $(F + GK)x_k + Gc_k \le \mathbf{1}$ and $x_{k+1} = \Phi x_k + Bc_k \in \mathcal{F}_N$. Furthermore, the proof of Theorem 2.4 shows that if $c_k = c^*_{0|k}$ (where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ in step (ii) of Algorithm 2.1), then the sequence

$$\mathbf{c}_{k+1} = (c^*_{1|k}, \ldots, c^*_{N-1|k}, 0) \tag{2.20}$$

is necessarily feasible for the optimization (2.17) at time $k+1$, and therefore Algorithm 2.1 is recursively feasible.

The candidate feasible sequence in (2.20) can be thought of as the extension to time $k+1$ of the optimal sequence at time k. It is in fact the sequence that generates, via (2.13a), the input sequence

$$\{u_{1|k}, \ldots, u_{N-1|k}, Kx_{N|k}\}$$

at time $k+1$. For this reason, it is sometimes referred to as the *tail of the solution of the MPC optimization problem at time k*, or simply the *tail*. As well as demonstrating recursive feasibility, the tail is often used to construct a suboptimal solution at time $k+1$ based on the optimal solution at time k. This enables a comparison of the optimal costs at successive time steps, which is instrumental in the analysis of the closed-loop stability properties of MPC laws.

Theorem 2.4 shows that the feasible sets corresponding to increasing values of N are nested, so that the feasible set $\mathcal{F}_N$ necessarily grows as N is increased. In practice the length of the mode 1 horizon is likely to be limited by the growth in computation that is required to solve Algorithm 2.1 (this is discussed in Sect. 2.8). However, given that $\mathcal{F}_N$ increases as N grows, the question arises as to whether there exists a finite value of N such that $\mathcal{F}_N$ is equal to the maximal feasible set defined by

$$\mathcal{F}_\infty \doteq \bigcup_{N=1}^{\infty} \mathcal{F}_N.$$

Here $\mathcal{F}_\infty$ is defined as the set of initial conditions that can be steered to $\mathcal{X}_T$ over an infinite horizon subject to constraints. However, $\mathcal{F}_\infty$ is independent of the choice of $\mathcal{X}_T$; this is a consequence of the fact that, for any bounded positively invariant set $\mathcal{X}_T$, the system (2.1a) can be steered from any initial state in $\mathcal{X}_T$ to the origin subject to the constraints (2.2) in finite time, as demonstrated by the following result.

Theorem 2.5 *Let* $\mathcal{F}_N^0 \doteq \big\{x_0 : \exists\{u_0, \ldots, u_{N-1}\}$ *such that* $Fx_i + Gu_i \leq \mathbf{1}$, $i = 0, \ldots, N-1$, *and* $x_N = 0\big\}$. *If* $\mathcal{X}_T$ *in (2.19) is positively invariant for (2.13b), (2.14b) and (2.2), where* Φ *is strictly stable and* $(\Phi, F + GK)$ *is observable, then* $\mathcal{F}_\infty = \bigcup_{N=1}^{\infty} \mathcal{F}_N = \bigcup_{N=1}^{\infty} \mathcal{F}_N^0$.

Proof First, note that any positively invariant set $\mathcal{X}_T$ must contain the origin because Φ is strictly stable. Second, strict stability of Φ and boundedness of $\mathcal{X}_T$ (which follows from observability of $(\Phi, F + GK)$) also implies that, for any $\epsilon > 0$, the set $\mathcal{B}_\epsilon \doteq \{x : \|x\| \leq \epsilon\}$ is reachable from any point in $\mathcal{X}_T$ in a finite number of steps (namely for all $x_0 \in \mathcal{X}_T$ there exists a sequence $\{u_0, \ldots, u_{n-1}\}$ such that $Fx_i + Gu_i \leq \mathbf{1}$ for $i = 0, \ldots, n-1$ and $x_n \in \mathcal{B}_\epsilon$) since $\|\Phi^n x\| \leq \epsilon$, for all $x \in \mathcal{X}_T$ for some finite n. Third, since (A, B) is controllable and $(0, 0)$ lies in the interior of the constraint set $\{(x, u) : Fx + Gu \leq \mathbf{1}\}$, there must exist $\epsilon > 0$ such that the origin is reachable in n_x steps from any point in $\mathcal{B}_\epsilon$, i.e. $\mathcal{B}_\epsilon \subseteq \mathcal{F}_{n_x}^0$. Combining these observations we obtain $\{0\} \subseteq \mathcal{X}_T \subseteq \mathcal{F}_{n+n_x}^0$ and hence $\mathcal{F}_N^0 \subseteq \mathcal{F}_N \subseteq \mathcal{F}_{n+n_x+N}^0$ for some finite n and all $N \geq 0$. From this we conclude that $\bigcup_{N=1}^{\infty} \mathcal{F}_N = \bigcup_{N=1}^{\infty} \mathcal{F}_N^0$. □

A consequence of Theorem 2.5 is that replacing the terminal set $\mathcal{X}_T$ by any bounded positively invariant set (or in fact any CPI set) in (2.18) results in the same set $\mathcal{F}_\infty$. Therefore $\mathcal{F}_\infty$ is identical to the maximal CPI set or *infinite time reachability set* [10, 11], which by definition is the largest possible feasible set for any stabilizing control law for the dynamics (2.1a) and constraints (2.2). In general $\mathcal{F}_N$ does not necessarily tend to a finite limit[1] as $N \to \infty$, but the following result shows that under certain conditions $\mathcal{F}_\infty$ is equal to $\mathcal{F}_N$ for finite N.

---

[1]If for example the system (2.1a) is open-loop stable and $F = 0$, then clearly the MCPI set is the entire state space and $\mathcal{F}_N$ grows without bound as N increases. In general the MCPI set is finite if and only if the system (A, B, F, G), mapping input u_k to output $Fx_k + Gu_k$ has no transmission zeros inside the unit circle (see, e.g. [11, 12]).

Theorem 2.6 *If $\mathcal{F}_{N+1} = \mathcal{F}_N$ for finite $N > 0$, then $\mathcal{F}_\infty = \mathcal{F}_N$.*

Proof An alternative definition of $\mathcal{F}_{N+1}$ (which is nonetheless equivalent to (2.18)) is that $\mathcal{F}_{N+1}$ is the set of states x for which there exists a control input u such that $Fx + Gu \leq \mathbf{1}$ and $Ax + Bu \in \mathcal{F}_N$. If $\mathcal{F}_{N+1} = \mathcal{F}_N$, then it immediately follows from this definition that $\mathcal{F}_{N+2} = \mathcal{F}_{N+1}$. Applying this argument repeatedly we get $\mathcal{F}_{N+i} = \mathcal{F}_N$, for all $i = 1, 2, \ldots$ and hence $\mathcal{F}_\infty = \mathcal{F}_N$. □

Example 2.2 Figure 2.3 shows the feasible sets $\mathcal{F}_N$ of Algorithm 2.1 for the system model and constraints of Example 2.1, for a range of values of mode 1 horizon N. Here the terminal set $\mathcal{X}_T$ is the maximal positively invariant set $\mathcal{X}^{\text{MPI}}$ of Fig. 2.2; this is shown in Fig. 2.3 as the feasible set for $N = 0$. As expected the feasible sets $\mathcal{F}_N$ for increasing N are nested. For this example, the maximal CPI set is given by $\mathcal{F}_\infty = \mathcal{F}_N$ for $N = 26$ and the minimal description of $\mathcal{F}_\infty$ involves 100 inequalities. ◊

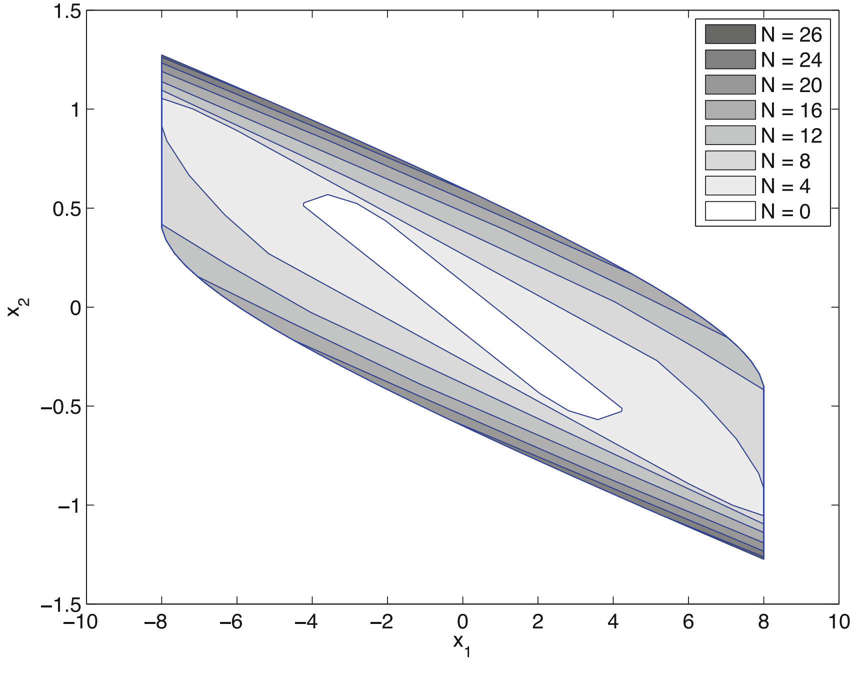

Fig. 2.3 The feasible sets $\mathcal{F}_N$, $N = 4, 8, 12, 16, 20, 24, 26$ and the terminal set $\mathcal{F}_0 = \mathcal{X}_T$ for the example of (2.16a), (2.16b). The maximal controlled invariant set is $\mathcal{F}_\infty = \mathcal{F}_{26}$

2.6 Stability and Convergence

This section introduces the main tools for analysing closed-loop stability under the MPC law of Algorithm 2.1 for the ideal case of no model uncertainty or unmodeled disturbances. The control law is nonlinear because of the inequality constraints in the optimization (2.17), and the natural framework for the stability analysis is therefore Lyapunov stability theory. Using the feasible but suboptimal tail sequence that was introduced in Sect. 2.5, we show that the optimal value of the cost function in (2.17) is non-increasing along trajectories of the closed-loop system. This provides guarantees of asymptotic convergence of the state and Lyapunov stability under Assumption 2.2. Where possible, we keep the discussion in this section non-technical and refer to the literature on stability theory for technical details.

The feasibility of the tail of the optimal sequence $\mathbf{c}_k^*$ implies that the sequence $\mathbf{c}_{k+1}$ defined in (2.20) is feasible but not necessarily an optimal solution of (2.17) at time $k+1$. Using (2.20) it is easy to show that the corresponding cost $J(x_{k+1}, \mathbf{c}_{k+1})$ is equal to $J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2$. After optimization at time $k+1$, we therefore have

$$J^*(x_{k+1}) \leq J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2. \tag{2.21}$$

Summing both sides of this inequality over all $k \geq 0$ gives the closed-loop performance bound

$$\sum_{k=0}^{\infty} \bigl(\|x_k\|_Q^2 + \|u_k\|_R^2\bigr) \leq J^*(x_0) - \lim_{k\to\infty} J^*(x_k). \tag{2.22}$$

The quantity appearing on the LHS of this inequality is the cost evaluated along the closed-loop trajectories of (2.1) under Algorithm 2.1. Since $J^*(x_k)$ is non-negative for all k, the bound (2.22) implies that the closed-loop cost can be no greater than the initial optimal cost value, $J^*(x_0)$.

Given that the optimal cost is necessarily finite if (2.17) is feasible, and since each term in the sum on the LHS of (2.22) is non-negative, the closed-loop performance bound in (2.22) implies the following convergence result

$$\lim_{k\to\infty} \bigl(\|x_k\|_Q^2 + \|u_k\|_R^2\bigr) = 0 \tag{2.23}$$

along the trajectories of the closed-loop system. We now give the basic results concerning closed-loop stability.

Theorem 2.7 *If (2.17) feasible at* $k = 0$*, then the state and input trajectories of (2.1a) under Algorithm 2.1 satisfy* $\lim_{k\to\infty}(x_k, u_k) = (0, 0)$.

Proof This follows from (2.23) and Assumption 2.2 since $R \succ 0$ implies $u_k \to 0$ as $k \to \infty$; hence from the observability of (Q, A) and $\|x_k\|_Q \to 0$ we conclude that $x_k \to 0$ as $k \to \infty$. □

Theorem 2.8 *Under the control law of Algorithm 2.1, the origin* $x = 0$ *of the system (2.1a) is asymptotically stable and its region of attraction is equal to the feasible set* $\mathcal{F}_N$*. If* $Q \succ 0$*, then* $x = 0$ *is exponentially stable.*

Proof The conditions on Q and R in Assumption 2.2 ensure that the optimal cost $J^*(x_k)$ is a positive-definite function of x_k since $J^*(x_k) = 0$ if and only if $x_k = 0$, and $J^*(x_k) > 0$ whenever $x_k \neq 0$. Therefore (2.21) implies that $J^*(x_k)$ is a Lyapunov function which demonstrates that $x = 0$ is a stable equilibrium (in the sense of Lyapunov) of the closed-loop system [13]. Combined with the convergence result of Theorem 2.7, this shows that $x = 0$ is an asymptotically stable equilibrium point, and since Theorem 2.7 applies to all feasible initial conditions, the region of attraction is $\mathcal{F}_N$.

To show that the rate of convergence is exponential if $Q \succ 0$ we first note that the optimal value of (2.17) is a continuous piecewise quadratic function of x_k [14]. Therefore, $J^*(x_k)$ can be bounded above and below for all $x_k \in \mathcal{F}_N$ by

$$\alpha \|x_k\|^2 \le J^*(x_k) \le \beta \|x_k\|^2 \tag{2.24}$$

where α and β are necessarily positive scalars since $J^*(x_k)$ is positive-definite. If the smallest eigenvalue of Q is $\underline{\lambda}(Q)$, then from (2.24) and (2.21) we get

$$\|x_k\|^2 \le \frac{1}{\alpha} \left| 1 - \frac{\underline{\lambda}(Q)}{\beta} \right|^k J^*(x_0)$$

for all $k = 0, 1, \ldots$, and hence $x = 0$ is exponentially stable. □

Example 2.3 For the same system dynamics, constraints and cost as in Example 2.1 the predicted and closed-loop state trajectories under the MPC law of Algorithm 2.1 with $N = 6$ and initial state $x(0) = (-7.5, 0.5)$ are shown in Fig. 2.4. Figure 2.5 gives the corresponding predicted and closed-loop input trajectories. The jump in the predicted input trajectory at $N = 6$ is due to the switch to the mode 2 feedback law at that time step.

Table 2.1 gives the variation with mode 1 horizon N of predicted cost J_0^* and closed-loop cost $J_{\mathrm{cl}}(x_0) \doteq \sum_{k=0}^{\infty} (\|x_k\|_Q^2 + \|u_k\|_R^2)$ for $x(0) = (-7.5, 0.5)$. The infinite-dimensional optimal performance is obtained with $N = N_\infty$, where $N_\infty = 11$ for this initial condition, so there is no further decrease in predicted cost for values of $N > 11$. However, because of the receding horizon implementation, the closed-loop response of the MPC law for $N = 6$ is indistinguishable from the ideal optimal response for this initial condition. ◊

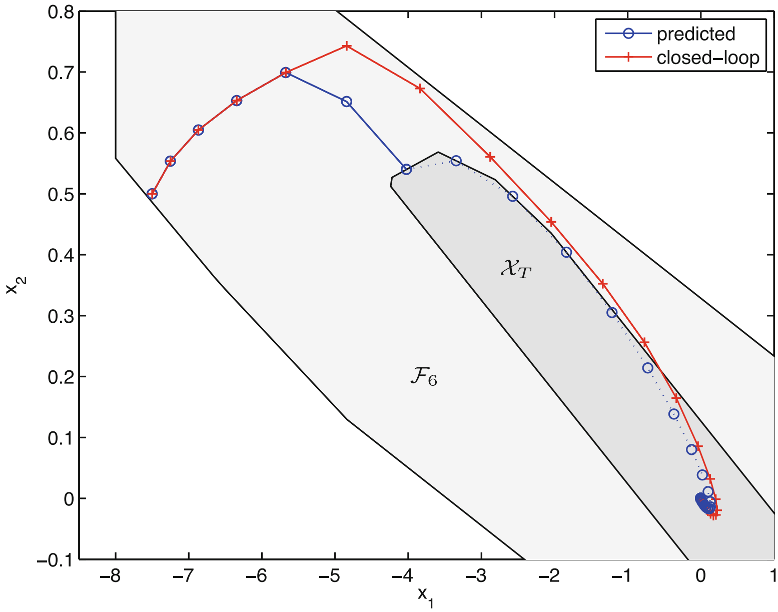

Fig. 2.4 Predicted and closed-loop state trajectories for Algorithm 2.1 with $N = 6$

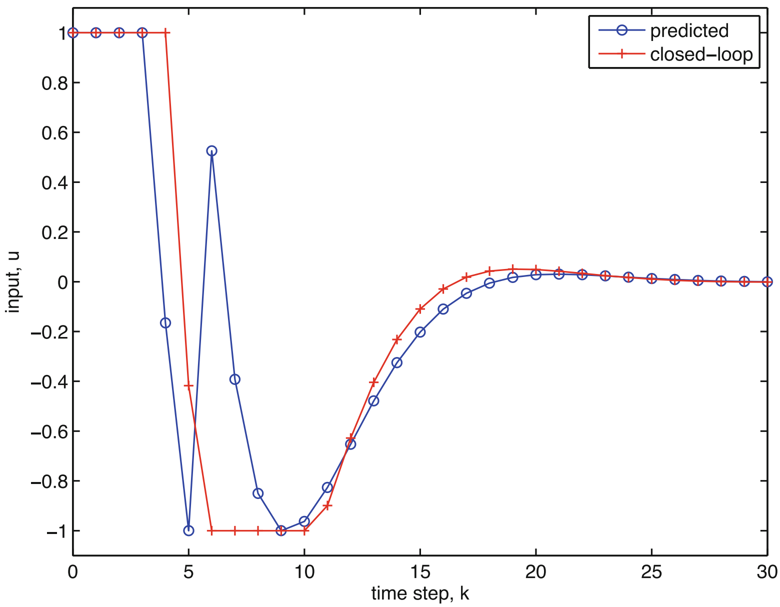

Fig. 2.5 Predicted and closed-loop input trajectories for Algorithm 2.1 with $N = 6$

Table 2.1 Variation of predicted and closed-loop cost with N for $x_0 = (-7.5, 0.5)$ in Example 2.3

N	6	7	8	11	>11
$J^*(x_0)$	364.2	357.0	356.3	356.0	356.0
$J_{cl}(x_0)$	356.0	356.0	356.0	356.0	356.0

2.7 Autonomous Prediction Dynamics

The dual-mode prediction dynamics (2.14a) and (2.14b) can be expressed in a more compact autonomous form that incorporates both prediction modes [15, 16]. This alternative prediction model, which includes the degrees of freedom in predictions within the state of an autonomous prediction system, enables the constraints on predicted trajectories to be formulated as constraints on the prediction system state at the start of the prediction horizon. With this approach the feasible sets for the model state and the degrees of freedom in predictions are determined simultaneously by computing an invariant set (rather than a controlled invariant set) for the autonomous system state. This can result in significant reductions in computation for the case that the system model is uncertain since, as discussed in Chap. 5, it greatly simplifies handling the the effects of uncertainty over the prediction horizon. In this section we show that an autonomous formulation is also convenient in the case of nominal MPC.

An autonomous prediction system that generates the predictions of (2.13a), (2.13b) and (2.14a), (2.14b) can be expressed as

$$z_{i+1|k} = \Psi z_{i|k}, \quad i = 0, 1, \ldots \tag{2.25}$$

where the initial state $z_{0|k} \in \mathbb{R}^{n_x + Nn_u}$ consists of the state x_k of the model (2.1a) appended by the vector $\mathbf{c}_k$ of degrees of freedom,

$$z_{0|k} = \begin{bmatrix} x_k \\ c_{0|k} \\ \vdots \\ c_{N-1|k} \end{bmatrix}.$$

The state transition matrix in (2.25) is given by

$$\Psi = \begin{bmatrix} \Phi & BE \\ 0 & M \end{bmatrix} \tag{2.26a}$$

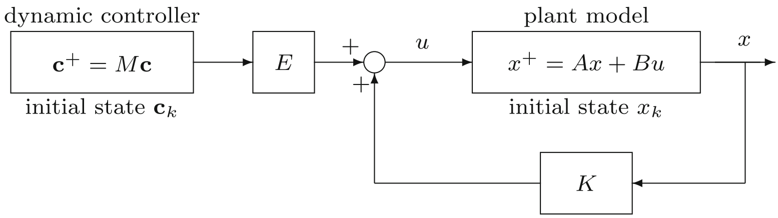

Fig. 2.6 Block diagram representation of the autonomous prediction systems (2.25) and (2.26). The free variables in the state and input predictions at time k are contained in the initial controller state $\mathbf{c}_k$; the signals marked x and u are the i steps ahead predicted state and control input, and x^+, $\mathbf{c}^+$ denote their successor states

where $\Phi = A + BK$ and

$$E = \begin{bmatrix} I_{n_u} & 0 & \cdots & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & I_{n_u} & 0 & \cdots & 0 \\ 0 & 0 & I_{n_u} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{n_u} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2.26b}$$

The state and input predictions of (2.13a), (2.13b) and (2.14a), (2.14b) are then given by

$$u_{i|k} = \begin{bmatrix} K & E \end{bmatrix} z_{i|k} \tag{2.27a}$$

$$x_{i|k} = \begin{bmatrix} I_{n_x} & 0 \end{bmatrix} z_{i|k} \tag{2.27b}$$

for $i = 0, 1, \ldots$. The prediction systems (2.25) and (2.26) can be interpreted as a dynamic feedback law applied to (2.1a), with the controller state at the beginning of the prediction horizon containing the degrees of freedom, $\mathbf{c}_k$, in predictions (Fig. 2.6).

2.7.1 Polytopic and Ellipsoidal Constraint Sets

The constraints (2.2) applied to the predictions of (2.27a), (2.27b) are equivalent to the following constraints on the initial prediction system state $z_k = z_{0|k}$:

$$\begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^i z_k \le \mathbf{1}, \quad i = 0, 1, \ldots \tag{2.28}$$

Clearly this implies an infinite number of constraints that apply across an infinite prediction horizon. However, analogously to the definition of terminal invariant sets in Sect. 2.4, a feasible set for z_k satisfying (2.28) can be constructed by determining a positively invariant set for the dynamics $z_{k+1} = \Psi z_k$ and constraints

$\begin{bmatrix} F + GK & GE \end{bmatrix} z_k \leq \mathbf{1}$. Theorem 2.3 shows that the maximal positively invariant set for these dynamics and constraints is given by

$$\mathcal{Z} \doteq \{z : \begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^i z \leq \mathbf{1}, \quad i = 0, 1, \ldots, \nu_z\} \tag{2.29}$$

where ν_z is a positive integer such that $\begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^{\nu_z+1} z \leq \mathbf{1}$, for all z satisfying $\begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^i z \leq \mathbf{1}$, $i = 0, 1, \ldots, \nu_z$. Since $\mathcal{Z}$ is the MPI set, every state z_k for which (2.28) is satisfied must lie in $\mathcal{Z}$. Given that a mode 1 prediction horizon of N steps is implicit in the augmented prediction dynamics (2.25), the projection of $\mathcal{Z}$ onto the x-subspace is therefore equal to the feasible set $\mathcal{F}_N$ defined in (2.18), i.e.

$$\mathcal{F}_N = \left\{ x : \exists \, \mathbf{c} \text{ such that } \begin{bmatrix} F + GK & GE \end{bmatrix} \Psi^i \begin{bmatrix} x \\ \mathbf{c} \end{bmatrix} \leq \mathbf{1}, \ i = 0, 1, \ldots, \nu_z \right\}.$$

The value of ν_z defining the MPI set in (2.29) grows as the mode 1 prediction horizon N is increased. Furthermore, it can be seen from (2.26) that every eigenvalue of Ψ is equal either to 0 or to an eigenvalue of Φ, so if one or more of the eigenvalues of Φ lies close to the unit circle in the complex plane, then ν_z in (2.29) could be large even for short horizons N. The equivalence of (2.27a), (2.27b) with (2.13a), (2.13b) and (2.14a), (2.14b) implies that the online MPC optimization in (2.17) is equivalent to

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \ J(x_k, \mathbf{c}_k) \ \ \text{subject to} \ \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{Z}. \tag{2.30}$$

which is a quadratic programming problem with $\nu_z n_C$ constraints.

A large value of ν_z could therefore make the implementation of Algorithm 2.1 computationally demanding. If this is the case, and in particular for applications with very high sampling rates, it may be advantageous to replace the polyhedral invariant set $\mathcal{Z}$ with an ellipsoidal invariant set, $\mathcal{E}_z$:

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \ J(x_k, \mathbf{c}_k) \ \ \text{subject to} \ \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{E}_z. \tag{2.31}$$

This represents a simplification of the online optimization to a problem that involves just a single constraint, thus allowing for significant computational savings. Furthermore, using an ellipsoidal set that is positively invariant for the autonomous prediction dynamics (2.25) and constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z \leq \mathbf{1}$, the resulting MPC law retains the recursive feasibility and stability properties of Algorithm 2.1. Approximating the MPI set $\mathcal{Z}$ (which is by definition maximal) using a smaller ellipsoidal set necessarily introduces suboptimality into the resulting MPC law; but as discussed in Sect. 2.8, the degree of suboptimality is in many cases negligible.

The invariant ellipsoidal set $\mathcal{E}_z$ can be computed offline by solving an appropriate convex optimization problem. The design of these sets is particularly convenient computationally because the conditions for invariance with respect to the linear autonomous dynamics (2.25) and linear constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z_k \leq \mathbf{1}$

may be written in terms of linear matrix inequalities (LMIs), which are necessarily convex and can be handled using semidefinite programming (SDP) [17]. Linear matrix inequalities and the offline optimization of $\mathcal{E}_z$ are considered in more detail in Sect. 2.7.3; here we simply summarize the conditions for invariance of $\mathcal{E}_z$ in the following theorem:

Theorem 2.9 *The ellipsoidal set defined by $\mathcal{E}_z \doteq \{z : z^T P_z z \le 1\}$ for $P_z \succ 0$ is positively invariant for the dynamics $z_{k+1} = \Psi z_k$ and constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z_k \le \mathbf{1}$ if and only if P_z satisfies*

$$P_z - \Psi^T P_z \Psi \succeq 0 \tag{2.32}$$

and

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & GE \end{bmatrix} \\ \begin{bmatrix} (F + GK)^T \\ (GE)^T \end{bmatrix} & P_z \end{bmatrix} \succeq 0, \quad e_i^T H e_i \le 1, \quad i = 1, 2, \ldots, n_C \tag{2.33}$$

for some symmetric matrix H, where e_i is the ith column of the identity matrix.

Proof The inequality in (2.32) implies $z^T \Psi^T P \Psi z \le z^T P z \le 1$, which is a sufficient condition for invariance of the ellipsoidal set $\mathcal{E}_z$ under $z_{k+1} = \Psi z_k$. Conversely, (2.32) is also necessary for invariance since if $P_z - \Psi^T P_z \Psi \not\succeq 0$, then there would exist z satisfying $z^T \Psi^T P_z \Psi z > z^T P_z z$ and $z^T P_z z = 1$, which would imply that $\Psi z \notin \mathcal{E}_z$ for some $z \in \mathcal{E}_z$.

We next show that (2.33) provides necessary and sufficient conditions for satisfaction of the constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z \le \mathbf{1}$, for all $z \in \mathcal{E}_z$. To simplify notation, let $\tilde{F} \doteq \begin{bmatrix} F + GK & GE \end{bmatrix}$ and let $\tilde{F}_i$ denote the ith row of $\tilde{F}$. Since

$$\max_z \bigl\{ \tilde{F}_i z \ \text{ subject to } \ z^T P_z z \le 1 \bigr\} = \bigl(\tilde{F}_i P_z^{-1} \tilde{F}_i^T \bigr)^{1/2}$$

it follows that $\tilde{F} x \le \mathbf{1}$, for all $x \in \mathcal{E}_z$ if and only if $\tilde{F}_i P_z^{-1} \tilde{F}_i^T \le 1$ for each row $i = 1, \ldots, n_C$. These conditions can be expressed equivalently in terms of a condition on a positive-definite diagonal matrix:

$$\begin{bmatrix} H_{1,1} - \tilde{F}_1 P_z^{-1} \tilde{F}_1^T & & \\ & \ddots & \\ & & H_{n_C, n_C} - \tilde{F}_{n_C} P_z^{-1} \tilde{F}_{n_C}^T \end{bmatrix} \succeq 0$$

for some scalars $H_{i,i} \le 1$, $i = 1, \ldots, n_C$, and this in turn is equivalent to

$$H - \tilde{F} P^{-1} \tilde{F}^T \succeq 0$$

for some symmetric matrix H with $e_i^T H e_i \le 1$, for all i. Using Schur complements (as discussed in Sect. 2.7.3), this condition is equivalent to

$$\begin{bmatrix} H & \tilde{F} \\ \tilde{F}^T & P_z \end{bmatrix} \succeq 0, \quad e_i^T H e_i \le 1, \quad i = 1, \dots, n_C$$

which implies the necessity and sufficiency of (2.33). □

2.7.2 *The Predicted Cost and MPC Algorithm*

Given the autonomous form of the prediction dynamics of (2.25) it is possible to use a Lyapunov equation similar to (2.5) to evaluate the predicted cost $J(x_k, \mathbf{c}_k)$ of (2.11) along the predicted trajectories of (2.27a), (2.27b). The stage cost (namely the part of the cost incurred at each prediction time step) has the general form

$$\begin{aligned} \|x\|_Q^2 + \|u\|_R^2 &= \|x\|_Q^2 + \|Kx + c\|_R^2 = x^T(Q + K^T R K)x + \mathbf{c}^T E^T R E \mathbf{c} \\ &= \|z\|_{\hat{Q}}^2, \quad \hat{Q} = \begin{bmatrix} Q + K^T R K & K^T R E \\ E^T R K & E^T R E \end{bmatrix}. \end{aligned}$$

Hence $J(x_k, \mathbf{c}_k)$ can be written as

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) = \sum_{i=0}^{\infty} \|z_{i|k}\|_{\hat{Q}}^2 = \|z_{0|k}\|_W^2$$

where, by Lemma 2.1, W is the (positive-definite) solution of the Lyapunov equation

$$W = \Psi^T W \Psi + \hat{Q}. \tag{2.34}$$

The special structure of Ψ and $\hat{Q}$ in this Lyapunov equation implies that its solution also has a specific structure, as we describe next.

Theorem 2.10 *If K is the optimal unconstrained linear feedback gain for the dynamics of (2.1a), then the cost (2.11) for the predicted trajectories of (2.27a), (2.27b) can be written as*

$$J(x_k, \mathbf{c}_k) = x_k^T W_x x_k + \mathbf{c}_k^T W_c \mathbf{c}_k$$
$$W_c = \begin{bmatrix} B^T W_x B + R & 0 & \cdots & 0 \\ 0 & B^T W_x B + R & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & B^T W_x B + R \end{bmatrix} \tag{2.35}$$

where W_x is the solution of the Riccati equation (2.9).

Proof Let $W = \begin{bmatrix} W_x & W_{xc} \\ W_{cx} & W_c \end{bmatrix}$, then substituting for W, Ψ and $\hat{Q}$ in (2.34) gives

$$W_x = \Phi^T W_x \Phi + Q + K^T R K \tag{2.36a}$$

$$W_{cx} = M^T W_{cx} \Phi + E^T (B^T W_x \Phi + RK) \tag{2.36b}$$

$$W_c = (BE)^T W_x (BE) + (BE)^T W_{xc} M + M^T W_{cx} BE + M^T W_c M + E^T R E \tag{2.36c}$$

The predicted cost for $\mathbf{c}_k = 0$ is $\|x_k\|^2_{W_x}$, and since K is the unconstrained optimal linear feedback gain, it follows from (2.36a) and Theorem 2.1 that W_x is the solution of the Riccati equation (2.9). Furthermore, from Theorem 2.1 we have $K = -(B^T W_x B + R)^{-1} B^T W_x A$, so that $B^T W_x \Phi + RK = 0$ and hence (2.36b) gives $W_{cx} - M^T W_{cx} \Phi = 0$, which implies that $W_{cx} = 0$. Therefore,

$$W = \begin{bmatrix} W_x & 0 \\ 0 & W_c \end{bmatrix}, \tag{2.37}$$

and from (2.36c) we have $W_c - M^T W_c M = E^T (B^T W_x B + R) E$. Hence from the structure of M and E in (2.26b), W_c is given by (2.35). □

Corollary 2.1 *The unconstrained LQ optimal control law is given by the feedback law $u = Kx$, where $K = -(B^T W_x B + R)^{-1} B^T W_x A$ and W_x is the solution of the Riccati equation (2.9).*

Proof Theorem 2.1 has already established that the unconstrained optimal linear feedback gain is as given in the corollary. The question remains as to whether it is possible to obtain a smaller cost by perturbing this feedback law. Equation (2.35) implies that this cannot be the case because the minimum cost is obtained for $\mathbf{c}_k = 0$. This argument applies for arbitrary N and hence for perturbation sequences of any length. □

Using the autonomous prediction system formulation of this section, Algorithm 2.1 can be restated as follows:

Algorithm 2.2 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \|\mathbf{c}_k\|^2_{W_c} \ \text{subject to} \ \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{S} \tag{2.38}$$

where $\mathcal{S} = \mathcal{Z}$ defined in (2.29) ($\nu_z n_C$ linear constraints), or $\mathcal{S} = \mathcal{E}_z$ defined by the solution of (2.32) and (2.33) (a single quadratic constraint).

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ for problem (2.38). ◁

Theorem 2.11 *Under the MPC law of Algorithm 2.2, the origin $x = 0$ of system (2.1a) is an asymptotically stable equilibrium with a region of attraction equal to the set of states that are feasible for the constraints in (2.38).*

Proof The constraint set in (2.38) is by assumption positively invariant. Therefore, the tail $\mathbf{c}_{k+1} = M\mathbf{c}^*_k$ provides a feasible but suboptimal solution for (2.38) at time $k + 1$. Stability and asymptotic convergence of x_k to the origin is then shown by applying the arguments of the proofs of Theorems 2.7 and 2.8 to the optimal value of the cost $J(x_k, \mathbf{c}^*_k)$ at the solution of (2.38). □

2.7.3 Offline Computation of Ellipsoidal Invariant Sets

In order to determine the invariant ellipsoidal set $\mathcal{E}_z$ for the autonomous prediction dynamics (2.25), the matrices P_z and H must be considered as variables in the conditions of Theorem 2.9. These conditions then constitute Linear Matrix Inequalities (LMIs) in the elements of P_z and H. Linear matrix inequalities are used extensively throughout this book; for an introduction to the properties of LMIs and LMI-based techniques that are commonly used in systems analysis and control design problems, we refer the reader to [17].

In its most general form a linear matrix inequality is a condition on the positive definiteness of a linear combination of matrices, where the coefficients of this combination are considered as variables. Thus a (strict) LMI in the variable $x \doteq (x_1, \ldots, x_n) \in \mathbb{R}^n$ can be expressed

$$M(x) \doteq M_0 + M_1 x_1 + \ldots + M_n x_n \succ 0 \tag{2.39}$$

where $M_0, \ldots, M_n$ are given matrices.[2] The convenience of LMIs lies in the convexity of (2.39) (see also Questions 1–3 on page 233). This property makes it possible to include conditions, such as those defining an invariant ellipsoidal set in Theorem 2.9, in convex optimization problems that can be solved efficiently using semidefinite programming.

[2] A non-strict LMI is similarly defined by $M(x) \succeq 0$. Any non-strict LMI can be expressed equivalently as a combination of a linear equality constraint and a strict LMI (see, e.g. [17]). However, none of the non-strict LMIs encountered in this chapter or in Chap. 5 carry implicit equality constraints, and hence non-strict LMIs may be assumed to be either strictly feasible or infeasible. We therefore make use of both strict and non-strict LMIs with the understanding that $M(x) \succeq 0$ can be replaced with $M(x) \succ 0$ for the purposes of numerical implementation.

A suitable criterion for selecting P_z is to maximize the region of attraction of Algorithm 2.2, namely the feasible set for the constraint $z_k^T P_z z_k \leq 1$. This region is equal to the projection of $\mathcal{E}_z = \{z : z^T P_z z \leq 1\}$ onto the x-subspace:

$$\{x : \exists\, \mathbf{c} \text{ such that } x^T P_{xx} x + 2\mathbf{c}^T P_{cx} x + \mathbf{c}^T P_{cc} \mathbf{c} \leq 1\}$$

where the matrices P_{xx}, P_{xc}, P_{cx}, P_{cc} are blocks of P_z partitioned according to

$$P_z = \begin{bmatrix} P_{xx} & P_{xc} \\ P_{cx} & P_{cc} \end{bmatrix}. \tag{2.40}$$

By considering the minimum value of $z^T P_z z$ over all $\mathbf{c}$ for given x, it is easy to show that the projection of $\mathcal{E}_z$ onto the x-subspace is given by

$$\mathcal{E}_x \doteq \left\{x : x^T (P_{xx} - P_{xc} P_{cc}^{-1} P_{xc}) x \leq 1\right\}.$$

Inverting the partitioned matrix P_z we obtain

$$P_z^{-1} = S \doteq \begin{bmatrix} S_{xx} & S_{xc} \\ S_{cx} & S_{cc} \end{bmatrix},$$

where

$$S_{xx} = \left(P_{xx} - P_{xc} P_{cc}^{-1} P_{xc}\right)^{-1},$$

and hence the volume of the projected ellipsoidal set $\mathcal{E}_x$ is proportional to $1/\det(S_{xx}^{-1}) = \det(S_{xx})$. The volume of the region of attraction of Algorithm 2.2 is therefore maximized by the optimization

$$\underset{S, P_z, H}{\text{maximize}} \ \det(S_{xx}) \text{ subject to (2.32), (2.33)} \tag{2.41}$$

Maximizing the objective in (2.41) is equivalent to maximizing $\log\det(S_{xx})$, which is a concave function of the elements of S (see, e.g. [18]). But this is not yet a semidefinite programming problem since (2.32) and (2.33) are LMIs in P_z rather than S. These constraints can however be expressed as Linear Matrix Inequalities in S using Schur complements.

In particular, the positive definiteness of a partitioned matrix

$$\begin{bmatrix} U & V^T \\ V & W \end{bmatrix} \succ 0$$

where U, V, W are real matrices of conformal dimensions, is equivalent to positive definiteness of the Schur complements

$$U \succ 0 \text{ and } W - VU^{-1}V^T \succ 0,$$

or

$$W \succ 0 \text{ and } U - V^T W^{-1} V \succ 0$$

(the proof of this result is discussed in Question 1 in Chap. 5 on page 233). Therefore, after pre- and post-multiplying (2.32) by S, using Schur complements we obtain the following condition:

$$\begin{bmatrix} S & \Psi S \\ S\Psi^T & S \end{bmatrix} \succeq 0, \tag{2.42}$$

which is an LMI in S. Similarly, pre- and post-multiplying the matrix inequality in (2.33) by $\left[\begin{smallmatrix} I & 0 \\ 0 & S \end{smallmatrix}\right]$ yields the condition

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & GE \end{bmatrix} S \\ S \begin{bmatrix} (F+GK)^T \\ (GE)^T \end{bmatrix} & S \end{bmatrix} \succeq 0 \tag{2.43}$$

which is an LMI in S and H. Therefore $\mathcal{E}_z$ can be computed by solving the SDP problem

$$\begin{aligned} \underset{S,H}{\text{maximize}} \ \log\det(S_{xx}) \ \text{ subject to } \ & (2.42), (2.43) \\ & \text{and } e_i^T H e_i \le 1, \ i = 1, \ldots, n_C. \end{aligned} \tag{2.44}$$

Example 2.4 For the system model, constraints and cost of Example 2.1, Fig. 2.7 shows the ellipsoidal regions of attraction $\mathcal{E}_x$ of Algorithm 2.2 for values of N in the range 5–40 and compares these with the polytopic feasible set $\mathcal{F}_N$ for $N = 10$. As expected, the ellipsoidal feasible sets are smaller than the polytopic feasible sets of Fig. 2.3, but the difference in area is small; the area of $\mathcal{E}_x$ for $N = 40$ is 13.4 while that of $\mathcal{F}_{10}$ is 13.6, a difference of only 1 %. On the other hand 36 linear constraints are needed to define the polytopic set $\mathcal{Z}$ for $N = 10$ whereas $\mathcal{E}_z$ is a single (quadratic) constraint.

Figure 2.8 shows closed-loop state and input responses for Algorithm 2.2, comparing the responses obtained with the ellipsoidal constraint $z_k \in \mathcal{E}_z$ against the responses obtained with the linear constraint set $z_k \in \mathcal{Z}$ for $N = 10$. The difference in the closed-loop costs of the two controllers for the initial condition $x_0 = (-7.5, 0.5)$ is 17 %. ♢

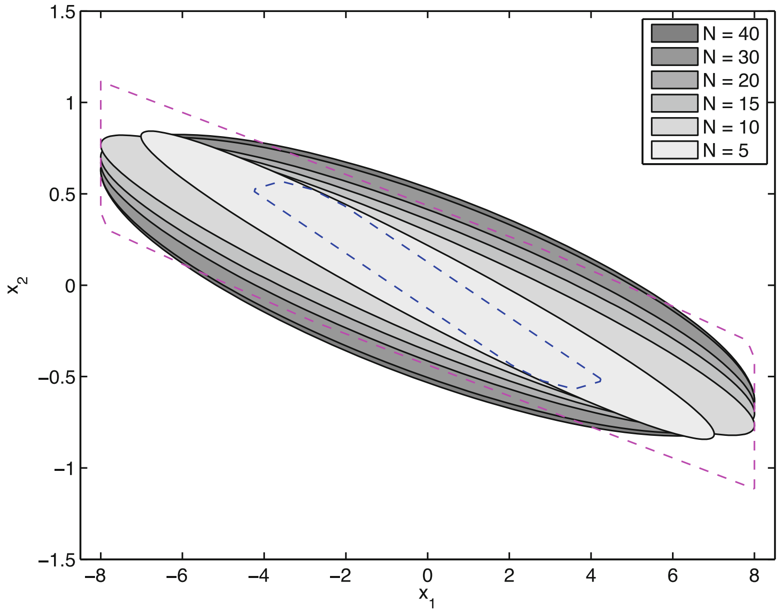

Fig. 2.7 The ellipsoidal regions of attraction of Algorithm 2.2 for $N = 5, 10, 15, 20, 30, 40$. The polytopic sets $\mathcal{F}_{10}$ and $\mathcal{X}_T$ are shown (*dashed lines*) for comparison

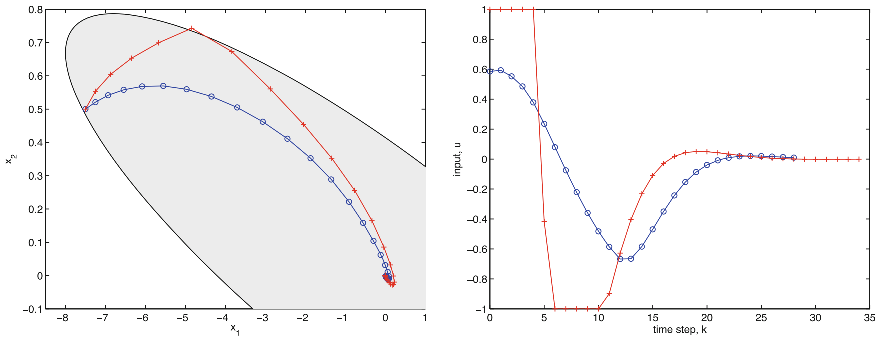

Fig. 2.8 Closed-loop responses of Algorithm 2.2 for the example of (2.16a), (2.16b) for the quadratic constraint $z_k \in \mathcal{E}_z$ with $N = 20$ (*blue* o) and the linear constraints $z_k \in \mathcal{Z}$ with $N = 10$ (*red* +). *Left* state trajectories and the feasible set $\mathcal{E}_x$ for $N = 20$. *Right* control inputs

2.8 Computational Issues

The optimization problem to be solved online in Algorithm 2.1 has a convex quadratic objective function and linear constraints, and is therefore a convex Quadratic Program (QP). Likewise if Algorithm 2.2 is formulated in terms of linear constraints, then this also requires the online solution of a convex QP problem. A variety of general QP solvers (based on active set methods [19] or interior point methods [20]), can therefore be used to perform the online MPC optimization required by these algorithms.

However algorithms for general quadratic programming problems do not exploit the special structure of the MPC problem considered here, and as a result their computational demand may exceed allowable limits. In particular they may not be applicable to problems with high sample rates, high-dimensional models, or long prediction horizons. For example the computational load of both interior point and active set methods grows approximately cubically with the mode 1 prediction horizon N.

The rate of growth with N of the required computation can be reduced however if the predicted model states are considered to be optimization variables. Thus redefining the vector of degrees of freedom as $d_k \in \mathbb{R}^{Nn_x+Nn_u}$:

$$d_k = (c_{0|k}, x_{1|k}, c_{1|k}, x_{2|k}, \ldots, c_{N-1|k}, x_{N|k})$$

and introducing the predicted dynamics of (2.14) as equality constraints results in an online optimization of the form

$$\underset{d_k}{\text{minimize}}\ d_k^T H_d d_k \ \text{ subject to }\ D_d d_k = h_h,\ C_c d_k \le h_c.$$

Although the number of optimization variables has increased from Nn_u to $Nn_u + Nn_x$, the key benefit is that the matrices H_d, D_d, C_c are sparse and highly structured. This structure can be exploited to reduce the online computation so that it grows only linearly with N (e.g. see [19, 20]).

An alternative to reducing the online computation is to use multiparametric programming to solve the optimization problem offline for initial conditions that lie in different regions of the state space. Thus, given that x_k is a known constant, the minimization of the cost of (2.35) is equivalent to the minimization of

$$J(d) = d^T H_0 d \tag{2.45}$$

where for simplicity, the vector of degrees of freedom **c** has been substituted by d and the cost is renamed as simply J. The minimization of J is subject to the linear constraints implied by the dynamics (2.14) and system constraints (2.2), together with the terminal constraints of (2.35); the totality of these constraints can be written as

$$C_0 d \le h_0 + V_0 x \tag{2.46}$$

Then adjoining the constraints (2.46) with the cost of (2.45) through the use of a vector of Lagrange multipliers λ, we obtain the first-order Karush–Kuhn–Tucker (KKT) conditions [19]

$$H_0 d + C_0^T \lambda = 0 \tag{2.47a}$$

$$\lambda^T (C_0 d - h_0 - V_0 x) = 0 \tag{2.47b}$$

$$C_0 d \le h_0 + V_0 x \tag{2.47c}$$

$$\lambda \ge 0 \tag{2.47d}$$

Now suppose that at the given x only a subset of (2.46) is active, so that gathering all these active constraints and the corresponding Lagrange multipliers we can write

$$\tilde{C}_0 d - \tilde{h}_0 - \tilde{V}_0 x = 0 \tag{2.48a}$$

$$\tilde{\lambda} \ge 0 \tag{2.48b}$$

In addition, the Lagrange multipliers corresponding to inactive constraints will be zero so that from (2.47) it follows that

$$d = -H_0^{-1} \tilde{C}_0^T \tilde{\lambda}. \tag{2.49}$$

The solution for $\tilde{\lambda}$ can be derived by substituting (2.49) into (2.48a) as

$$\tilde{\lambda} = -(\tilde{C}_0 H_0^{-1} \tilde{C}^T)^{-1} (\tilde{h}_0 + \tilde{V}_0 x). \tag{2.50}$$

and substituting this into (2.49) produces the optimal solution as

$$d = H_0^{-1} \tilde{C}_0^T - (\tilde{C}_0 H_0^{-1} \tilde{C}_0^T)^{-1} (\tilde{h}_0 + \tilde{V}_0 x). \tag{2.51}$$

Thus for given active constraints, the optimal solution is a known affine function of the state. Clearly the optimal solution must satisfy the constraints (2.46) as well as the Lagrange multipliers of (2.50) must satisfy (2.48a):

$$C_o [H_o^{-1} \tilde{C}_o^T - (\tilde{C}_o H_o^{-1} \tilde{C}_o^T)^{-1} (\tilde{h}_o + \tilde{V}_o x)] \le h_o + V_o x$$

and

$$-(\tilde{C}_o H_o^{-1} \tilde{C}_o^T)^{-1} (\tilde{h}_o + \tilde{V}_o x) > 0.$$

These two conditions give a characterization of the polyhedral region in which x must lie in order that (2.48a) is the active constraint set.

A procedure based on these considerations is given in [14] for partitioning the controllable set of Algorithms 2.1 and 2.2 into the union of a number of non-overlapping polyhedral regions. Then the MPC optimization can be implemented online by identifying the particular polyhedral region in which the current state lies. In this approach the associated optimal solution (2.51) is then recovered from a lookup table, and the first element of this is used to compute and implement the current optimal control input.

A disadvantage of this multiparametric approach is that the number of regions grows exponentially with the dimension of the state and the length of the mode 1 prediction horizon N, and this can make the approach impractical for anything other than small-scale problems with small values of N. Indeed in most other cases, the computational and storage demands of the multiparametric approach exceed those required by the QP solvers that exploit the MPC structure described above. Methods have been proposed (e.g. [21]) for improving the efficiency with which the polyhedral state-space partition is computed by merging regions that have the same control law, however the complexity of the polyhedral partition remains prohibitive in this approach.

Example 2.5 For the second-order system defined in (2.16a), (2.16b), with the cost and terminal constraints of Example 2.3 the MPC optimization problem (2.17) can be solved using multiparametric programming. For a mode 1 horizon of $N = 10$ this results in a partition of the state space into 243 polytopic regions (Fig. 2.9), each of which corresponds to a different active constraint set at the solution of the MPC optimization problem (2.17). ♢

A further alternative [15, 16] which results in significant reduction in the online computation replaces the polytopic constraints $z_k \in \mathcal{Z}$ defined (2.29) by the ellipsoidal constraint $z_k \in \mathcal{E}_z$ defined in (2.44) and thus addresses the optimization

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \|z_k\|_W^2 \quad \text{subject to} \ z_k^T P_z z_k \le 1, \quad z_k = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \tag{2.52}$$

As discussed in Sect. 2.7, this results in a certain degree of conservativeness because the ellipsoidal constraint $z_k \in \mathcal{E}_z$ gives an inner approximation to the polytopic constraint $z_k \in \mathcal{Z}$ of (2.29). The problem defined in (2.52) can be formulated as a second-order cone program (SOCP) in $Nn_u + 1$ variables.[3] If a generic solution method is employed, then this problem could turn out to be more computationally demanding than the QP that arises when the constraints are linear. However, the simple form of the cost and constraint in (2.38) allow for a particularly efficient solution, which is to be discussed next.

[3] Second-order cone programs are convex optimization problems that can be solved using interior point methods. See [22] for details and further applications of SOCP.

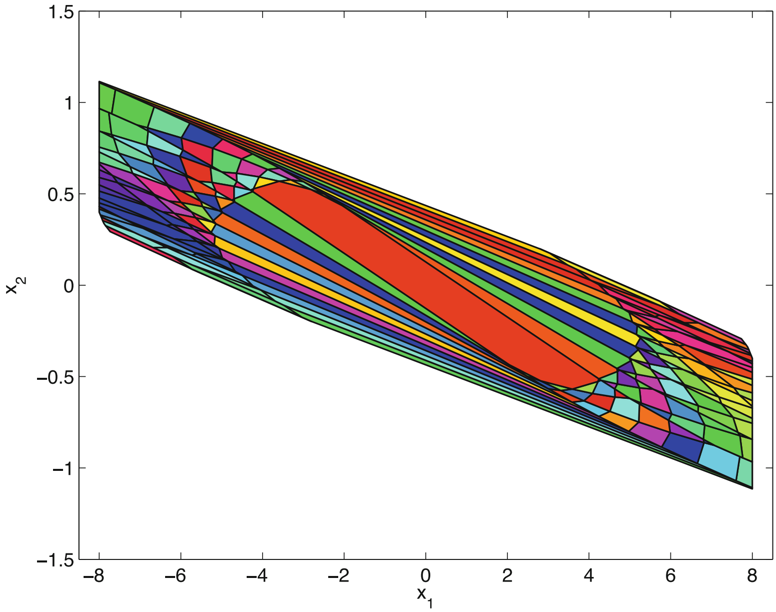

Fig. 2.9 The partition of the state space of the system of Example 2.5 into regions in which different constraint sets are active at the solution of the online MPC optimization problem

To exploit the structure of the cost and constraint in (2.52), we use the partitions of (2.37) and (2.40) to write $z_k^T W z_k = x_k^T W_x x + \mathbf{c}_k^T W_c \mathbf{c}_k$ and $z_k^T P_z z_k = x_k^T P_{xx} x_k + 2\mathbf{c}_k^T P_{cx} x_k + \mathbf{c}_k^T P_{cc} \mathbf{c}_k$, where use has been made of the fact that $P_{xc} = P_{cx}^T$. The minimizing value of $\mathbf{c}_k$ in (2.52) can only occur at a point at which the two ellipsoidal boundaries, $\partial \mathcal{E}_J \doteq \{z_k : z_k^T W z_k = \alpha\}$ and $\partial \mathcal{E}_z \doteq \{z_k : z_K^T P_z z_k = 1\}$, are tangential to one another for some constant $\alpha > 0$, namely when the gradients (with respect to $\mathbf{c}$) are parallel, i.e.

$$W_c \mathbf{c}_k = \mu(P_{cx} x_k + P_{cc} \mathbf{c}_k), \quad \mu \le 0 \tag{2.53}$$

for some scalar μ, or equivalently

$$\mathbf{c}_k = \mu M_\mu P_{cx} x_k, \quad M_\mu = (W_c - \mu P_{cc})^{-1}. \tag{2.54}$$

At the solution therefore, the inequality constraint in (2.52) will hold with equality so that μ can be obtained as the solution of $x_k^T P_{xx} x_k + 2\mathbf{c}_k^T P_{cx} + \mathbf{c}_k^T P_{cc} \mathbf{c}_k = 1$, which after some algebraic manipulation gives μ as a root of

$$\phi(\mu) = x_k^T P_{xc} \left(M_\mu W_c P_{cc}^{-1} W_c M_\mu - P_{cc}^{-1} \right) P_{cx} x_k + x_k^T P_{xx} x_k - 1 = 0. \tag{2.55}$$

Equation (2.55) is equivalent to a polynomial equation in μ which can be shown (using straightforward algebra) to have $2N$ roots, all corresponding to points of tangency of $\partial\mathcal{E}_J$ and $\partial\mathcal{E}_z$. However (2.52) has a unique minimum, and it follows that only one of these roots can be negative, as is required by (2.53).

By repeatedly differentiating $\phi(\mu)$ with respect to μ it is easy to show that the derivatives of this polynomial satisfy

$$\frac{d^r\phi}{d\mu^r} > 0 \ \ \forall\mu \leq 0.$$

This implies that the Newton–Raphson method, when initialised at $\mu = 0$, is guaranteed to converge to the unique negative root of (2.55), and that the rate of its convergence is quadratic.

Thus the optimal solution to (2.52) is obtained extremely efficiently by substituting the negative root of (2.55) into (2.54); in fact the computation required is equivalent to solving a univariate polynomial with monotonic derivatives. The price that must be paid for this gain in computational efficiency is a degree of suboptimality that results from the use of the ellipsoidal constraint $z_k \in \mathcal{E}_z$, which provides only an inner approximation to the actual polytopic constraint of (2.29). However, simulation results [16] show that in most cases the degree of suboptimality is not significant. Furthermore predicted performance can be improved by a subsequent univariate search over $\alpha \in [0, 1]$ with $z_k = (x_k, \alpha\mathbf{c}_k^*)$ where $\mathbf{c}_k^*$ is the solution of (2.52). To retain the guarantee of closed-loop stability this is performed subject to the constraints that the vector Ψz_k defining the tail of the predicted sequence at time k should lie in the ellipsoid $\mathcal{E}_z$ and subject to the constraint $Fx_k + Gu_k \leq 1$. This modification requires negligible additional computation.

2.9 Optimized Prediction Dynamics

The MPC algorithms described thus far parameterize the predicted inputs in terms of a projection onto the standard basis vectors e_i, so, for example

$$\mathbf{c}_k = \sum_{i=0}^{N-1} c_{i|k} e_{i+1}$$

in the case that if $n_u = 1$. As a consequence the degrees of freedom have a direct effect on the predictions only over the N-step mode 1 prediction horizon, which therefore has to be taken to be sufficiently long to ensure that constraints are met during the transients of the prediction system response. Combined with the additional requirement that the terminal constraint is met at the end of the mode 1 horizon for as large a set of initial conditions as possible, this places demands on N that can make the computational load of MPC prohibitive for applications with high sampling rates.

To overcome this problem an extra mode can be introduced into the predicted control trajectories, as is done for example in triple mode MPC [23]. This additional mode introduces degrees of freedom into predictions after the end of the mode 1 horizon but allows efficient handling of the constraints at these prediction instants, thus allowing the mode 1 horizon to be shortened without adversely affecting optimality and the size of the feasible set. Alternatively in the context of dual-mode predictions it is possible to consider parameterizing predicted control trajectories as an expansion over a finite set of basis functions. Exponential basis functions, which allow the use of arguments based on the tail for analysing stability and convergence (e.g. [24]), are most commonly employed in MPC, a special case being expansion over Laguerre functions (e.g. [25]).

A framework that encompasses projection onto a general set of exponential basis functions was developed in [26]. In this approach, the matrices E and M appearing in the transition matrix Ψ of the augmented prediction dynamics (2.25) are not chosen as prescribed by (2.26b), but instead are replaced by variables, denoted A_c and C_c that are optimized offline as we discuss later in this section. With this modification the prediction dynamics are given by

$$z_{i+1|k} = \Psi z_{i|k}, \quad i = 0, 1, \ldots \tag{2.56a}$$

where

$$z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Phi & BC_c \\ 0 & A_c \end{bmatrix} \tag{2.56b}$$

and the predicted state and control trajectories are generated by

$$u_{i|k} = \begin{bmatrix} K & C_c \end{bmatrix} z_{i|k} \tag{2.56c}$$

$$x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} z_{i|k}. \tag{2.56d}$$

As in Sect. 2.7, the predicted control law of (2.56c) has the form of a dynamic feedback controller, the initial state of which is given by $\mathbf{c}_k$. However in Sect. 2.7 the matrix M of (2.26) is nilpotent, so that $M^N \mathbf{c}_k = 0$ and hence $u_{i|k} = Kx_{i|k}$, for all $i = N, N+1, \ldots$. For the general case considered in (2.56), A_c is not necessarily nilpotent, which implies that the direct effect of the elements of $\mathbf{c}_k$ can extend beyond the initial N steps of the prediction horizon in this setting.

Following a development analogous to that of Sect. 2.7, the predicted cost (2.11) can be expressed as $J(x_k, \mathbf{c}_k) = \|z_{0|k}\|_W^2$ where W satisfies the Lyapunov matrix equation

$$W = \Psi^T W \Psi + \hat{Q}, \quad \hat{Q} = \begin{bmatrix} Q + K^T RK & K^T RC_c \\ C_c^T RK & C_c^T RC_c \end{bmatrix}. \tag{2.57}$$

By examining the partitioned blocks of this equation, it can be shown (using the same approach as the proof of Theorem 2.10) that its solution is block diagonal

$$W = \begin{bmatrix} W_x & 0 \\ 0 & W_c \end{bmatrix}$$

whenever K is the unconstrained optimal feedback gain. Here W_x is the solution of the Riccati equation (2.9) and W_c is the solution of the Lyapunov equation $W_c = A_c^T W A_c + C_c^T (B^T W_x B + R) C_c$. By Lemma 2.1, the solution is unique and satisfies $W_c \succ 0$ whenever A_c is strictly stable.

The constraints (2.2) applied to the predictions of (2.56) require that $z_{0|k}$ lies in the polytopic set

$$\mathcal{Z} = \{z : \begin{bmatrix} F + GK & GC_c \end{bmatrix} \Psi^i z \leq \mathbf{1}, \quad i = 0, 1, \ldots, \nu_z\}, \tag{2.58}$$

where $\begin{bmatrix} F + GK & GC_c \end{bmatrix} \Psi^{\nu_z+1} z \leq \mathbf{1}$, for all z satisfying $\begin{bmatrix} F + GK & GC_c \end{bmatrix} \Psi^i z \leq \mathbf{1}$, $i = 0, 1, \ldots, \nu_z$. By Theorem 2.3 this is the MPI set for the dynamics of (2.56) and constraints (2.2), and its projection onto the x-subspace is therefore equal to the feasible set for x_k for the prediction system (2.56) and constraints $\begin{bmatrix} F + GK & GC_c \end{bmatrix} z \leq \mathbf{1}$. The MPC law of Algorithm 2.2 with the cost matrix W defined in (2.57) and constraint set $\mathcal{Z}$ defined in (2.58) has the stability and convergence properties stated in Theorem 2.11.

Alternatively, and similarly to the discussion in Sect. 2.7, it is possible to replace the linear constraints $z_{0|k} \in \mathcal{Z}$ by a single quadratic constraint $z_{0|k} \in \mathcal{E}_z$ in order to reduce the online computational load of Algorithm 2.2. As in Sect. 2.7, we require that $\mathcal{E}_z = \{z : z^T P_z z \leq 1\}$ is positively invariant for the dynamics $z_{k+1} = \Psi z_k$ and constraints $\begin{bmatrix} F + GK & GC_c \end{bmatrix} z_k \leq \mathbf{1}$, which by Theorem 2.9 requires that there exists a symmetric matrix H such that P_z, A_c and C_c satisfy

$$P_z - \Psi^T P_z \Psi \succeq 0 \tag{2.59a}$$

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & GC_c \end{bmatrix} \\ \begin{bmatrix} (F + GK)^T \\ (GC_c)^T \end{bmatrix} & P_z \end{bmatrix} \succeq 0, \quad e_i^T H e_i \leq 1, \quad i = 1, \ldots n_C. \tag{2.59b}$$

Under these conditions the stability and convergence properties specified by Theorem 2.11 again apply.

Using $\mathcal{E}_z$ as the constraint set in the online optimization in place of $\mathcal{Z}$ reduces the region of attraction of the MPC law. However, to compensate for this effect it is possible to design the prediction system parameters A_c and C_c so as to maximize the projection of $\mathcal{E}_z$ onto the x-subspace. Analogously to (2.44), this is achieved by maximizing the determinant of $[I_{n_x} \;\; 0] P_z^{-1} [I_{n_x} \;\; 0]^T$ subject to (2.59a), (2.59b). Unlike the case considered in Sect. 2.7, this is performed with A_c and C_c as optimization variables. Viewed as inequalities in these variables, (2.59a), (2.59b) represent non-convex constraints. The problem can however be convexified provided the dimension of $\mathbf{c}_k$ is at least as large as that of n_x [26] using a technique introduced by [27] in the context of $\mathcal{H}_\infty$ control, as we discuss next.

Introducing variables $U, V \in \mathbb{R}^{n_x \times \nu_c}$ (where ν_c is the length of $\mathbf{c}_k$), $\Xi \in \mathbb{R}^{n_x \times n_x}$, $\Gamma \in \mathbb{R}^{n_u \times n_x}$ and symmetric $X, Y \in \mathbb{R}^{n_x \times n_x}$, we re-parameterize the problem by defining

$$P_z = \begin{bmatrix} X^{-1} & X^{-1}U \\ U^T X^{-1} & \bullet \end{bmatrix} \quad P_z^{-1} = \begin{bmatrix} Y & V \\ V^T & \bullet \end{bmatrix}, \quad \Xi = U A_c V^T, \quad \Gamma = C_c V^T \tag{2.60}$$

(where $\bullet$ indicates blocks of P_z and P_z^{-1} that are determined uniquely by X, Y, U, V). Since $P_z P_z^{-1} = I$, we also require that

$$UV^T = X - Y. \tag{2.61}$$

The constraints (2.59a), (2.59b) can then be expressed as LMIs in Ξ, Γ, X and Y. Specifically, using Schur complements, (2.59a) is equivalent to

$$\begin{bmatrix} P_z & P_z\Psi \\ \Psi^T P_z & P_z \end{bmatrix} \succeq 0,$$

and multiplying the LHS of this inequality by $\operatorname{diag}\{\Pi^T, \Pi^T\}$ on the left and $\operatorname{diag}\{\Pi, \Pi\}$ on the right, where $\Pi = \begin{bmatrix} Y & X \\ V^T & 0 \end{bmatrix}$, yields the equivalent condition

$$\begin{bmatrix} \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi Y + B\Gamma & \Phi X \\ \Xi + \Phi Y + B\Gamma & \Phi X \end{bmatrix} \\ \star & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} \succeq 0 \tag{2.62a}$$

(where the block marked $\star$ is omitted as the matrix is symmetric). Similarly, pre- and post-multiplying the matrix inequality in (2.59b) by $\operatorname{diag}\{I, \Pi^T\}$ and $\operatorname{diag}\{I, \Pi\}$, respectively, yields

$$\begin{bmatrix} H & \begin{bmatrix} (F + GK)Y + G\Gamma & (F + GK)X \end{bmatrix} \\ \star & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} \succeq 0, \quad e_i^T H e_i \le 1, \quad i = 1, \ldots, n_C. \tag{2.62b}$$

Therefore matrices P_z, A_c and C_c can exist satisfying (2.59a), (2.59b) only if the conditions (2.62a), (2.62b) are feasible. Moreover, (2.62a), (2.62b) are both necessary and sufficient for feasibility of (2.59a), (2.59b) if $\nu_c \geq n_x$ since (2.61) then imposes no additional constraints on X and Y (in the sense that U and V then exist satisfying (2.61), for all $X, Y \in \mathbb{R}^{n_x \times n_x}$). The volume of the projection of $\mathcal{E}_z$ onto the x-subspace is proportional to $\det(Y)$, which is maximized by solving the convex optimization:

$$\underset{\Xi,\Gamma,X,Y}{\text{maximize}} \ \log\det(Y) \ \text{ subject to (2.62a), (2.62b).} \tag{2.63}$$

Finally, we note that the conditions (2.62a), (2.62b) do not depend on the value of ν_c, and since there is no advantage to be gained using a larger value, we set $\nu_c = n_x$. From the solution of (2.63), A_c and C_c are given uniquely by

$$A_c = U^{-1}\Xi V^{-T}, \quad C_c = \Gamma V^{-T}.$$

while P_z can be recovered from (2.60).

A remarkable property of the optimized prediction dynamics is that the maximal projection of $\mathcal{E}_z$ onto the x-subspace is as large as the maximal positively invariant ellipsoidal set under any linear state feedback control law [26]. The importance of this is that it overcomes the trade-off that exists in the conventional MPC formulations of Sects. 2.7 and 2.5 between performance and the size of the feasible set. Thus, in the interests of enlarging the terminal invariant set (and hence the overall region of attraction), it may be tempting to de-tune the terminal control law. But this has an adverse effect on predicted performance, and potentially also reduces closed-loop performance. Such loss of performance is however avoided if the optimized prediction dynamics are used since K can be chosen to be the unconstrained LQ optimal gain, without any detriment to the size of the region of attraction.

Example 2.6 The maximal ellipsoidal region of attraction of Algorithm 2.2 for the same system model, constraints and cost as Example 2.1 is shown in Fig. 2.10. Since this is obtained by optimizing the prediction dynamics using (2.63), the number of degrees of freedom in the resulting prediction system (i.e. the length of $\mathbf{c}_k$ in (2.56)) is the same as n_x, which here is 2. The area of this maximal ellipsoid is 13.5, whereas the area of the ellipsoidal region of attraction obtained from (2.44) for the non-optimized prediction system (2.25) and the same number of degrees of freedom in predictions (i.e. $N = 2$) is just 2.3.

Figure 2.10 also shows the polytopic feasible set for x_k in Algorithm 2.2 when the optimized prediction dynamics are used to define the polytopic constraint set $\mathcal{Z}$ in (2.58). Despite having only 2 degrees of freedom, the optimized prediction dynamics result in a polytopic feasible set covering 97 % of the area of the maximal feasible set $\mathcal{F}_\infty$, which for this example is equal to the polytopic feasible set for the non-optimized dynamics with $N = 26$ degrees of freedom (also shown in Fig. 2.10). For the initial condition $x_0 = (-7.5, 0.5)$, the closed-loop cost of Algorithm 2.2 with the optimized prediction dynamics containing 2 degrees of freedom and polytopic constraint set $\mathcal{Z}$ is 357.7, which from Table 2.1 is only 0.5 % suboptimal relative to the ideal optimal cost with $N = 11$. ◊

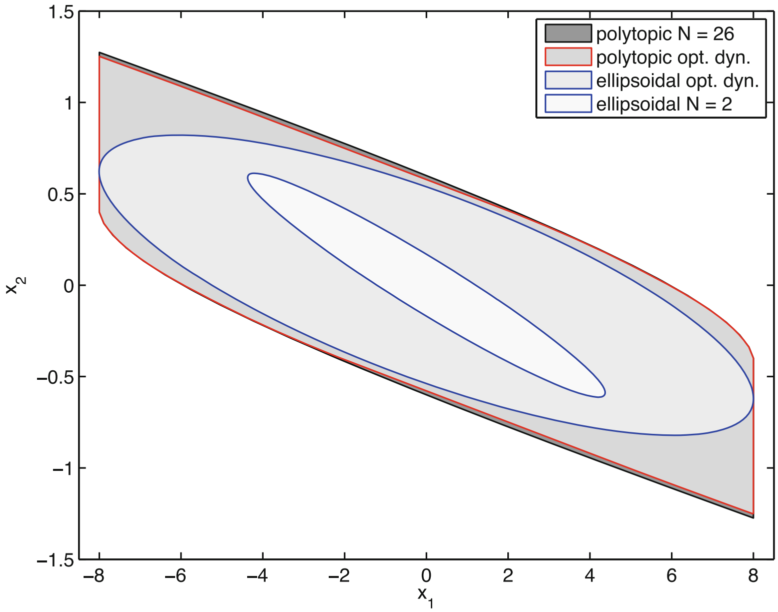

Fig. 2.10 Ellipsoidal region of attraction for optimized dynamics (with 2 degrees of freedom) and ellipsoidal region of attraction for $N = 2$. Also shown are the maximal polytopic region of attraction ($\mathcal{F}_{26}$) and the polytopic region of attraction for the optimized dynamics

2.10 Early MPC Algorithms

Perhaps the earliest reference to MPC strategies is [28], although the ideas of rolling horizons and decision making based on forecasts had been used earlier in different contexts (e.g. production scheduling). There have since been thousands of MPC papers published in the open literature, including a plethora of reports on applications of MPC to industrial problems. Early contributions (e.g. [29, 30]) were based on finite horizon predictive costs and as such did not carry guarantees of closed-loop stability.

The most cited of the early papers on predictive control is the seminal work [31, 32] on Generalized Predictive Control (GPC). This uses an input–output model to express the vector of output predictions as an affine function of the vector of predicted inputs

$$\mathbf{y}_k = \begin{bmatrix} y_{1|k} \\ \vdots \\ y_{N|k} \end{bmatrix} = C_G \Delta \mathbf{u}_k + \mathbf{y}_k^f, \quad \Delta \mathbf{u}_k = \begin{bmatrix} \Delta u_{0|k} \\ \vdots \\ \Delta u_{N_u-1|k} \end{bmatrix}$$

Here N_u denotes an input prediction horizon which is chosen to be less than or equal to the prediction horizon N. The matrix C_G is the block striped (Toeplitz) lower triangular matrix comprising the coefficients of the system step response, $C_G \Delta \mathbf{u}_k$ denotes the predicted forced response at time k, and $\mathbf{y}_k^f$ denotes the free response at time k due to non-zero initial conditions. The notation Δu is used to denote the control increments (i.e. $\Delta u_{i|k} = u_{i|k} - u_{i-1|k}$). Posing the problem in terms of control increments implies the automatic inclusion in the feedback loop of integral action which rejects (in the steady state) constant additive disturbances.

The GPC algorithm minimizes a cost, subject to constraints, which penalizes predicted output errors (deviations from a constant reference vector r) and predicted control increments

$$J_k = (\mathbf{r} - \mathbf{y}_k)^T \hat{Q}(\mathbf{r} - \mathbf{y}_k) + \Delta \mathbf{u}_k^T \hat{R} \Delta \mathbf{u}_k \tag{2.64}$$

where $\mathbf{r} = [r^T \ \cdots \ r^T]^T$, $\hat{Q} = \text{diag}\{Q, \ldots, Q\}$ and $\hat{R} = \text{diag}\{R, \ldots, R\}$. By setting the derivative of this cost with respect to $\Delta \mathbf{u}_k$ equal to zero, the unconstrained optimum vector of predicted control increments can be derived as

$$\Delta \mathbf{u}_k = \left(C_G^T \hat{Q} C_G + \hat{R} \right)^{-1} C_G^T \hat{Q}(\mathbf{r} - \mathbf{y}_k^f) \tag{2.65}$$

The optimal current control move $\Delta u_{0|k}$ is then computed from the first element of this vector, and the control input $u_k = \Delta u_{0|k} + u_{k-1}$ is applied to the plant.

GPC has proven effective in a wide range of applications and is the basis of a number of commercially successful MPC algorithms. There are several reasons for the success of the approach, principal among these are: the simplicity and generality of the plant model, and the lack of sensitivity of the controller to variable or unknown plant dead time and unknown model order; the fact that the approach lends itself to self-tuning and adaptive control, output feedback control and stochastic control problems; and the ability of GPC to approximate various well-known control laws through appropriate definition of the cost (2.64), for example LQ optimal control, minimum variance and dead-beat control laws. For further discussion of these aspects of GPC and its industrial applications we refer the reader to [31–34].

Although widely used in industry, the original formulation of GPC did not guarantee closed-loop stability except in limiting cases of the input and output horizons (for example, in the limit as both the prediction and control horizons tend to infinity, or when the control horizon is $N_u = 1$, the prediction horizon is $N = \infty$ and the open-loop system is stable). However, the missing stability guarantee can be established by imposing a suitable terminal constraint on predictions.

Terminal equality constraints that force the predicted tracking errors to be zero at all prediction times beyond the N-step prediction horizon were proposed for receding horizon controllers in the context of continuous time, time-varying unconstrained systems in [35], time invariant discrete time unconstrained systems [36], and nonlinear constrained systems [37]. This constraint effectively turns the cost of (2.64) into an infinite horizon cost which can be shown to be monotonically non-increasing using an argument based on the prediction tail. As a result it can be shown that tracking errors are steered asymptotically to zero. The terminal equality constraint need only to be applied over n_x prediction steps after the end of an initial N-step horizon. Under the assumption that $N > n_x$, the general solution of the equality constraints will contain, implicitly, $(N - n_x)n_u$ degrees of freedom and these can be used to minimize the resulting predicted cost (i.e. the cost of (2.64) after the expression for the general solution of the equality constraints has been substituted into (2.64)). A closely related algorithm to GPC that addresses the case of constrained systems is Stable GPC (SGPC) [38], which establishes closed-loop stability by ensuring that optimal predicted cost is a Lyapunov function for the closed-loop system. Related approaches [36, 39] use terminal equality constraints explicitly, however SGPC implements the equality constraints implicitly while preserving an explicit representation of the degrees of freedom in predictions.

The decision variables in the SGPC predicted control trajectories appear as perturbations of a stabilizing feedback law, and in terms of a left factorization of transfer function matrices, the predicted control sequence is given by

$$u_k = \tilde{Y}^{-1}(z^{-1})\left(c_k - z^{-1}\tilde{X}(z^{-1})y_{k+1}\right). \tag{2.66}$$

Here z is the z-transform variable (z^{-1} can be thought of as the backward shift operator, namely $z^{-1}f_k = f_{k-1}$), and $\tilde{X}(z^{-1})$, $\tilde{Y}(z^{-1})$ are polynomial solutions (expressed in powers of z^{-1}) of the matrix Bezout identity

$$\tilde{Y}(z^{-1})A(z^{-1}) + z^{-1}\tilde{X}(z^{-1})B(z^{-1}) = I. \tag{2.67}$$

For simplicity, we use u_k instead of Δu_k and consider the regulation rather than the setpoint tracking problem (i.e. we take $r = 0$). Here $B(z^{-1})$, $A(z^{-1})$ are the polynomial matrices (in powers of z^{-1}) defining right coprime factors of the system transfer function matrix, $G(z^{-1})$, where

$$y_{k+1} = G(z^{-1})u_k = B(z^{-1})A^{-1}(z^{-1})u_k \tag{2.68}$$

The determination of the coprime factors can be achieved through the computation of the Smith–McMillan form of the transfer function matrix, $G(z^{-1}) = L(z^{-1})S(z^{-1})R(z^{-1})$ where $S(z^{-1}) = \mathcal{E}(z^{-1})\Psi^{-1}(z^{-1})$ with both $\mathcal{E}(z^{-1})$ and $\Psi(z^{-1})$ being diagonal polynomial matrix functions of z^{-1}. The right coprime factors can then be chosen as $B(z^{-1}) = L(z^{-1})\mathcal{E}(z^{-1})$, $A(z^{-1}) = R^{-1}(z^{-1})\Psi(z^{-1})$. Alternatively, $B(z^{-1})$, $A(z^{-1})$ can be computed through an iterative procedure, which we describe now.

Assuming that $G(z^{-1})$ is given as

$$G(z^{-1}) = \frac{1}{d\left(z^{-1}\right)}N(z^{-1})$$

we need to find the solution, $A(z^{-1})$, $B(z^{-1})$, of the Bezout identity

$$N(z^{-1})A(z^{-1}) = B(z^{-1})d(z^{-1}) \tag{2.69}$$

for which (2.67) admits a solution for $\tilde{X}(z^{-1})$, $\tilde{Y}(z^{-1})$. This solution can be shown to be unique under the assumption that the coefficient of z^0 in $A(z^{-1})$ is the identity, and that $A(z^{-1})$ and $B(z^{-1})$ are of minimal degree. Equation (2.69) defines a set of under-determined linear conditions on the coefficients of $B(z^{-1})$, $A(z^{-1})$. Thus the coefficients of $B(z^{-1})$, $A(z^{-1})$ can be expressed as an affine function of a matrix, say R, where R defines the degrees of freedom which are to be given up so that (2.67) admits a solution. The determination of R constitutes a nonlinear problem which, nevertheless, can be solved to any desired degree of accuracy by solving (2.67) iteratively. The iteration consists of using the least squares solution for R of (2.67) to update the choice for the coefficients of $A(z^{-1})$, $B(z^{-1})$; these updated values are then used in (2.67) to update the solution for $\tilde{Y}(z^{-1})$, $\tilde{X}(z^{-1})$, and so on. Each cycle of this iteration reduces the norm of the error in the solution of (2.67) and the iterative process can be terminated when the norm of the error is below a practically desirable threshold.

Substituting (2.68) into (2.66), pre-multiplying by $\tilde{Y}(z^{-1})$ and using the Bezout identity (2.67) provides the prediction model:

$$\begin{aligned} y_{k+1} &= B(z^{-1})c_k + y^f_{k+1} \\ u_k &= A(z^{-1})c_k + u^f_k. \end{aligned} \tag{2.70}$$

Here y^f_k and u^f_k denote the components of the predicted output and input trajectories corresponding to the free response of the model due to non-zero initial conditions. Consider now the dual coprime factorizations $B(z^{-1})A^{-1}(z^{-1}) = \tilde{A}^{-1}(z^{-1})\tilde{B}(z^{-1})$, $X(z^{-1})Y^{-1}(z^{-1}) = \tilde{Y}^{-1}(z^{-1})\tilde{X}(z^{-1})$ satisfying the Bezout identity

$$\begin{bmatrix} z^{-1}\tilde{X}(z^{-1}) & \tilde{Y}(z^{-1}) \\ \tilde{A}(z^{-1}) & -\tilde{B}(z^{-1}) \end{bmatrix} \begin{bmatrix} B(z^{-1}) & Y(z^{-1}) \\ A(z^{-1}) & -z^{-1}X(z^{-1}) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.71}$$

Detailed calculation, based on simulating forward in time the relationships $\tilde{Y}(z^{-1})u_k = c_k - z^{-1}\tilde{X}(z^{-1})y_{k+1}$ and $\tilde{A}(z^{-1})y_{k+1} = \tilde{B}(z^{-1})u_k$, leads to the following affine relationship from the vector of predicted controller perturbations, $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ (with $c_{i|k} = 0$, for all $i \geq \nu$), to the vectors of predicted outputs, $\mathbf{y}_k = (y_{1|k}, \ldots, y_{N|k})$, and inputs, $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$:

$$\begin{bmatrix} C_{z^{-1}\tilde{X}} & C_{\tilde{Y}} \\ C_{\tilde{A}} & -C_{\tilde{B}} \end{bmatrix} \begin{bmatrix} \mathbf{y}_k \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \mathbf{c}_k \\ 0 \end{bmatrix} - \begin{bmatrix} H_{z^{-1}\tilde{X}} & C_{\tilde{Y}} \\ H_{\tilde{A}} & -H_{\tilde{B}} \end{bmatrix} \begin{bmatrix} \mathbf{y}^p_k \\ \mathbf{u}^p_k \end{bmatrix} \tag{2.72}$$

where $N = \nu + n_A$, $\mathbf{y}^p_k = (y_{k-n_X-1}, \ldots, y_k)$ and $\mathbf{u}^p_k = (u_{k-n_Y}, \ldots, u_{k-1})$ denote vectors of past input and output values and n_A, n_X, n_Y are the degrees of the polynomials $A(z^{-1})$, $X(z^{-1})$, $Y(z^{-1})$. The C and H matrices are block Toeplitz convolution matrices, which are defined for any given matrix polynomial $F(z^{-1}) = F_0 + F_1 z^{-1} + \cdots + F_m z^{-m}$ by

$$C_F \doteq \begin{bmatrix} F_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ F_1 & F_0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ F_m & F_{m-1} & \cdots & F_0 & 0 & \cdots & 0 \\ 0 & F_m & \cdots & F_1 & F_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_m & F_{m-1} & \cdots & F_0 \end{bmatrix}, \quad H_F \doteq \begin{bmatrix} F_m & F_{m-1} & \cdots & F_1 \\ 0 & F_m & \cdots & F_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

where the row-blocks of C_F and H_F consist, respectively, of N and m blocks.

The solution of (2.72) for the vectors, $\mathbf{y}_k$ and $\mathbf{u}_k$, of output and input predictions is affine in the vector of the degrees of freedom $\mathbf{c}_k$, and hence the predicted cost is quadratic in $\mathbf{c}_k$. In particular the Bezout identity (2.71) implies an explicit expression for the inverse of the matrix on the LHS of (2.72), which in turn implies the solution

$$\begin{bmatrix} \mathbf{y}_k \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} C_B \\ C_A \end{bmatrix} \mathbf{c}_k - \begin{bmatrix} C_B & C_Y \\ C_A & -C_{z^{-1}X} \end{bmatrix} \begin{bmatrix} H_{z^{-1}\tilde{X}} & H_{\tilde{Y}} \\ H_{\tilde{A}} & -H_{\tilde{B}} \end{bmatrix} \begin{bmatrix} \mathbf{y}_k^p \\ \mathbf{u}_k^p \end{bmatrix}.$$

The second term on the RHS of this expression corresponds to the free responses of the output and input predictions, and, on account of the structure of the convolution matrices in (2.72) and the Bezout identity (2.71), these free responses are zero at the end of the prediction horizon consisting of $N = \nu + N_A$ steps. From this observation and the finite impulse response of the filters $B(z^{-1})$ and $A(z^{-1})$ in (2.70), it follows that SGPC imposes an implicit terminal equality constraint, namely that both the predicted input and output vectors reach the steady value of zero at the end of the horizon of $N = \nu + N_A$ prediction time steps, and this gives the algorithm a guarantee of closed-loop stability.

Equality terminal constraints can be overly stringent but it is possible to modify SGPC so that the predicted control law of (2.66) imitates what is obtained using the predicted control law, $u_{i|k} = Kx_{i|k} + c_{i|k}$, of the closed-loop paradigm. This can be achieved through the use of the Bezout identity

$$\tilde{Y}(z^{-1})A(z^{-1}) + z^{-1}\tilde{X}(z^{-1})B(z^{-1}) = A_{cl}(z^{-1}) \tag{2.73}$$

where $A_{cl}(z^{-1})$ is such that $B(z^{-1})$ and $A_{cl}(z^{-1})$ define right coprime factors of the closed-loop transfer function matrix (under the control law $u = Kx + c$). The fact that the same $B(z^{-1})$ can be used for both the open and closed-loop transfer function matrices can be argued as follows. Let $\hat{B}(z^{-1})$, $\hat{A}(z^{-1})$ be the right coprime factors of $(zI - A)^{-1}B$ such that $B\hat{A}(z^{-1}) = (zI - A)\hat{B}(z^{-1})$. The consistency condition for this equation is $N(zI - A)\hat{B}(z^{-1}) = 0$ where N is the full-rank left annihilator of B (satisfying the condition $NB = 0$). This is however is also the consistency condition for the equation

$$BA_{cl}(z^{-1}) = (zI - A - BK)^{-1}\,\hat{B}(z^{-1}),$$

which implies that $\hat{B}(z^{-1})$ can also be used in the right coprime factorization of $(zI-A-BK)^{-1}B$. Thus the same $B(z^{-1}) = C\hat{B}(z^{-1})$ can be used for both the open and closed-loop transfer function matrices given that these transfer function matrices are obtained by the pre-multiplication by C of $(zI-A)^{-1}B$ and $(zI-A-BK)^{-1}B$, respectively. The property that a common $B(z^{-1})$ can be used in the factorization of the open and closed-loop transfer function matrices can also be used to prove that the control law of (2.66) guarantees the internal stability of the closed-loop system [40] (when $\tilde{Y}(z^{-1})$, $\tilde{X}(z^{-1})$ satisfy either of (2.67) or (2.73)).

SGPC introduced a Youla parameter into the MPC problem and this provides an alternative way to that described in Sect. 2.9 to endow the prediction structure with control dynamics. This can be achieved by replacing the polynomial matrices $\tilde{Y}(z^{-1})$, $\tilde{X}(z^{-1})$, respectively by

$$\tilde{M}(z^{-1}) = \tilde{Y}(z^{-1}) - z^{-1}Q(z^{-1})B(z^{-1})$$
$$\tilde{N}(z^{-1}) = \tilde{X}(z^{-1}) + A(z^{-1})Q(z^{-1})$$

where $Q(z^{-1})$ represents a free parameter (which can be chosen to be any polynomial matrix, or stable transfer function matrix). If $\tilde{Y}(z^{-1})$ and $\tilde{X}(z^{-1})$ satisfy the Bezout identity (either (2.67) or (2.73)), then so will $\tilde{M}(z^{-1})$ and $\tilde{N}(z^{-1})$, which therefore can be used in the control law of (2.66) in place of $\tilde{Y}(z^{-1})$ and $\tilde{X}(z^{-1})$. The advantage of this is that the degrees of freedom in $Q(z^{-1})$ can be used to enhance the robustness of the closed-loop system to model parameter uncertainty or to enlarge the region of attraction of the algorithm [38].

At first sight it may appear that the relationships above will not hold in the presence of constraints. However this is not so, because the perturbations c_k have been introduced in order to ensure that constraints are respected and therefore the predicted trajectories are generated by the system operating within its linear range. These prediction equations can be used to express the vector of predicted outputs and inputs as functions of the vector of predicted degrees of freedom, $\mathbf{c}_k = (c_{0k}, \ldots, c_{N-1|k}, c_\infty, c_\infty, \ldots)$ where c_∞ denotes the constant value of c which ensures that the steady-state predicted output is equal to the desired setpoint vector r and the vector $\mathbf{c}_k$ contains Nn_u degrees of freedom. Clearly for a regulation problem with $r = 0$, c_∞ would be chosen to be zero. SGPC then proceeds to minimize the cost of (2.65) over the degrees of freedom $(c_{0k}, \ldots, c_{N-1|k})$ subject to constraints and implements the control move indicated by (2.66).

The algorithms discussed in this section are based on output feedback and are appropriate in cases where the assumption that the states are measurable and available for the purposes of feedback does not hold true. In instances like this one can, instead, revert to a state-space system representation constructed using current and past inputs and outputs as states (e.g. [41]) or a state-space description of the combination of the system dynamics together with the dynamics of a state observer (e.g. [42], which established invariance using low-complexity polytopes, namely polytopes with $2n_x$ vertices).

2.11 Exercises

1 A first-order system with the discrete time model

$$x_{k+1} = 1.5x_k + u_k$$

is to be controlled using a predictive controller that minimizes the predicted performance index

$$J(x_k, u_{0|k}, u_{1|k}) = \sum_{i=0}^{1} \left(x_{i|k}^2 + 10u_{i|k}^2 \right) + q x_{2|k}^2$$

where q is a positive constant.

(a) Show that the unconstrained predictive control law is $u_k = -0.35x_k$ if $q = 1$.
(b) The unconstrained optimal control law with respect to the infinite horizon cost $\sum_{k=0}^{\infty}(x_k^2 + 10u_k^2)$ is $u_k = -0.88x_k$. Determine the value of q so that the unconstrained predictive control law coincides with this LQ optimal control law.
(c) The predicted cost is to be minimized subject to input constraints

$$-0.5 \le u_{i|k} \le 1.$$

If the predicted inputs are defined as $u_{i|k} = -0.88x_{i|k}$, for all $i \ge 2$, show that the MPC optimization problem is guaranteed to be recursively feasible if $u_{i|k}$ satisfies these constraints for $i = 0, 1$ and 2.

2 (a) A discrete time system is defined by

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix} x_k, \quad y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

where α is a constant. Show that $-1 \le y_k \le 1$, for all $k \ge 0$ if and only if $|\alpha| < 1$ and

$$\begin{bmatrix}-1\\-1\end{bmatrix} \le x_0 \le \begin{bmatrix}1\\1\end{bmatrix}.$$

(b) A model predictive control strategy is to be designed for the system

$$x_{k+1} = \begin{bmatrix}\beta & 1\\0 & \alpha\end{bmatrix} x_k + \begin{bmatrix}1\\0\end{bmatrix} u_k, \quad y_k = \begin{bmatrix}1 & 0\end{bmatrix} x_k, \quad -1 \le u_k \le 1$$

where α and β are constants, with $|\alpha| < 1$. Assuming that, for $i \ge N$, the i steps ahead predicted input is defined as

$$u_{i|k} = \begin{bmatrix}-\beta & 0\end{bmatrix} x_{i|k},$$

show that:

(i) $\displaystyle\sum_{i=0}^{\infty}(y_{i|k}^2 + u_{i|k}^2) = \sum_{i=0}^{N-1}(y_{i|k}^2 + u_{i|k}^2) + (\beta^2 + 1)x_{N|k}^T \begin{bmatrix}1 & 0\\0 & \frac{1}{1-\alpha^2}\end{bmatrix} x_{N|k}.$

(ii) $-1 \le u_{i|k} \le 1$ for all $i \ge N$ if

$$\begin{bmatrix}-1\\-1\end{bmatrix} \le |\beta|\, x_{N|k} \le \begin{bmatrix}1\\1\end{bmatrix}.$$

(c) Comment on the suggestion that an MPC law based on minimizing the cost in (b)(i) subject to $-1 \le u_{i|k} \le 1$ for $i = 0, \ldots, N-1$ and the terminal constraint $x_{N|k} = 0$ would be stable. Why would it be preferable to use the terminal inequality constraints of (b)(ii) instead of this terminal equality constraint.

3 A system has the model

$$x_{k+1} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} x_k + \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} u_k, \quad y_k = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} x_k.$$

(a) Show that, if $u_k = \frac{1}{\sqrt{2}} y_k$, then

$$\sum_{k=0}^{\infty} \tfrac{1}{2}\left(y_k^2 + u_k^2\right) = \|x_0\|^2.$$

(b) A predictive control law is defined at each time step k by $u_k = u_{0|k}^*$, where $(u_{0|k}^*, \ldots, u_{N-1|k}^*)$ is the minimizing argument of

$$\min_{u_{0|k},\ldots,u_{N-1|k}} \sum_{i=0}^{N-1} \tfrac{1}{2}\left(y_{i|k}^2 + u_{i|k}^2\right) + \|x_{N|k}\|^2.$$

Show that the closed-loop system is stable.

(c) The system is now subject to the constraint $-1 \le y_k \le 1$, for all k. Will the closed-loop system necessarily be stable if the optimization in part (b) includes the constraints $-1 \le y_{i|k} \le 1$, for $i = 1, 2, \ldots, N+1$?

4 A discrete time system is described by the model $x_{k+1} = Ax_k + Bu_k$ with

$$A = \begin{bmatrix} 0.3 & -0.9 \\ -0.4 & -2.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

where $u_k = Kx_k$ for $K = \begin{bmatrix} 0.244 & 1.751 \end{bmatrix}$, and for all $k = 0, 1 \ldots$ the state x_k is subject to the constraints

$$\left| \begin{bmatrix} 1 & -1 \end{bmatrix} x_k \right| \le 1.$$

(a) Describe a procedure based on linear programming for determining the largest invariant set compatible with constraints $|\begin{bmatrix} 1 & -1 \end{bmatrix} x| \le 1$.

(b) Demonstrate by solving a linear program that the maximal invariant set is defined by

$$\{x : Fx \le \mathbf{1} \text{ and } F\Phi x \le \mathbf{1}\},$$

where $F = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\Phi = \begin{bmatrix} 0.42 & -0.025 \\ -0.16 & -0.35 \end{bmatrix}$.

5 Consider the system of Question 4 with the cost $\sum_{k=0}^{\infty} \left(\|x_k\|_Q^2 + \|u_k\|_R^2 \right)$, with $Q = I$ and $R = 1$.

(a) For $K = \begin{bmatrix} 0.244 & 1.751 \end{bmatrix}$, solve the Lyapunov matrix equation (2.5) to find W and hence verify using Theorem 2.1 that K is the optimal unconstrained feedback gain.

(b) Use the maximal invariant set given in Question 4(b) to prove that $x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} \Psi^i z_k$ satisfies the constraints $|\begin{bmatrix} 1 & -1 \end{bmatrix} x_{i|k}| \le 1$, for all $i \ge 0$ if $\begin{bmatrix} F & 0 \end{bmatrix} \Psi^i z_k \le \mathbf{1}$ for $i = 0, 1, \ldots, N+1$, where

$$F = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} A + BK & BE \\ 0 & M \end{bmatrix}, \quad z_k = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times N}, \quad M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}.$$

(c) Show that the predicted cost is given by

$$J(x_k, \mathbf{c}_k) = \|x_k\|_W^2 + \rho \|\mathbf{c}_k\|^2, \quad W = \begin{bmatrix} 1.33 & 0.58 \\ 0.58 & 4.64 \end{bmatrix}, \quad \rho = 6.56.$$

(d) For the initial condition $x_0 = (3.8, 3.8)$, the optimal predicted cost,

$$J_N^*(x_0) \doteq \min_{\mathbf{c}\in\mathbb{R}^N} J(x_0, \mathbf{c}) \ \text{ subject to } \begin{bmatrix} F & 0 \end{bmatrix} \Psi^i \begin{bmatrix} x_0 \\ \mathbf{c} \end{bmatrix} \leq \mathbf{1}, \ i = 1, \ldots, N+1$$

varies with N as follows:

N	8	9	10	11
$J_N^*(x_0)$	∞	826.6	826.6	826.6

(the problem is infeasible for $N \leq 8$). Suggest why $J_N^*(x_0)$ is likely to be equal to 826.6, for all $N > 9$ and state the likely value of the infinite horizon cost for the closed loop state and control sequence starting from x_0 under $u_k = Kx_k + c_{0|k}^*$ if $N = 9$.

6 For the system and constraints of Question 4 with $K = \begin{bmatrix} 0.244 & 1.751 \end{bmatrix}$:

(a) Taking $N = 2$, solve the optimization (2.41) to determine, for the prediction dynamics $z_{k+1} = \Psi z_k$, the ellipsoidal invariant set $\{z : z^T P_z z \leq 1\}$ that has the maximum area projection onto the x-subspace. Hence show that the greatest scalar α such that $x_0 = (\alpha, \alpha)$ satisfies $z_0^T P_z z_0 \leq 1$ for $z_0 = (x_0, \mathbf{c}_0)$, for some $\mathbf{c}_0 \in \mathbb{R}^2$, is $\alpha = 1.79$.

(b) Show that, for $N = 2$, the greatest α such that $x_0 = (\alpha, \alpha)$ is feasible for the constraints $\begin{bmatrix} F & 0 \end{bmatrix} \Psi^i z_0 \leq \mathbf{1}$, $i = 0, \ldots, N+1$, for $z_0 = (x_0, \mathbf{c}_0)$, for some $\mathbf{c}_0 \in \mathbb{R}^2$, is $\alpha = 2.41$. Explain why this value is necessarily greater than the value of α in (a).

(c) Determine the optimized prediction dynamics by solving (2.63) and verify that

$$C_c = \begin{bmatrix} -1.22 & -0.45 \end{bmatrix}, \quad A_c = \begin{bmatrix} 0.96 & 0.32 \\ -0.015 & -0.063 \end{bmatrix},$$

and also that the maximum scaling α such that $x_0 = (\alpha, \alpha)$ is feasible for $z_0^T P_z z_0 \leq 1$ for $z_0 = (x_0, \mathbf{c}_0)$, for some $\mathbf{c}_0 \in \mathbb{R}^2$, is $\alpha = 2.32$.

(d) Using the optimized prediction dynamics computed in part (c), define

$$\hat{\Psi} = \begin{bmatrix} A + BK & BC_c \\ 0 & A_c \end{bmatrix}$$

and show that $x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} \hat{\Psi}^i z_k$ satisfies constraints $|\begin{bmatrix} 1 & -1 \end{bmatrix} x_{i|k}| \leq 1$, for all $i \geq 0$ if $\begin{bmatrix} F & 0 \end{bmatrix} \hat{\Psi}^i z_k \leq \mathbf{1}$ for $i = 0, \ldots, 5$. Hence show that the maximum scaling α such that $x_0 = (\alpha, \alpha)$ satisfies these constraints for some $\mathbf{c}_0 \in \mathbb{R}^2$ is $\alpha = 3.82$.

(e) Show that the optimal value of the predicted cost for the prediction dynamics and constraints determined in (d) with $x_0 = (3.8, 3.8)$ is $J^*(x_0) = 1686$.

Explain why this value is greater than the predicted cost in Question 5(d) for $N = 9$. What is the advantage of the MPC law based on the optimized prediction dynamics?

7 With $K = \begin{bmatrix}0.067 & 2\end{bmatrix}$, the model of Question 4 gives

$$A + BK = \begin{bmatrix} 0.33 & 0.1 \\ -0.33 & -0.1 \end{bmatrix}.$$

(a) Explain the significance of this for the size of the feasible initial condition set of an MPC law which is subject to the state constraints $|\begin{bmatrix}1 & 1\end{bmatrix} x| \le 1$ rather than the constraints of Question 4?
(b) Explain why the feasible set of the MPC algorithm in Question 5(d) (which is subject to the constraints $|\begin{bmatrix}1 & -1\end{bmatrix} x| \le 1$) is finite for all N.

8 GPC can be cast in terms of state-space models, through which the predicted output sequence $\mathbf{y}_k = (y_{1|k}, \ldots, y_{N|k})$ can be expressed as an affine function of the predicted input sequence $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N_u-1|k})$ as $\mathbf{y}_k = C_x x_k + C_u \mathbf{u}_k$. Using this expression show that the unconstrained optimum for the minimization of the regulation cost $J_k = \mathbf{y}_k^T \hat{Q} \mathbf{y}_k + \mathbf{u}_k^T \hat{R} \mathbf{u}_k$, with $\hat{Q} = \operatorname{diag}\{Q, \ldots, Q\}$ and $\hat{R} = \operatorname{diag}\{R, \ldots, R\}$, is given by

$$\mathbf{u}_k^* = -\left(\hat{R} + C_u^T \hat{Q} C_u\right)^{-1} C_u^T \hat{Q} C_x x_k.$$

Hence show that for

$$A = \begin{bmatrix} 0.83 & -0.46 \\ -0.05 & 0.86 \end{bmatrix}, \quad B = \begin{bmatrix} 0.26 \\ 0.55 \end{bmatrix}, \quad C = \begin{bmatrix} 0.67 & 0.71 \end{bmatrix},$$

and in the absence of constraints, GPC results in an unstable closed loop system for all prediction horizons $N \le 9$ and input horizons $N_u \le N$. Confirm that the open-loop system is stable but that its zero is non-minimum phase. Construct an argument which explains the instability observed above.

9 (a) Compute the transfer function of the system of Question 8 and show that the polynomials

$$\tilde{X}(z^{-1}) = 21.0529 z^{-1} - 32.2308, \quad \tilde{Y}(z^{-1}) = 19.8907 z^{-1} + 1$$

are solutions of the Bezout identity (2.67).
(b) It is proposed to use SGPC to regulate the system of part (a) about the origin (i.e. the reference setpoint is taken to be $r = 0$) using two degrees of freedom, $\mathbf{c}_k = (c_{0|k}, c_{1|k})$, in the predicted state and input sequences, the implicit assumption being that $c_{i|k} = 0$, for all $i \ge 2$. Form the 4×4 convolution matrices $C_{z^{-1}\tilde{X}}$, $C_{\tilde{Y}}$, $C_{\tilde{A}}$, $C_{\tilde{B}}$ and confirm that

$$\begin{bmatrix} C_{z^{-1}\tilde{X}} & C_{\tilde{Y}} \\ C_{\tilde{A}} & -C_{\tilde{B}} \end{bmatrix}^{-1} = \begin{bmatrix} C_A & C_Y \\ C_B & -C_{z^{-1}X} \end{bmatrix}.$$

Hence show that the prediction equation giving the vectors of predicted outputs $\mathbf{y}_k = (y_{1|k}, \ldots, y_{4|k})$ and inputs $\mathbf{u}_k = (u_{0|k}, \ldots, u_{3|k})$ is

$$\begin{bmatrix} \mathbf{y}_k \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} C_B \\ C_A \end{bmatrix} \begin{bmatrix} \mathbf{c}_k \\ 0_{2\times 1} \end{bmatrix} - \begin{bmatrix} 12.6 & -19.9 & 11.9 \\ & 0_{3\times 3} & \\ 21.1 & -32.2 & 19.9 \\ -13.31 & 21.1 & -12.6 \\ & 0_{2\times 3} & \end{bmatrix} \begin{bmatrix} \mathbf{y}_k^p \\ \mathbf{u}_k^p \end{bmatrix}.$$

(c) Show that the predicted sequences in (b) implicitly satisfy a terminal constraint. Hence explain why the closed-loop system under SGPC is necessarily stable.

10 For the data of Question 9 plot the frequency response of the modulus of $K(z^{-1})/\big(1 + G(z^{-1})K(z^{-1})\big)$ where

$$K(z^{-1}) = \frac{\tilde{X}(z^{-1}) + A(z^{-1})Q(z^{-1})}{\tilde{Y}(z^{-1}) - z^{-1}B(z^{-1})Q(z^{-1})}$$

for the following two cases:

(a) $Q(z^{-1}) = 0$
(b) $Q(z^{-1}) = -11.7z^{-1} + 43$

Hence suggest what might be the benefit of introducing a Youla parameter into SGPC in terms of robustness to additive uncertainty in the system transfer function.

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Part II
Robust MPC

Chapter 3
Open-Loop Optimization Strategies for Additive Uncertainty

The essential components of the classical predictive control algorithms considered in Chap. 2 also underpin the design of algorithms for robust MPC. Guarantees of closed-loop properties such as stability and convergence rely on appropriately defined terminal control laws, terminal sets and cost functions. Likewise, to ensure that constraints can be met in the future, the initial plant state must belong to a suitable controllable set. However the design of these constituents and the analysis of their effects on the performance of MPC algorithms becomes more complex in the case where the system dynamics are subject to uncertainty. The main difficulty is that properties such as invariance, controlled invariance (including recursive feasibility) and monotonicity of the predicted cost must be guaranteed for all possible uncertainty realizations. In many cases this leads to computation which grows rapidly with the problem size and the prediction horizon.

Uncertainty is a ubiquitous feature of control applications. It can arise as a result of the presence of additive disturbances in the system model and can also be multiplicative in nature, for example as a result of an imprecise knowledge of the model parameters. In either case it is essential that certain properties (including closed-loop stability and performance) are preserved despite the presence of uncertainty, and this is the main preoccupation of robust MPC (RMPC). In this chapter and in Chap. 4, consideration will be given to the additive case, whereas the multiplicative case will be examined in Chap. 5. These topics will be re-examined in later chapters in the context of stochastic MPC.

Within the range of approaches that have been proposed for robust MPC, there is a fundamental difference between strategies in which optimization is performed over open-loop prediction strategies and those that optimize the parameters of predicted feedback laws. This chapter discusses open-loop optimization algorithms; these are often conceptually simpler and generally have lower computational complexity than their counterparts employing closed-loop strategies. However the techniques introduced here for determining feasibility of robust constraints and closed-loop stability analysis carry over to the closed-loop strategies considered in Chap. 4. The chapter begins with a discussion of robust constraint handling, then describes cost functions

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and stability analyses, before continuing to describe alternative approaches based on the online optimization of tubes that bound predicted state and input trajectories. The chapter concludes with a discussion of early robust MPC algorithms.

3.1 The Control Problem

We consider the system model that is obtained when a disturbance term representing additive model uncertainty is introduced into the linear dynamics of (2.1):

$$x_{k+1} = Ax_k + Bu_k + Dw_k. \tag{3.1}$$

Here D is a matrix of known parameters and $w_k \in \mathbb{R}^{n_w}$ is a vector of disturbance inputs, unknown at time k.

As in Chap. 2, the system matrices A, B are assumed known, $u_k \in \mathbb{R}^{n_u}$ is the control input at time k and the state $x_k \in \mathbb{R}^{n_x}$ is known at time k. The disturbance w_k is assumed to belong to a known set $\mathcal{W}$, namely, at each time instant k, we require that

$$w_k \in \mathcal{W}.$$

The disturbance set $\mathcal{W}$ is assumed to be full dimensional (i.e. not restricted to a subspace of $\mathbb{R}^{n_w}$) and D is assumed to be full rank, with $\text{rank}(D) = n_w$. We also assume that $\mathcal{W}$ contains the origin in its interior. Clearly, if the origin did not lie in the interior of $\mathcal{W}$, then the model uncertainty in (3.1) could be represented as the sum of a known, constant disturbance and an unknown, time-varying disturbance belonging to a known disturbance set that does contain the origin in its interior. A constant disturbance would result in a constant offset in the evolution of the state of (3.1), which could be accounted for by a translation of the origin of state space; with this modification it can always be assumed that the origin lies in the interior of $\mathcal{W}$.

The disturbance set $\mathcal{W}$ is further assumed to be a convex polytopic set. Any compact convex polytope may be represented by its vertices, for example

$$\mathcal{W} = \text{Co}\{w^{(j)},\ j = 1, \ldots, m\}, \tag{3.2}$$

where $w^{(j)}$, $j = 1, \ldots, m$ are the vertices (extreme points) of $\mathcal{W}$ and $\text{Co}\{\cdot\}$ denotes the convex hull. An alternative description in terms of linear inequalities is given by

$$\mathcal{W} = \{w : Vw \le \mathbf{1}\}, \tag{3.3}$$

for some matrix $V \in \mathbb{R}^{n_V \times n_w}$ that specifies the hyperplanes bounding $\mathcal{W}$. Here V is assumed to be minimal in the sense that n_V is the smallest number hyperplanes that define the boundary of $\mathcal{W}$. The linear inequality representation (3.3) is usually more parsimonious than the vertex representation (3.2). This is because the number, m, of vertices of $\mathcal{W}$ must be at least as large as the minimal number, n_V, of rows of V because $\mathcal{W}$ is by assumption full dimensional, and m is typically much greater than n_V. However both representations are employed in different RMPC formulations, and depending on the context we will make use of either (3.2) or (3.3).

In this chapter, we introduce some notation specific to sets.

- The *Minkowski sum* of a pair of sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$ is denoted $\mathcal{X} \oplus \mathcal{Y}$, and is defined as the set (see Fig. 3.1)

$$\mathcal{X} \oplus \mathcal{Y} \doteq \{z \in \mathbb{R}^n : z = x + y, \text{ for any } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}.$$

- The *Pontryagin difference* of two sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$, denoted $\mathcal{X} \ominus \mathcal{Y}$, is the set

$$\mathcal{X} \ominus \mathcal{Y} = \{z \in \mathbb{R}^n : z + y \in \mathcal{X}, \text{ for all } y \in \mathcal{Y}\},$$

  so that $\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y}$ if and only if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$.
- The image of $\mathcal{X} \subset \mathbb{R}^n$ under a matrix $H \in \mathbb{R}^{m \times n}$ is defined as the set

$$H\mathcal{X} \doteq \{z \in \mathbb{R}^m : z = Hx, \text{ for any } x \in \mathcal{X}\}.$$

  Thus, for example $H\mathcal{X} = \text{Co}\{Hx^{(j)}, \ j = 1, \ldots, q\}$ if $\mathcal{X}$ is a compact convex polytope described in terms of its vertices as $\mathcal{X} = \text{Co}\{x^{(j)}, \ j = 1, \ldots, q\}$.
- For a closed set $\mathcal{X} \subset \mathbb{R}^n$ and $F \in \mathbb{R}^{m \times n}$, $h \in \mathbb{R}^m$, we use $F\mathcal{X} \leq h$ to denote the conditions

$$\max_{x \in \mathcal{X}} Fx \leq h.$$

  Here and throughout this chapter the maximization of a vector-valued function is to be performed elementwise, so that $\max_{x \in \mathcal{X}} Fx$ denotes the vector in $\mathbb{R}^m$ with ith element equal to $\max_{x \in \mathcal{X}} F_i x$, where F_i is the ith row of the matrix F.

As in Chap. 2 the system is assumed to be subject to mixed constraints on the states and control inputs

$$Fx_k + Gu_k \leq \mathbf{1} \tag{3.4}$$

and performance is again judged in terms of a quadratic cost of the form of (2.3). However, in the uncertain case considered here, the quantity

$$\sum_{i=0}^{\infty} \left(\|x_{k+i}\|_Q^2 + \|u_{k+i}\|_R^2\right) \tag{3.5}$$

evaluated along trajectories of (3.1) is uncertain at time k since it depends on the disturbance sequence $\{w_k, w_{k+1}, \ldots\}$, which is unknown at time k.

To account for this, we consider two alternative definitions of the cost when evaluating predicted performance: the nominal cost corresponding to the case of no model uncertainty (i.e. $w_k = 0$ for all k); and the worst-case cost over all admissible

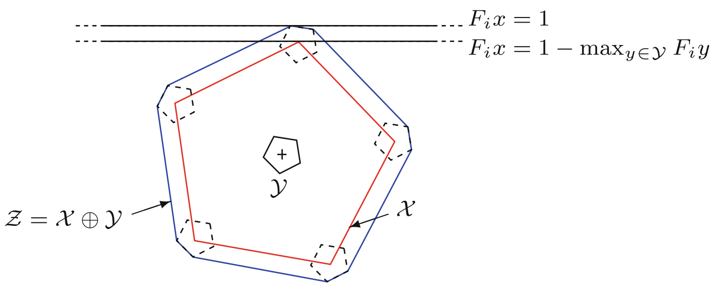

Fig. 3.1 A graphical representation of the Minkowski sum $\mathcal{X} \oplus \mathcal{Y}$ of a pair of closed convex polytopic sets $\mathcal{X}$, $\mathcal{Y}$, and of the linear constraint $F_i x \le \mathbf{1}$ applied to $\mathcal{X} \oplus \mathcal{Y}$. If $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$, then $F_i \mathcal{Z} \le \mathbf{1}$ is equivalent to $F_i \mathcal{X} \le \mathbf{1} - h_i$ for $h_i = \max_{y \in \mathcal{Y}} F_i \mathcal{Y}$

disturbance sequences ($w_k \in \mathcal{W}$ for all k). In the latter case, we introduce into the stage cost of (3.5) an additional quadratic term which is negative definite in w_k to ensure a finite worst-case cost; this results in the $\mathcal{H}_\infty$ cost of classical robust control.

Robust MPC algorithms employing open-loop and closed-loop optimization strategies possess fundamentally different properties. To make this distinction clear we introduce the following definition.

Definition 3.1 (*Open-loop and closed-loop strategies*) In an open-loop optimization strategy the sequence of control inputs, $\{u_{0|k}, u_{1|k}, \ldots\}$, predicted at time k is independent of the realization of the disturbance sequence $\{w_{0|k}, w_{1|k}, \ldots\}$. In a closed-loop optimization strategy the predicted control sequence $\{u_{0|k}, u_{1|k}, \ldots\}$ is a function of the realization of the disturbance sequence $\{w_{0|k}, w_{1|k}, \ldots\}$.

By a slight abuse of terminology, we refer to the closed-loop paradigm introduced in Sect. 2.3 as an open-loop optimization strategy when it is used in the context of RMPC. In (2.13) the control inputs predicted at time k are specified by the closed-loop paradigm for $i = 0, 1, \ldots$ as

$$u_{i|k} = K x_{i|k} + c_{i|k} \tag{3.6}$$

where K is a fixed stabilizing feedback gain, and $c_{i|k}$ for $i = 0, \ldots, N-1$ are optimization variables with $c_{i|k} = 0$ for $i \ge N$. Clearly $u_{i|k}$ depends on the state $x_{i|k}$ and hence on the realization of $w_{0|k}, \ldots, w_{i-1|k}$. Strictly speaking therefore (3.6) is a closed-loop strategy. However the optimization variables $c_{0|k}, \ldots, c_{N-1|k}$ do not depend on the realization of future uncertainty and hence they appear in the state predictions generated by (3.1) as an open-loop control sequence applied to pre-stabilized dynamics:

$$x_{i+1|k} = \Phi x_{i|k} + B c_{i|k} + D w_{i|k} \tag{3.7}$$

for $i = 0, 1, \ldots$, where $\Phi = A + BK$. Therefore the closed-loop paradigm is equivalent to an open-loop strategy applied to this system. Throughout this chapter, as in

Chap. 2, we assume that K is the optimal feedback gain in the absence of inequality constraints on the system states and control inputs.

A closed-loop optimization strategy optimizes predicted performance over a class of feedback policies which is parameterized by the degrees of freedom in the optimization problem. The approach benefits from information on future disturbances which, though unknown to an observer at current time, will be available to the controller when the future control input is applied to the plant. We note here that robust MPC based on closed-loop optimization has the distinct advantage that it can provide the optimal achievable performance in respect of both the region of attraction and the worst-case performance objective. Closed-loop optimization strategies are considered in detail in Chap. 4 and their advantages over open-loop strategies are discussed in Sect. 4.1.

3.2 State Decomposition and Constraint Handling

The future values of the state of the system (3.1) are uncertain because of the unknown future disturbances acting on the system. However, given knowledge of a set containing all realizations of the disturbance input, sequences of sets can be determined that necessarily contain the future state and control input, and this is the basis of methods for guaranteeing robust satisfaction of constraints. Because of the linearity of the model (3.1), the component of the predicted state that is generated by the disturbance input evolves independently of the optimization variables when an open-loop optimization strategy is employed. Since the constraints (3.4) are also linear, the worst-case disturbances with respect to these constraints do not depend on the optimization variables and can therefore be determined offline. This leads to a computationally convenient method of handling constraints for open-loop optimization strategies. In fact the resulting constraints on predicted states and inputs are of the same form as those of the nominal (uncertainty-free) MPC problem, and are simply tightened to account for the uncertainty in predictions.

This section discusses the application of the constraints (3.4) to the predictions of the model (3.1) under the open-loop strategy (3.6). We first decompose the predicted state into nominal and uncertain components, denoted s and e, respectively. Thus, let $x_{i|k} = s_{i|k} + e_{i|k}$, where the nominal and uncertain components evolve for $i = 0, 1, \ldots$ according to

$$s_{i+1|k} = \Phi s_{i|k} + Bc_{i|k}, \tag{3.8a}$$

$$e_{i+1|k} = \Phi e_{i|k} + Dw_{i|k}, \tag{3.8b}$$

with initial conditions $e_{0|k} = 0$ and $s_{0|k} = x_k$.

As in Sect. 2.7, it is convenient to augment the predicted model state with the degrees of freedom in predictions by defining an augmented state variable $z \in \mathbb{R}^{n_z}$, $n_z = n_x + Nn_u$,

$$z = \begin{bmatrix} s \\ \mathbf{c} \end{bmatrix},$$

where $\mathbf{c}$ is the vector of optimization variables, with $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ at time k. Then the nominal predicted state is given by $s_{i|k} = \begin{bmatrix} I_{n_x} & 0 \end{bmatrix} z_{i|k}$, where $z_{i+1|k}$ evolves for $i = 0, 1, \ldots$ according to the autonomous dynamics

$$z_{i+1|k} = \Psi z_{i|k}. \tag{3.9}$$

Here, the matrix Ψ and the initial condition $z_{0|k}$ are defined by

$$\Psi = \begin{bmatrix} \Phi & BE \\ 0 & M \end{bmatrix} \text{ and } z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}$$

with E and M given by (2.26b). In terms of the augmented state $z_{i|k}$ we obtain the predicted future values of the state and control input generated by (3.6) and (3.7) as

$$x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} z_{i|k} + e_{i|k} \tag{3.10a}$$

$$u_{i|k} = \begin{bmatrix} K & E \end{bmatrix} z_{i|k} + K e_{i|k}. \tag{3.10b}$$

The predicted state and input sequences in (3.10) satisfy the constraints (3.4) if and only if the following condition is satisfied for all $i = 0, 1, \ldots$,

$$F x_{i|k} + G u_{i|k} \leq \mathbf{1} \quad \forall \{w_{0|k}, \ldots, w_{i-1|k}\} \in \mathcal{W} \times \cdots \times \mathcal{W}.$$

Therefore the constraints (3.4) are equivalent to the following conditions

$$\bar{F} \Psi^i z_{0|k} \leq \mathbf{1} - h_i, \quad i = 0, 1, \ldots \tag{3.11}$$

where

$$\bar{F} = \begin{bmatrix} F + GK & GE \end{bmatrix}$$

and the vectors h_i are defined for all $i \geq 0$ by

$$h_0 \doteq 0 \tag{3.12a}$$

$$h_i \doteq \max_{\{w_{0|k}, \ldots, w_{i-1|k}\} \in \mathcal{W} \times \cdots \times \mathcal{W}} (F + GK) e_{i|k}, \quad i = 1, 2, \ldots \tag{3.12b}$$

From (3.8b) we obtain $e_{i|k} = w_{i-1|k} + \cdots + \Phi^{i-2} w_{1|k} + \Phi^{i-1} w_{0|k}$, and hence h_i in (3.12b) can be expressed

$$h_i = \sum_{j=0}^{i-1} \max_{w_j \in \mathcal{W}} (F + GK) \Phi^j D w_j, \quad i = 1, 2, \ldots \tag{3.13}$$

or equivalently by the recursion

$$h_i = h_{i-1} + \max_{w \in \mathcal{W}} (F + GK)\Phi^{i-1} Dw, \quad i = 1, 2, \ldots$$

Since the maximization in this expression applies elementwise, h_i is determined by the solution of in_u linear programs, each of which determines the maximizing value of w for an element of an individual term in (3.13).

Comparing (3.11) with (2.28) it can be seen that the robust constraints for additive model uncertainty are almost identical to the constraints for the case of no uncertainty that was considered in Chap. 2. The difference between these two cases is the vector h_i appearing in (3.11). The definition of h_i in (3.12) implies that this term simply tightens the constraint set by the minimum that is required to accommodate the worst-case value of $e_{i|k}$, namely the worst-case future uncertainty with respect to the constraints.

3.2.1 Robustly Invariant Sets and Recursive Feasibility

The conditions in (3.11) are given in terms of an infinite number of constraints, and from (3.13) these depend on the solution of an infinite number of linear programs. Clearly, this infinite set of conditions is not implementable, and it is necessary to consider whether (3.11) can be equivalently stated in terms of a finite number of constraints. A further question relates to recursive feasibility of (3.11). Specifically, whether there necessarily exists $\mathbf{c}_{k+1}$ so that, if the conditions of (3.11) are satisfied by $z_{0|k} = (x_k, \mathbf{c}_k)$, then they will also hold when $z_{0|k}$ is replaced by $z_{0|k+1} = (x_{k+1}, \mathbf{c}_{k+1})$.

Both of these issues can be addressed using the concept of robust positive invariance. We define this in the context of the uncertain dynamics and constraints given by

$$z_{i+1} = \Psi z_i + \bar{D} w_i, \quad w_i \in \mathcal{W} \tag{3.14a}$$

$$\bar{F} z_i \leq \mathbf{1}, \quad i = 0, 1, \ldots \tag{3.14b}$$

Note that the constraints of (3.11) are satisfied if and only if (3.14b) holds for all $w_i \in \mathcal{W}$, $i = 0, 1, \ldots$, if $\bar{D}$ is defined as

$$\bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}.$$

Definition 3.2 (*Robustly positively invariant set*) A set $\mathcal{Z} \subset \mathbb{R}^{n_z}$ is robustly positively invariant (RPI) under the dynamics (3.14a) and constraints (3.14b) if and only if $\bar{F} z \leq \mathbf{1}$ and $\Psi z + \bar{D} w \in \mathcal{Z}$ for all $w \in \mathcal{W}$, for all $z \in \mathcal{Z}$.

It is often desirable to determine the largest possible RPI set for a given system. This is defined analogously to the case of systems with no model uncertainty considered in Sect. 2.4.

Definition 3.3 (*Maximal robustly positively invariant set*) The maximal robustly positively invariant (MRPI) set under (3.14a) and (3.14b) is the union of all RPI sets under these dynamics and constraints.

Unlike the case of no model uncertainty, for which the maximal invariant set is necessarily non-empty if Φ is strictly stable, the MRPI set for (3.14a) and (3.14b) will be empty whenever the disturbance set $\mathcal{W}$ is sufficiently large. Since the constraints (3.14b) are equivalent to (3.11), it is clear that the MRPI set can be non-empty only if the constraint tightening parameters h_i defined in (3.12) satisfy $h_i < \mathbf{1}$ for all i. From the expression for h_i in (3.13), where Φ is by assumption a stable matrix (i.e. all of its eigenvalues lie inside the unit circle), it follows that h_i has a limit as $i \to \infty$ and hence, we require that

$$\lim_{i\to\infty} h_i < \mathbf{1}. \tag{3.15}$$

The conditions under which this inequality holds will be examined in Sect. 3.2.2. Assuming that (3.15) is satisfied, the following theorem (which is a simplified version of a result from [1]) shows that the MRPI set is defined in terms of a finite number of linear inequalities.

Theorem 3.1 *If (3.15) holds, then the MRPI set $\mathcal{Z}^{\text{MRPI}}$ for the dynamics defined by (3.14a) and the constraints (3.14b) can be expressed*

$$\mathcal{Z}^{\text{MRPI}} = \{z : \bar{F}\Psi^i z \le \mathbf{1} - h_i,\ i = 0, \ldots, \nu\} \tag{3.16}$$

where ν is the smallest positive integer such that $\bar{F}\Psi^{\nu+1} z \le \mathbf{1} - h_{\nu+1}$ for all z satisfying $\bar{F}\Psi^i z \le \mathbf{1} - h_i$, $i = 0, \ldots, \nu$. Furthermore ν is necessarily finite if Ψ is strictly stable and $(\Psi, \bar{F})$ is observable.

Proof For any nonnegative integer n, let $\mathcal{Z}^{(n)}$ denote the set

$$\mathcal{Z}^{(n)} \doteq \{z : \bar{F}\Psi^i z \le \mathbf{1} - h_i,\ i = 0, \ldots, n\}.$$

If $\bar{F}\Psi^{\nu+1} z \le \mathbf{1} - h_{\nu+1}$ for all $z \in \mathcal{Z}^{(\nu)}$, then, since $h_{i+1} \ge h_i + \bar{F}\Psi^i \bar{D} w$ for all $w \in \mathcal{W}$, the following conditions hold for all $z \in \mathcal{Z}^{(\nu)}$,

(a) $\bar{F}\Psi^i(\Psi z + \bar{D}w) \le \mathbf{1} - h_i$ for all $w \in \mathcal{W}$, for $i = 0, \ldots, \nu$,
(b) $\bar{F}z \le \mathbf{1}$.

These conditions imply that $\mathcal{Z}^{(\nu)}$ is RPI under the dynamics (3.14a) and constraints (3.14b), and hence $\mathcal{Z}^{(\nu)} \subseteq \mathcal{Z}^{\text{MRPI}}$. Furthermore $\mathcal{Z}^{\text{MRPI}}$ must be a subset of $\mathcal{Z}^{(n)}$ for all $n \ge 0$, since if $z \notin \mathcal{Z}^{(n)}$, then z cannot belong to any set that is RPI under (3.14a) and (3.14b). Hence $\mathcal{Z}^{\text{MRPI}} = \mathcal{Z}^{(\nu)}$ if $\bar{F}\Psi^{\nu+1} z \le \mathbf{1} - h_{\nu+1}$ for all $z \in \mathcal{Z}^{(\nu)}$.

It can moreover be concluded that $\mathcal{Z}^{\text{MRPI}} = \mathcal{Z}^{(\nu)}$ for some finite ν since $\mathcal{Z}^{\text{MRPI}}$ is necessarily bounded given that $(\Psi, \bar{F})$ is observable, and because $\mathcal{Z}^{(n+1)} = \mathcal{Z}^{(n)} \cap \{z : \bar{F}\Psi^{(n+1)}z \le \mathbf{1} - h_{n+1}\}$, where $\{z : \bar{F}\Psi^{(\nu+1)}z \le \mathbf{1} - h_{\nu+1}\}$ must contain any bounded set for some finite n since Ψ is strictly stable and since (3.15) implies that the elements of h_n are strictly less than unity for all $n \ge 0$. □

Since the conditions (3.11) hold if and only if (3.14b) holds for all $w_i \in \mathcal{W}$, $i = 0, 1, \ldots$, Theorem 3.1 implies that the constraints of (3.11) are equivalent to the condition that $z_{0|k} \in \mathcal{Z}^{\text{MRPI}}$, which is determined by a finite set of linear constraints. In addition, the MRPI set for the lifted dynamics provides the largest possible feasible set for the open-loop strategy (3.6) applied to (3.7). To see this, consider for example the projection of $\mathcal{Z}^{\text{MRPI}}$ in (3.16) onto the x-subspace,

$$\mathcal{F}_N \doteq \left\{x : \exists\, \mathbf{c} \text{ such that } \bar{F}\Psi^i \begin{bmatrix} x \\ \mathbf{c} \end{bmatrix} \le \mathbf{1} - h_i,\ i = 0, \ldots, \nu\right\}. \tag{3.17}$$

Analogously to the case of no model uncertainty considered in Sect. 2.7.1, $\mathcal{F}_N$ has an interpretation as the set of all feasible initial conditions for the predictions generated by (3.6) and (3.7) subject to constraints (3.4):

$$\begin{aligned} \mathcal{F}_N = \Big\{x_0 : \exists\, \{c_0, \ldots, c_{N-1}\} & \\ \text{such that } (F + GK)x_i + Gc_i \le \mathbf{1},\ i = 0, \ldots, N-1, & \\ \text{and } x_N \in \mathcal{X}_T,\ \forall \{w_0, \ldots, w_{N-1}\} \in \mathcal{W} \times \cdots \times \mathcal{W}\Big\}. & \end{aligned} \tag{3.18}$$

(Here the use of an open-loop strategy means that, for given x_0, the sequence $\mathbf{c} = \{c_0, \ldots, c_{N-1}\}$ must ensure that the constraints are satisfied for all possible disturbance sequences $\{w_0, \ldots, w_{N-1}\} \in \mathcal{W} \times \cdots \times \mathcal{W}$.) This interpretation of $\mathcal{F}_N$ implies that the predicted state N steps ahead must lie in a terminal set, $\mathcal{X}_T$, which is RPI for (3.1) under $u_k = Kx_k$. Furthermore, from (3.17) and the definition of Ψ, the terminal set $\mathcal{X}_T$ must be the intersection of $\mathcal{Z}^{\text{MRPI}}$ with the subspace on which $\mathbf{c} = 0$, so that

$$\begin{aligned} \mathcal{X}_T &= \left\{x : \big[(F + GK)\ GE\big]\Psi^i \begin{bmatrix} x \\ 0 \end{bmatrix} \le \mathbf{1} - h_i,\ i = 0, 1, \ldots\right\} \\ &= \left\{x_0 : (F + GK)x_i \le \mathbf{1} - h_i,\ x_{i+1} = \Phi x_i,\ i = 0, 1, \ldots\right\}. \end{aligned}$$

It follows that $\mathcal{X}_T$ is the maximal RPI set for (3.1) under $u_k = Kx_k$ subject to (3.4). Therefore $\mathcal{F}_N$ contains all initial conditions for which the constraints (3.4) can be satisfied over an infinite horizon with the open-loop strategy (3.6) and with $c_{i|k} = 0$ for all $i \ge N$.

Having established that the conditions in (3.11) can be expressed in terms of a finite number of constraints, and that these constraints allow the largest possible set of initial conditions $x_{0|k}$ under the open-loop strategy (3.6), we next show that these conditions are recursively feasible. This property, which is essential for ensuring the stability of a robust MPC strategy incorporating (3.11), can be established by defining a candidate vector $\mathbf{c}_{k+1}$ of optimization variables at time $k+1$ in terms of a vector $\mathbf{c}_k$ which satisfies, by assumption, the constraints (3.11) at time k. As in Sects. 2.5 and 2.7.2, we define, this candidate as $\mathbf{c}_{k+1} = M\mathbf{c}_k$, so that

$$c_{i|k+1} = \begin{cases} c_{i+1|k}, & i = 0, \ldots, N-2 \\ 0 & i \geq N-1 \end{cases}$$

Since $u_k = Kx_k + c_{0|k}$ implies $x_{k+1} = \Phi x_k + Bc_{0|k} + Dw_k$, we then obtain

$$s_{i|k+1} = s_{i|k} + \Phi^{i-1} D w_k, \quad i = 0, 1, \ldots$$

and hence

$$z_{0|k+1} = \Psi z_{0|k} + \bar{D} w_k.$$

But $\mathbf{c}_k$ satisfies (3.11) at time k if and only if $z_{0|k} \in \mathcal{Z}^{\text{MRPI}}$, so the requirement that $\mathbf{c}_{k+1} = M\mathbf{c}_k$ should satisfy constraints at time $k+1$ is equivalent to requiring $\Psi z + \bar{D} w \in \mathcal{Z}^{\text{MRPI}}$ for all $w \in \mathcal{W}$ and all $z \in \mathcal{Z}^{\text{MRPI}}$. This is ensured by the robust positive invariance of $\mathcal{Z}^{\text{MRPI}}$ under (3.14a) and (3.14b). It follows that there necessarily exists $\mathbf{c}_{k+1}$ such that $z_{0|k+1} \in \mathcal{Z}^{\text{MRPI}}$ whenever $z_{0|k} \in \mathcal{Z}^{\text{MRPI}}$, and hence this constraint set is recursively feasible.

3.2.2 Interpretation in Terms of Tubes

Tubes provides an intuitive geometric interpretation of robust constraint handling and are convenient for analysing the asymptotic behaviour of uncertain systems. In particular, a tube formulation allows the condition (3.15) on the infinite sequence of constraint tightening parameters h_i to be checked by solving a finite number of linear programs. Because of the model uncertainty in (3.7), the predicted states are described by a tube comprising a sequence of sets, each of which contains the state at a given future time instant for all realizations of future uncertainty. The use of tubes in control is not new (e.g. see [2, 3]), and they have been used in the context of MPC for a couple of decades (e.g. [4, 5]); their use in MPC has led to specialized techniques such as Tube MPC (TMPC) (e.g. [6]) which are discussed in more detail in Sect. 3.5.

Denoting the tube containing the predicted states as the sequence of sets $\{\mathcal{X}_{0|k}, \mathcal{X}_{1|k}, \ldots\}$, where $x_{i|k} = s_{i|k} + e_{i|k} \in \mathcal{X}_{i|k}$, $i = 0, 1, \ldots$, and using the decomposition (3.8) yields

$$\mathcal{X}_{i|k} = \{s_{i|k}\} \oplus \mathcal{E}_{i|k}$$

where $e_{i|k} \in \mathcal{E}_{i|k}$ for all i. Here the Minkowski sum $\{s_{i|k}\} \oplus \mathcal{E}_{i|k}$ simply translates each element $e_{i|k}$ of the set $\mathcal{E}_{i|k}$ to $s_{i|k} + e_{i|k}$. The sets that form the tube $\{\mathcal{E}_{0|k}, \mathcal{E}_{1|k}, \ldots\}$ evolve, by (3.8b), according to

$$\mathcal{E}_{i+1|k} = \Phi \mathcal{E}_{i|k} \oplus D\mathcal{W} \tag{3.19}$$

for all $i \geq 0$, with initial condition $\mathcal{E}_{0|k} = \{0\}$. Thus $\mathcal{E}_{i|k}$ can be expressed

$$\mathcal{E}_{i|k} = D\mathcal{W} \oplus \Phi D\mathcal{W} \oplus \cdots \oplus \Phi^{i-1} D\mathcal{W} = \bigoplus_{j=0}^{i-1} \Phi^j D\mathcal{W}. \tag{3.20}$$

The state tube $\{\mathcal{X}_{0|k}, \mathcal{X}_{1|k}, \ldots\}$ implies a tube for the predicted control input, $\{\mathcal{U}_{0|k}, \mathcal{U}_{1|k}, \ldots\}$, where $u_{i|k} \in \mathcal{U}_{i|k}$ for all i. In accordance with (3.6), $\mathcal{U}_{i|k}$ is given for $i = 0, 1, \ldots$ by

$$\mathcal{U}_{i|k} = \{K s_{i|k} + c_{i|k}\} \oplus K \mathcal{E}_{i|k}.$$

In this setting, the constraints (3.4) are therefore equivalent to

$$\bar{F}\left(\{\Psi^i z_{0|k}\} \oplus \begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{E}_{i|k}\right) \leq \mathbf{1}, \quad i = 0, 1, \ldots \tag{3.21}$$

Comparing (3.21) with (3.11) it can be seen that the amount by which the constraints on the nominal predictions $z_{i|k}$ must be tightened in order that the constraints (3.4) are satisfied for all uncertainty realizations is

$$h_i = \max_{e_{i|k} \in \mathcal{E}_{i|k}} (F + GK) e_{i|k},$$

which is in agreement with (3.13). This is illustrated in Fig. 3.2.

A consequence of (3.21) is that the requirement, which must be met in order for the MRPI set to be non-empty, that $h_i \leq (1 - \epsilon)\mathbf{1}$ for all $i \geq 0$ and some $\epsilon > 0$ is equivalent to

$$(F + GK)\mathcal{E}_{i|k} \leq (1 - \epsilon)\mathbf{1}, \quad i = 0, 1, \ldots \tag{3.22}$$

for some $\epsilon > 0$. However (3.20) implies that $\mathcal{E}_{i+1|k} = \mathcal{E}_{i|k} \oplus \Phi^i D\mathcal{W}$ and hence $\mathcal{E}_{i|k}$ is necessarily a subset of $\mathcal{E}_{i+1|k}$. Therefore the conditions in (3.22) are satisfied if and only if they hold asymptotically as $i \to \infty$. This motivates the consideration of the *minimal robust invariant set*, which defines the asymptotic behaviour of $\mathcal{E}_{i|k}$ as $i \to \infty$.

Definition 3.4 (*Minimal robustly positively invariant set*) The minimal robustly invariant (mRPI) set under (3.8b) is the RPI set contained in every closed RPI set of (3.8b).

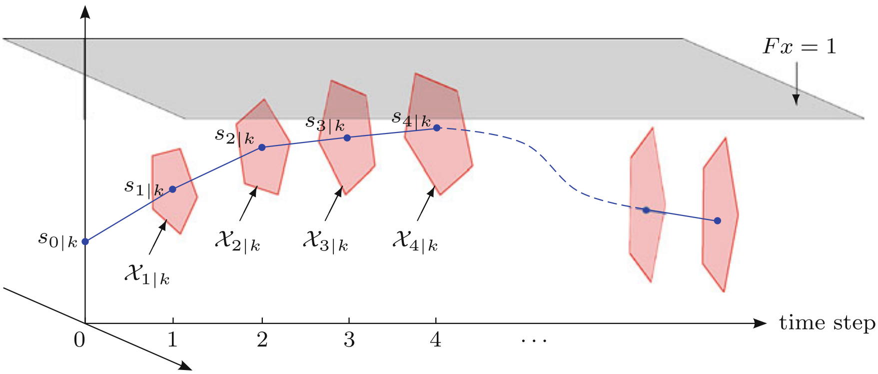

Fig. 3.2 An illustration of the state tube and constraints for the case that $G = 0$ and x is 2-dimensional

Because of the linearity of the dynamics in (3.8b), each set $\mathcal{E}_{i|k}$, $i = 0, 1, \ldots$ generated by (3.19) with initial condition $\mathcal{E}_{0|k} = \{0\}$ must be contained in an RPI set of (3.8b). Given also that $\mathcal{E}_{i|k} \subset \mathcal{E}_{i+1|k}$ and that, in the limit as $i \to \infty$, $\mathcal{E}_{i|k}$ in (3.20) is clearly RPI, the mRPI set for (3.8b) is given by

$$\mathcal{X}^{\text{mRPI}} \doteq \bigoplus_{j=0}^{\infty} \Phi^j D\mathcal{W}. \tag{3.23}$$

Unfortunately, unlike the maximal RPI set, the minimal RPI cannot generally be expressed either in terms of a finite number of linear inequalities or as the convex hull of a finite number of vertices. This is a consequence of the fact that, unless $\Phi^i = 0$ for some finite i, $\mathcal{E}_{i|k}$ is a proper subset of $\mathcal{E}_{i+1|k}$ for all $i \geq 0$. As a result it is not in general possible either to compute $\mathcal{X}^{\text{mRPI}}$ or to obtain an exact asymptotic value for h_i as $i \to \infty$. Instead it is necessary to characterize the asymptotic behaviour of $\mathcal{E}_{i|k}$ by finding an outer bound, $\hat{\mathcal{X}}^{\text{mRPI}}$, satisfying $\hat{\mathcal{X}}^{\text{mRPI}} \supseteq \mathcal{X}^{\text{mRPI}}$. Given a bounding set $\hat{\mathcal{X}}^{\text{mRPI}}$, a corresponding upper bound, $\hat{h}_\infty$, can be computed that satisfies $\hat{h}_\infty \geq h_i$ for all i, thus providing a sufficient condition for (3.15).

The bound $\hat{h}_\infty$ can be computed using several different approaches, however the method presented here is based on the mRPI set approximation of [7]. This approximation is derived from the observation that, if there exist a positive integer r and scalar $\rho \in [0, 1)$ satisfying

$$\Phi^r D\mathcal{W} \subseteq \rho D\mathcal{W} \tag{3.24}$$

then

$$\bigoplus_{j=0}^{\infty} \Phi^j D\mathcal{W} \subseteq \bigoplus_{j=0}^{r-1} \Phi^j D\mathcal{W} \oplus \rho \bigoplus_{j=0}^{r-1} \Phi^j D\mathcal{W} \oplus \rho^2 \bigoplus_{j=0}^{r-1} \Phi^j D\mathcal{W} \cdots$$

But $\bigoplus_{j=0}^{r-1} \Phi^j D\mathcal{W}$ is necessarily convex (since $\mathcal{W}$ is by assumption convex and hence $\Phi^j D\mathcal{W}$ is also convex), and for any convex set $\mathcal{X}$ and scalar $\alpha > 0$ we have $\mathcal{X} \oplus \alpha\mathcal{X} = (1+\alpha)\mathcal{X}$. It therefore follows that

$$\bigoplus_{j=0}^{\infty} \Phi^j D\mathcal{W} \subseteq (1 + \rho + \rho^2 + \cdots) \bigoplus_{j=0}^{r-1} \Phi^j D\mathcal{W}$$
$$= \frac{1}{1-\rho} \bigoplus_{j=0}^{r-1} \Phi^j D\mathcal{W}$$

Defining

$$\hat{\mathcal{X}}^{\mathrm{mRPI}} \doteq \frac{1}{1-\rho} \bigoplus_{j=0}^{r-1} \Phi^j D\mathcal{W} \tag{3.25}$$

it can be concluded that $\mathcal{X}^{\mathrm{mRPI}} \subseteq \hat{\mathcal{X}}^{\mathrm{mRPI}}$.

The mRPI set approximation given by (3.25) has the desirable properties that it approaches the actual mRPI set arbitrarily closely if ρ is chosen to be sufficiently small. In addition, for any $\rho > 0$ there necessarily exists a finite r satisfying (3.24) since Φ is strictly stable by assumption. Most importantly, $\hat{\mathcal{X}}^{\mathrm{mRPI}}$ is defined in terms of a finite number of inequalities (or vertices), and this allows the corresponding bound $\hat{h}_\infty$ to be determined as

$$\hat{h}_\infty \doteq \frac{1}{1-\rho} \sum_{j=0}^{r-1} \max_{w_j \in \mathcal{W}} (F + GK)\Phi^j w_j = \frac{1}{1-\rho} h_r$$

which gives a sufficient condition for (3.15) as

$$\hat{h}_\infty = \frac{1}{1-\rho} h_r < \mathbf{1}, \tag{3.26}$$

for any r and ρ satisfying (3.24).

In order to check whether the condition (3.24) is satisfied by a given disturbance set $\mathcal{W} = \{w : Vw \leq \mathbf{1}\}$, matrix Φ and scalars r and ρ, note that $\theta \in \rho D\mathcal{W}$ if and only if $VD^\dagger\theta \leq \rho\mathbf{1}$, where $D^\dagger$ is the Moore-Penrose pseudoinverse of D (i.e. $D^\dagger \doteq (D^T D)^{-1} D^T$). Therefore (3.24) is equivalent to

$$\max_{w \in \mathcal{W}} VD^\dagger \Phi^r Dw \leq \rho\mathbf{1}. \tag{3.27}$$

This can be checked by solving n_V linear programs.

Example 3.1 A simple supply chain model contains a supplier, a production facility and a distributor (Fig. 3.3). At the beginning of the kth discrete-time interval, a

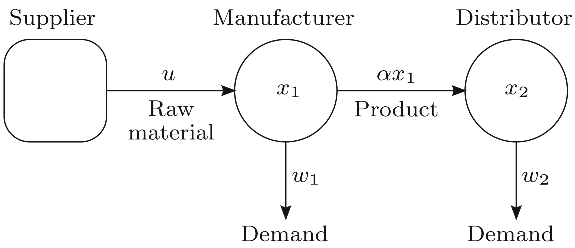

Fig. 3.3 A simple supply chain model

quantity u_k of raw material is delivered by the supplier to the manufacturer. Of this material, an amount $w_{1,k}$ is transferred to other manufacturers, and the remainder is added to the amount $x_{1,k}$ held in storage by the manufacturer. A fraction, $\alpha x_{1,k}$ of this is converted into product and transferred to the distributor, who stores an amount $x_{2,k}$ and supplies $w_{2,k}$ to customers. The value $\alpha = 0.5$ is assumed to be known, while the demand quantities $w_{1,k}$ and $w_{2,k}$ are unknown at time k but have known bounds.

The system can be represented by a model of the form (3.1) with state $x_k = (x_{1,k}, x_{2,k})$, control input u_k and disturbance input $w_k = (w_{1,k}, w_{2,k})$. As a result of limits on the supply rate, the storage capacities and the demand, we obtain the following input, state and disturbance constraints:

$$0 \le u_k \le 0.5, \quad (0, 0) \le x_k \le (1, 1), \quad (0.1, 0.1) \le w_k \le (0.2, 0.2).$$

The viability with respect to these constraints of the control strategy (3.6) can be determined using the robust invariant sets discussed in Sect. 3.2.1. To this end, we first determine a (non-zero) setpoint about which to regulate x_k. Given that only bounds on w_k are available, and in the absence of any statistical information on w, it is reasonable to define the setpoint in terms of the equilibrium values of states and inputs (denoted x^0 and u^0) that correspond to a constant disturbance (w^0) at the centroid of the disturbance set. Therefore, defining

$$w^0 = (0.15, 0.15), \quad u^0 = 0.3, \quad x^0 = (0.3, 0.5),$$

the system model can be expressed in terms of the transformed variables $x^\delta = x - x^0$, $u^\delta = u - u^0$ and $w^\delta = w - w^0$ as $x^\delta_{k+1} = Ax^\delta_k + Bu^\delta_k + Dw^\delta_k$ with

$$A = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and with constraints

$$\begin{gathered} -0.3 \le u^\delta_k \le 0.2, \quad (-0.3, -0.5) \le x^\delta_k \le (0.7, 0.5) \\ (-0.05, -0.05) \le w^\delta_k \le (0.05, 0.05). \end{gathered} \tag{3.28}$$

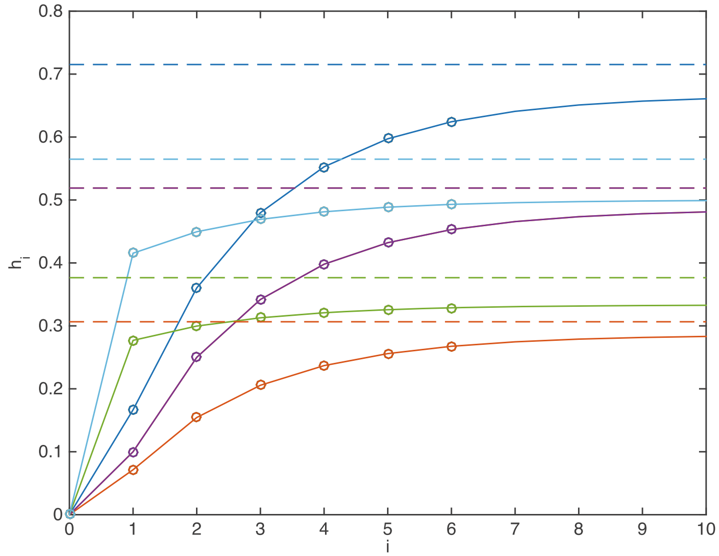

Fig. 3.4 The elements of the constraint tightening parameters h_i, for $i = 0, \ldots, 6$ (*circles*) and the elements of $\hat{h}_\infty = (1-\rho)^{-1} h_r$ (*dashed lines*) with $r = 6$ and $\rho = 0.127$ in Example 3.1. The *solid lines* show the evolution of h_i for $i > 6$ to give an indication of the asymptotic value, h_∞

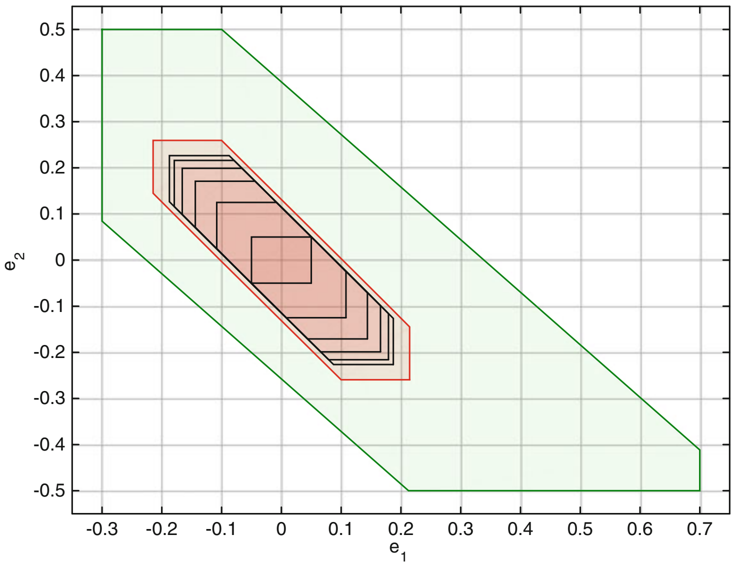

Fig. 3.5 The sets $\mathcal{E}_i$ in Example 3.1 for $i = 1, 2, 3, 4, 5, 6$ (*black lines*); the minimal robust invariant set approximation $\hat{\mathcal{X}}^{\text{mRPI}} = (1-\rho)^{-1}\mathcal{E}_r$ for $r = 6$, $\rho = 0.127$ (*red line*); and the constraint set $\{e : (F + GK)e \le \mathbf{1}\}$ (*green line*)

Using the open-loop strategy (3.6), with K chosen as the unconstrained LQ-optimal feedback gain for the nominal system and the cost (3.5) with $Q = I$ and $R = 0.01$, we obtain $K = [-0.89\ -0.78]$, which fixes the eigenvalues of $\Phi = A + BK$ at 0.61 and 0.005.

To determine whether the set of feasible initial conditions is non-empty, we first check whether (3.15) is satisfied by checking the sufficient condition (3.26). We therefore need values of r and $\rho < 1$ such that $\Phi^r D\mathcal{W} \subseteq \rho D\mathcal{W}$; these can be found by computing the minimum value of ρ that satisfies (3.27) for a given value of r, and then increasing r until ρ is judged to be sufficiently small. Here it is expected that a value of ρ around 0.1 will be small enough for $\hat{h}_\infty$ to provide an accurate estimate of h_∞. Taking $r = 6$, for which the minimum value of ρ satisfying (3.27) is $\rho = 0.127$, we obtain $\hat{h}_\infty = (0.72, 0.31, 0.52, 0.52, 0.38, 0.56)$ (Fig. 3.4)—note that h_i is a 6-dimensional vector because there are $n_C = 6$ individual constraints in (3.28). Hence $\hat{h}_\infty < \mathbf{1}$, which implies that the minimal RPI set approximation $\hat{\mathcal{X}}^{\text{mRPI}}$ is a proper subset of $\{e : (F + GK)e \le \mathbf{1}\}$ (as shown in Fig. 3.5). Theorem 3.1 therefore indicates that the maximal RPI set defined in (3.16) is non-empty for all $N \ge 0$.

The MRPI sets $\mathcal{Z}^{\text{MRPI}}$ for this system under the open-loop strategy (3.6) can be computed for given N using Theorem 3.1. The set $\mathcal{F}_N$ of all feasible initial conditions, which by (3.18) is equal to the projection of the corresponding MRPI set onto the x-subspace, is shown in Fig. 3.6 for a range of values of N. For this example there is no increase in the x-subspace projection of the MRPI set for $N > 4$, since $\mathcal{F}_4$ is equal to the maximal stabilizable set $\mathcal{F}_\infty$. ◊

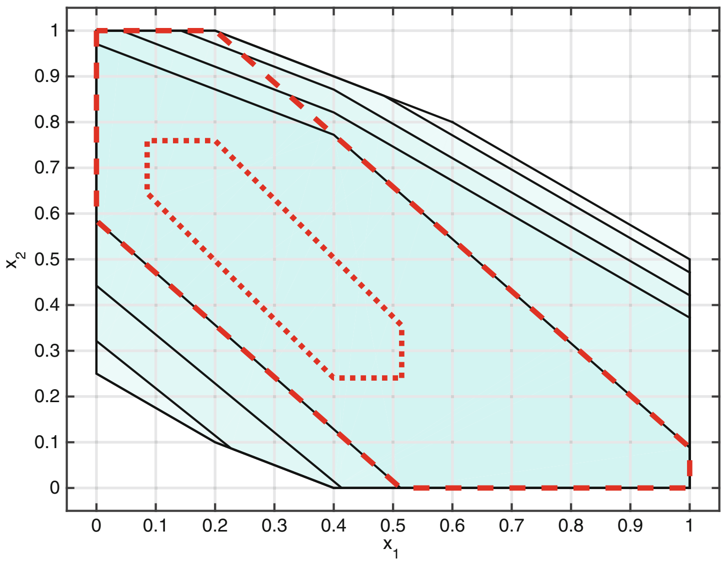

Fig. 3.6 The feasible initial condition sets $\mathcal{F}_N$, $N = 0, 1, 3, 4$, for Example 3.1; also shown are the sets $\{x : (F + GK)(x - x^0) \le \mathbf{1}\}$ (*dashed line*) and $\hat{\mathcal{X}}^{\text{mRPI}} \oplus \{x^0\}$ (*dotted line*)

3.3 Nominal Predicted Cost: Stability and Convergence

When faced with the problem of defining a performance objective for robust MPC given only the knowledge of a nominal value and a bounding set for the model uncertainty, a nominal cost is a simple and obvious choice that can provide desirable closed-loop stability properties. This section defines a robust MPC strategy that combines a nominal cost with the robust constraints formulated in Sect. 3.2. We analyse closed-loop stability and convergence using a technique based on l_2 stability theory.

We use the term *nominal cost* when referring to a predicted performance index evaluated along the predicted trajectories that are obtained when the model uncertainty is equal to its nominal value. As in Sect. 3.2, the nominal value of the additive uncertainty w in the model (3.1) is taken to be $w = 0$. Then, assuming the open-loop strategy (3.6) and a quadratic cost index of the form (2.10), the nominal predicted cost $J(s_{0|k}, \{c_{0|k}, \ldots, c_{N-1|k}\}) = J(s_{0|k}, \mathbf{c}_k)$ is defined as

$$J(s_{0|k}, \mathbf{c}_k) \doteq \sum_{i=0}^{\infty} \bigl(\|s_{i|k}\|_Q^2 + \|v_{i|k}\|_R^2 \bigr). \tag{3.29}$$

where $\{s_{i|k}, i = 0, 1, \ldots\}$ is the nominal predicted state trajectory governed by (3.8a) with $s_{0|k} = x_k$, and $v_{i|k} = K s_{i|k} + c_{i|k}$. Throughout this section K is assumed to be the optimal unconstrained feedback gain for the nominal cost (3.29). However the methods discussed here are also applicable to the case that K is non-optimal but $A + BK$ is stable (see Question 4 on p. 114).

Expressing $s_{i|k}$ in terms of the augmented model state employed in Sect. 3.2, we obtain $s_{i|k} = [I \ 0] z_{i|k}$, where $z_{i|k}$ is generated by the autonomous dynamics (3.9) in an identical manner to the autonomous prediction system considered in Sect. 2.7. Therefore, using Theorem 2.10, the cost (3.29) is given by

$$J(s_{0|k}, \mathbf{c}_k) = \sum_{i=0}^{\infty} \|z_{i|k}\|_{\hat{Q}}^2 = \|z_{0|k}\|_W^2 \tag{3.30a}$$

$$\hat{Q} = \begin{bmatrix} Q + K^T R K & K^T R E \\ E^T R K & E^T R E \end{bmatrix}, \tag{3.30b}$$

and, by Lemma 2.1, the matrix W in the expression for $J(s_{0|k}, \mathbf{c}_k)$ can be determined by solving the Lyapunov equation (2.34). Furthermore, given that K is the unconstrained optimal feedback gain, Theorem 2.10 implies that W is block diagonal and hence

$$J(s_{0|k}, \mathbf{c}_k) = \|s_{0|k}\|_{W_x}^2 + \|\mathbf{c}_k\|_{W_c}^2,$$

where W_x is the solution of the Riccati equation (2.9), and where W_c is block diagonal: $W_c = \mathrm{diag}\{B^T W_x B + R, \ldots, B^T W_x B + R\}$. Combining the nominal predicted cost with the constraints constructed in Sect. 3.2 we obtain the following robust MPC strategy, which requires the online solution of a quadratic program with Nn_u variables and $n_C(\nu+1)$ constraints.

Algorithm 3.1 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \|\mathbf{c}_k\|^2_{W_c} \ \text{subject to } \bar{F}\Psi^i \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \leq \mathbf{1} - h_i, \ i = 0, \ldots, \nu \tag{3.31}$$

where ν satisfies the conditions of Theorem 3.1.

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ for problem (3.31). ◁

The assumption in step (i) that ν satisfies the conditions of Theorem 3.1 implies that the constraint set $\{z = (x_k, \mathbf{c}_k) : \bar{F}\Psi^i z \leq \mathbf{1} - h_i, \ i = 0, \ldots, \nu\}$ is robustly positive invariant, and, as discussed in Sect. 3.2.1, this ensures that the optimization (3.31) is recursively feasible. Therefore $c^*_{0|k}$ exists for all k and the state of the closed-loop system under Algorithm 3.1 is governed for $k = 0, 1, \ldots$ by

$$x_{k+1} = \Phi x_k + Bc^*_{0|k} + Dw_k. \tag{3.32}$$

However the MPC optimization (3.31) is equivalent to the minimization of the nominal cost (3.29) and, unlike the case in which there is no model uncertainty, there is no guarantee that the optimal value J^*_k will be monotonically non-increasing when the system is subject to unknown disturbances.

We therefore use an alternative method of analysing closed-loop stability. First we demonstrate that the sequence $\{\|c^*_{0|k}\|, \|c^*_{1|k}\|, \ldots\}$ is square-summable, and we then use l_2 stability theory to show that the closed-loop system imposes a finite l_2 gain between the disturbance sequence $\{w_0, w_1, \ldots\}$ and the sequence $\{x_0, x_1, \ldots\}$ of closed-loop plant states. Finally, we use this result to conclude that x_k converges asymptotically to the minimal RPI set $\mathcal{X}^{\text{mRPI}}$.

The discussion of recursive feasibility in Sect. 3.2.1 demonstrates that $\mathbf{c}_{k+1} = M\mathbf{c}^*_k$ is feasible but suboptimal for (3.31). Therefore $\mathbf{c}^*_{k+1}$ necessarily satisfies $\|\mathbf{c}^*_{k+1}\|_{W_c} \leq \|M\mathbf{c}^*_k\|_{W_c}$, which, from the definitions of M and W_c implies that

$$\|\mathbf{c}^*_{k+1}\|^2_{W_c} \leq \|\mathbf{c}^*_k\|^2_{W_c} - \|c^*_{0|k}\|^2_{R+B^TW_xB}. \tag{3.33}$$

From this bound we obtain the following result.

Lemma 3.1 *Let $\underline{\lambda}(R)$ denote the smallest eigenvalue of R in the cost (3.29), then*

$$\sum_{k=0}^{\infty} \|c^*_{0|k}\|^2 \leq \frac{1}{\underline{\lambda}(R)} \|\mathbf{c}^*_0\|^2_{W_c}. \tag{3.34}$$

Proof Summing both the sides of (3.33) over $k = 0, 1, \ldots$ and using the bound $\|c\|^2_{R+B^T W_x B} \geq \underline{\lambda}(R)\|c\|^2$, where $\underline{\lambda}(R) > 0$ due to $R \succ 0$, yields (3.34). □

We next give a version of a result from l_2 stability theory which states that the l_2 gains from the inputs $c^*_{0|k}$ and w_k to the state of the closed-loop system (3.32) are necessarily finite since Φ is assumed to be strictly stable.

Lemma 3.2 *All trajectories of the closed-loop system (3.32) satisfy the bound*

$$\sum_{k=0}^{\infty} \|x_k\|^2 \leq \|x_0\|^2_P + \gamma_1^2 \sum_{k=0}^{\infty} \|c^*_{0|k}\|^2 + \gamma_2^2 \sum_{k=0}^{\infty} \|w_k\|^2 \tag{3.35}$$

for some matrix $P \succ 0$ and some scalars γ_1, γ_2, provided Φ is strictly stable.

Proof There necessarily exists $P \succ 0$ satisfying $P - \Phi^T P \Phi \succ I_{n_x}$ since Φ is strictly stable (see, e.g. [8], Sect. 5.9). Using Schur complements it follows that there exists $P \succ 0$ and positive scalars γ_1, γ_2 satisfying

$$\begin{bmatrix} P & \Phi^T P \\ P\Phi & P \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ PB & PD \end{bmatrix} \begin{bmatrix} \gamma_1^{-2} I_{n_u} & 0 \\ 0 & \gamma_2^{-2} I_{n_w} \end{bmatrix} \begin{bmatrix} 0 & B^T P \\ 0 & D^T P \end{bmatrix} \succ \begin{bmatrix} I_{n_x} & 0 \\ 0 & 0 \end{bmatrix}.$$

Using Schur complements again, we therefore have

$$\begin{bmatrix} P & 0 & 0 \\ 0 & \gamma_1^2 I_{n_u} & 0 \\ 0 & 0 & \gamma_2^2 I_{n_w} \end{bmatrix} - \begin{bmatrix} \Phi^T \\ B^T \\ D^T \end{bmatrix} P \begin{bmatrix} \Phi & B & D \end{bmatrix} \succeq \begin{bmatrix} I_{n_x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Pre- and post-multiplying both sides of this inequality by $(x_k, c^*_{0|k}, w_k)$ and using (3.32) we obtain

$$\|x_k\|^2_P + \gamma_1^2 \|c^*_{0|k}\|^2 + \gamma_2^2 \|w_k\|^2 - \|x_{k+1}\|^2_P \geq \|x_k\|^2,$$

and summing both sides of this inequality over $k = 0, 1, \ldots$ gives (3.35). □

A consequence of inequalities (3.34) and (3.35) is that the closed-loop system under Algorithm 3.1 inherits the bound on the l_2 gain from w to x that is imposed by the linear feedback law $u = Kx$ in the absence of constraints. These inequalities can also be used to analyse the asymptotic convergence of x_k. Thus, let $x_k = s_k + e_k$, where s_k and e_k are the components of the state of the closed-loop system (3.32) that satisfy

$$s_{k+1} = \Phi s_k + B c^*_{0|k} \tag{3.36a}$$

$$e_{k+1} = \Phi e_k + D w_k \tag{3.36b}$$

with $s_0 = x_0$ and $e_0 = 0$. Then from (3.35) and Lemma 3.2 applied to (3.36a) we obtain

$$\sum_{k=0}^{\infty} \|s_k\|^2 \leq \|x_0\|_P^2 + \frac{\gamma_1^2}{\underline{\lambda}(R)} \|\mathbf{c}_0^*\|_{W_c}^2 \tag{3.37}$$

and it follows that $s_k \to 0$ as $k \to \infty$. Moreover, since $x_k = s_k + e_k$ where e_k lies for all k in the mRPI set $\mathcal{X}^{\mathrm{mRPI}}$ defined in (3.23), the asymptotic convergence of s_k can be used to demonstrate convergence of x_k to $\mathcal{X}^{\mathrm{mRPI}}$. This final point is explained in more detail in the proof of Theorem 3.2, which summarizes the results of this section.

Theorem 3.2 *For the system (3.1) with the control law of Algorithm 3.1:*

(a) the feasible set $\mathcal{F}_N$ defined in (3.18) is robustly positively invariant;
(b) for any $x_0 \in \mathcal{F}_N$, the closed-loop evolution of the state of (3.1) satisfies

$$\sum_{k=0}^{\infty} \|x_k\|^2 \leq \|x_0\|_P^2 + \frac{\gamma_1^2}{\underline{\lambda}(R)} \|\mathbf{c}_0^*\|_{W_c}^2 + \gamma_2^2 \sum_{k=0}^{\infty} \|w_k\|^2 \tag{3.38}$$

for some matrix $P \succ 0$ and scalars γ_1, γ_2;
(c) x_k converges asymptotically to the minimal RPI set $\mathcal{X}^{\mathrm{mRPI}}$ of (3.23).

Proof The RPI property of the feasible set $\mathcal{F}_N$ follows from the definition of the constraint set in (3.31) as a RPI set, whereas the bound (3.38) is a direct consequence of bounds (3.34) and (3.35). Furthermore, by (3.37) $\|s_k\|$ is square-summable, so for any given $\epsilon > 0$ there must exist finite n such that $s_k \in \mathcal{B}_\epsilon = \{s : \|s\| \leq \epsilon\}$ for all $k \geq n$, and therefore

$$x_k \in \mathcal{E}_k \oplus \mathcal{B}_\epsilon, \ \forall k \geq n,$$

where $\mathcal{E}_k$ is the bounding set for e_k defined by $\mathcal{E}_{k+1} = \Phi\mathcal{E}_k \oplus D\mathcal{W}$ with $\mathcal{E}_0 = \{0\}$. Since $\mathcal{E}_k \subseteq \mathcal{X}^{\mathrm{mRPI}}$ for all k, we have

$$x_k \in \mathcal{X}^{\mathrm{mRPI}} \oplus \mathcal{B}_\epsilon, \ \forall k \geq n,$$

and it can be concluded that x_k converges to $\mathcal{X}^{\mathrm{mRPI}}$ as $k \to \infty$. □

3.4 A Game Theoretic Approach

Robustly stabilizing controllers that guarantee limits on the response to additive disturbances can be designed using the linear quadratic game theory of optimal control [9, 10]. By choosing the control input so as to minimize a predicted cost that assumes the worst-case future model uncertainty, this approach is able to impose a specified bound on the l_2 gain from the disturbance input to a given system output. This strategy has its roots in game theory [11], which interprets the control input and the disturbance input as opposing players, each of which seeks to influence the behaviour of the system, one by minimizing and the other maximizing the performance index. For this reason it is also known as a min-max approach [12].

The idea has been exploited in unconstrained MPC (e.g. [13, 14]) but the concern here is with the linear discrete-time constrained case [15, 16]. Control laws that aim to optimize the worst-case performance with respect to model uncertainty can be overly cautious. On the other hand, game theoretic approaches to MPC based on the minimization of worst case predicted performance can avoid the possibility of poor sensitivity to disturbances which could be exhibited by MPC laws based on the nominal cost considered in Sect. 3.3. At the same time, the approach retains the feasibility and asymptotic convergence properties of robust MPC.

According to the game theoretic approach, the cost of (2.3) is replaced by

$$\check{J}(x_0, \{u_0, u_1, \ldots\}) \doteq \max_{\{w_0, w_1, \ldots\}} \sum_{i=0}^{\infty} \bigl(\|x_i\|_Q^2 + \|u_i\|_R^2 - \gamma^2 \|w_i\|^2\bigr). \tag{3.39}$$

The scalar parameter γ appearing in this cost limits the l_2 gain between the disturbance input w and the output $y = (Q^{1/2}x, R^{1/2}u)$ to γ. If there are no constraints on inputs and states, then the maximizing feedback law for w in (3.39) and the feedback law for u that minimizes $\check{J}$ are given by [17]

$$u = Kx, \quad w = Lx \tag{3.40}$$

where

$$\begin{bmatrix} K \\ L \end{bmatrix} = -\left(\begin{bmatrix} B^T \\ D^T \end{bmatrix} \check{W}_x \begin{bmatrix} B & D \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & -\gamma^2 I \end{bmatrix}\right)^{-1} \begin{bmatrix} B^T \\ D^T \end{bmatrix} \check{W}_x A \tag{3.41}$$

and $\check{W}_x$ is the unique positive definite solution of the Riccati equation

$$\begin{aligned} \check{W}_x &= A^T \check{W}_x A + Q \\ &\quad - A^T \check{W}_x \begin{bmatrix} B & D \end{bmatrix} \left(\begin{bmatrix} B^T \\ D^T \end{bmatrix} \check{W}_x \begin{bmatrix} B & D \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & -\gamma^2 I \end{bmatrix}\right)^{-1} \begin{bmatrix} B^T \\ D^T \end{bmatrix} \check{W}_x A. \end{aligned} \tag{3.42}$$

This result can be derived in a similar manner to the Riccati equation and optimal feedback gain for the unconstrained linear quadratic control problem with no model uncertainty in Theorem 2.1.

Under some mild assumptions on the system model (3.1) and the cost weights in (3.39) (see, e.g. [17] for details), the solution to the Riccati equation (3.42) exists whenever γ is sufficiently large that $\gamma^2 I - D^T \check{W}_x D$ is positive-definite, and moreover the resulting closed-loop system matrix $\Phi = A + BK$ is strictly stable. Clearly, it is important to have knowledge of the corresponding lower limit on γ^2, and it may therefore be more convenient to compute $\check{W}_x$ simultaneously with the minimum value of γ^2 using semidefinite programming. For example, minimizing γ^2 subject to the LMI

$$\begin{bmatrix} \begin{bmatrix} S & 0 \\ \star & \gamma^2 I \end{bmatrix} & \begin{bmatrix} (AS+BY)^T \\ D^T \end{bmatrix} & \begin{bmatrix} SQ^{1/2} & Y^T R^{1/2} \\ 0 & 0 \end{bmatrix} \\ \star & S & 0 \\ \star & \star & I \end{bmatrix} \succeq 0 \tag{3.43}$$

in variables S, Y and γ^2, yields the corresponding solutions for $\check{W}_x = S^{-1}$, $K = Y\check{W}_x$ and $L = (\gamma^2 I - D^T \check{W}_x D)^{-1} D^T \check{W}_x (A + BK)$. Throughout this section we assume that $\gamma^2 I - D^T \check{W}_x D \succ 0$ holds.

In order to formulate a predictive control law based on the cost (3.39), we adopt the open-loop strategy (3.6), with K defined via (3.41) as the optimal unconstrained state feedback gain for (3.39). By determining the maximizing disturbance sequence for any given x_k and optimization variables $\mathbf{c}_k$, the predicted cost, which we denote as $\check{J}(x_k, \mathbf{c}_k)$, along trajectories of (3.7) can be obtained as an explicit function of x_k and $\mathbf{c}_k$. This is consistent with the definition of an open-loop optimization strategy because it enables the entire sequence $\mathbf{c}_k = \{c_{0|k}, \ldots, c_{N-1|k}\}$ that achieves the minimum worst-case cost to be determined as a function of x_k.

The following lemma expresses $\check{J}(x_k, \mathbf{c}_k)$ as a quadratic function of x_k and $\mathbf{c}_k$ by considering the worst-case unconstrained disturbances in (3.39). Clearly, the resulting worst-case cost may be conservative since it ignores the information that the disturbance w_k lies in $\mathcal{W}$. An alternative cost definition that accounts for the constraints on the disturbance w by introducing extra optimization variables is discussed at the end of this section.

Lemma 3.3 *The worst-case cost (3.39) for the open-loop strategy (3.6) is given by*

$$\check{J}(x_k, \mathbf{c}_k) = \|x_k\|^2_{\check{W}_x} + \|\mathbf{c}_k\|^2_{\check{W}_c} \tag{3.44}$$

where $\check{W}_x$ is the solution of the Riccati equation (3.42) and $\check{W}_c$ is block diagonal:

$$\check{W}_c = \begin{bmatrix} B^T \check{W}'_x B + R & & 0 \\ & \ddots & \\ 0 & & B^T \check{W}'_x B + R \end{bmatrix} \tag{3.45a}$$

$$\check{W}'_x = \check{W}_x + \check{W}_x D(\gamma^2 I - D^T \check{W}_x D)^{-1} D^T \check{W}_x. \tag{3.45b}$$

Proof Let $z_{0|k} = (x_k, \mathbf{c}_k)$ and consider evaluating the cost (3.39) along the predicted trajectories generated by the dynamics $z_{i+1|k} = \Psi z_{i|k} + \bar{D} w_{i|k}, i = 0, 1, \ldots$ Clearly the cost (3.39) must be quadratic in $(x_k, \mathbf{c}_k)$. Furthermore, the minimizing control law and maximizing disturbance are given by the linear feedback laws (3.40) in the absence of constraints and so $\check{J}(0,0)=0$ must be the minimum value of $\check{J}(x,\mathbf{c})$ over all x and $\mathbf{c}$. Therefore the cost must have the form $\check{J}(x_k,\mathbf{c}_k)=\|z_{0|k}\|^2_{\check{W}}$ for some matrix $\check{W}$, and hence

$$\begin{aligned}\|z_{0|k}\|^2_{\check{W}} &= \max_{\{w_{0|k},w_{1|k},\ldots\}} \sum_{i=0}^{\infty}\bigl(\|z_{i|k}\|^2_{\hat{Q}}-\gamma^2\|w_{i|k}\|^2\bigr)\\ &= \max_{w_{0|k}}\Bigl\{\|z_{0|k}\|^2_{\hat{Q}}-\gamma^2\|w_{0|k}\|^2+\max_{\{w_{1|k},w_{2|k},\ldots\}}\sum_{i=1}^{\infty}\bigl(\|z_{i|k}\|^2_{\hat{Q}}-\gamma^2\|w_{i|k}\|^2\bigr)\Bigr\}\\ &= \|z_{0|k}\|^2_{\hat{Q}}+\max_{w_{0|k}}\bigl\{\|\Psi z_{0|k}+\bar{D}w_{0|k}\|^2_{\check{W}}-\gamma^2\|w_{0|k}\|^2\bigr\} \qquad (3.46)\end{aligned}$$

with $\hat{Q}$ defined as in (3.30b). The maximizing disturbance is therefore $w_{0|k}=(\gamma^2 I-\bar{D}^T\check{W}\bar{D})^{-1}\bar{D}^T\check{W}\Psi z_{0|k}$, so that $\|z_{0|k}\|^2_{\check{W}}=z_{0|k}^T(\hat{Q}+\Psi^T\check{W}'\Psi)z_{0|k}$ where

$$\check{W}'=\check{W}+\check{W}\bar{D}(\gamma^2 I-\bar{D}^T\check{W}\bar{D})^{-1}\bar{D}^T\check{W}.$$

Invoking (3.46) for all $z_{0|k}$ then gives

$$\check{W}=\Psi^T\check{W}'\Psi+\hat{Q},$$

and the block-diagonal form of $\check{W}$ together with the expressions for its diagonal blocks in (3.45a, 3.45b) then follow from the definition of Ψ in terms of the unconstrained optimal feedback gain K. □

The cost of Lemma 3.3 can be used to form the objective of a min-max RMPC algorithm based on an open-loop optimization strategy. As in Algorithm 3.1, we use the set $\mathcal{Z}^{\text{MRPI}}$ constructed in Sect. 3.2.1 to invoke constraints robustly and to guarantee recursive feasibility. Given the linearity of these constraints and the quadratic nature of the cost (3.44), the online optimization is again a quadratic programming problem with Nn_u variables and $n_C(\nu+1)$ constraints.

Algorithm 3.2 At each time instant $k=0,1,\ldots$:

(i) Perform the optimization

$$\underset{\mathbf{c}_k}{\text{minimize}}\ \|\mathbf{c}_k\|^2_{\check{W}_c}\ \text{ subject to } \bar{F}\Psi^i\begin{bmatrix}x_k\\ \mathbf{c}_k\end{bmatrix}\le \mathbf{1}-h_i,\ i=0,\ldots,\nu \qquad (3.47)$$

where ν satisfies the conditions of Theorem 3.1.

(ii) Apply the control law $u_k=Kx_k+c^*_{0|k}$, where $\mathbf{c}^*_k=(c^*_{0|k},\ldots,c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ for problem (3.47). ◁

The control theoretic properties of the nominal robust MPC law of Algorithm 3.1 apply to this min-max robust MPC strategy since the same method is used to construct the constraint set in each case, and because their respective cost matrices W_c and $\check{W}_c$ have the same structure. These properties can be summarized as follows.

Corollary 3.1 *(a) Recursive feasibility of the optimization (3.47) is ensured by robust invariance of $\mathcal{Z}^{\text{MRPI}}$ under the dynamics $z_{0|k+1} = \Psi z_{0|k} + \bar{D} w_k$, $w_k \in \mathcal{W}$.*

(b) The bound (3.34) (with W_c replaced by $\check{W}_c$) holds along closed-loop trajectories of (3.1) under Algorithm 3.2 as a result of the block-diagonal structure of $\check{W}_c$.

(c) Lemma 3.2 applies to the closed-loop trajectories under Algorithm 3.2. since $\Phi = A + BK$ is by assumption strictly stable.

(d) From (a)–(c) it follows that the conclusions of Theorem 3.2 apply to Algorithm 3.2; thus the state of (3.1) converges asymptotically to the minimal RPI set $\mathcal{X}^{\text{mRPI}}$ (3.23) associated with the control law that is defined by the solution of the Riccati equation (3.42).

In addition to these properties, the closed-loop system has a disturbance l_2 gain that is bounded from above by γ, as we show next.

Theorem 3.3 *For $x_0 \in \mathcal{F}_N$ and any nonnegative integer n, the control law of Algorithm 3.2 guarantees that the closed-loop trajectories of (3.1) satisfy*

$$\sum_{k=0}^{n} \left(\|x_k\|_Q^2 + \|u_k\|_R^2\right) \le \|x_0\|_{\check{W}_x}^2 + \|\mathbf{c}_0^*\|_{\check{W}_c}^2 + \gamma^2 \sum_{k=0}^{n} \|w_k\|^2. \tag{3.48}$$

Proof The effect of the actual disturbance at time k on the optimal value of the cost can be no greater than the worst case value predicted at time k:

$$\begin{aligned}
\check{J}(x_k, \mathbf{c}_k^*) &= \max_{\{w_{i|k}, w_{i+1|k}, \ldots\}} \sum_{i=0}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - \gamma^2 \|w_{i|k}\|^2\right) \\
&\ge \|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2 \|w_k\|^2 \\
&\quad + \max_{\{w_{i+1|k}, w_{i+2|k}, \ldots\}} \sum_{i=0}^{\infty} \left(\|x_{i|k+1}\|_Q^2 + \|u_{i|k+1}\|_R^2 - \gamma^2 \|w_{i+1|k}\|^2\right)
\end{aligned}$$

where $u_{i|k+1} = K x_{i|k+1} + c_{i+1|k}^*$ for all $i = 0, 1, \ldots$ Hence

$$\check{J}(x_k, \mathbf{c}_k^*) \ge \|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2 \|w_k\|^2 + \check{J}(x_{k+1}, M\mathbf{c}_k^*),$$

and since $(x_{k+1}, M\mathbf{c}_k^*)$ is feasible for (3.47), the minimization (3.47) at $k+1$ ensures that $\check{J}(x_{k+1}, M\mathbf{c}_k^*) \ge \check{J}(x_{k+1}, \mathbf{c}_{k+1}^*)$. For all k we therefore obtain

$$\check{J}_k^* \geq \|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2\|w_k\|^2 + \check{J}_{k+1}^*$$

where $\check{J}_k^* \doteq \check{J}(x_k, \mathbf{c}_k^*)$. Summing this inequality over $k = 0, 1, \ldots, n$ yields the bound (3.48) since $\check{W}_x$ and $\check{W}_c$ are positive-definite matrices. □

In essence (3.48) defines an achievable upper bound on the induced $\mathcal{H}_\infty$ norm of the response of the output $y = (Q^{1/2}x, R^{1/2}u)$ to additive disturbances. This aspect will be revisited in a more general setting in Chap. 5, where consideration is given to the case in which both additive and multiplicative uncertainty are present in the model.

Example 3.2 Applying the min-max approach of Algorithm 3.2 to the supply chain model of Example 3.1 we find that, for cost weights $Q = I$ and $R = 0.01$, the minimum achievable disturbance l_2 gain is $\gamma^2 = 8.15$. Setting $\gamma^2 = 10$ in (3.39) results in the optimal unconstrained feedback gain $K = [-1.27\ {-1.55}]$. This places the eigenvalues of Φ at 0.22 and 0.005, which indicates that the auxiliary control law $u = Kx$ is more aggressive than its counterpart for the nominal cost in Example 3.1. As a result, for $r = 6$, the minimum value of ρ satisfying (3.27) is considerably smaller at $\rho = 1.3 \times 10^{-3}$, and the minimal RPI set and its outer approximation are also smaller (Fig. 3.7). The areas of $\mathcal{X}^{\text{mRPI}}$ and $\hat{X}^{\text{mRPI}}$ are 0.0594 and 0.0595, respectively; for comparison the areas for Example 3.1 are 0.073 and 0.096.

The more aggressive feedback gain K for the min-max approach is more likely to conflict with the input constraints of this example, and this is reflected in the maximum element of $\hat{h}_\infty$ being closer to unity (here $\hat{h}_\infty = (0.67, 0.29, 0.29, 0.29, 0.29, 0.61, 0.91)$). As a consequence, the feasible set given by the projection, $\mathcal{F}_N$, of the maximal RPI set $\mathcal{Z}^{\text{MRPI}}$ onto the x-subspace is smaller than the feasible set for the same horizon N in Example 3.1. This can be seen by comparing Figs. 3.6 and 3.7. Figure 3.7 also indicates that a horizon of $N = 18$ is required in order to achieve the maximal feasible set $\mathcal{F}_\infty$, whereas the maximal feasible set (which for this example is identical for the nominal and min-max approaches) is obtained with $N = 4$ in Example 3.1. ◊

To account for the disturbance constraints (3.3) in the definition of the MPC performance index, we replace the worst-case cost (3.39) with

$$\check{J}(x_0, \{u_0, u_1, \ldots\}) = \max_{\substack{w_i \in \mathcal{W} \\ i=0,\ldots,N-1}} \sum_{i=0}^{N-1} \left(\|x_i\|_Q^2 + \|u_i\|_R^2 - \gamma^2\|w_i\|^2\right) + \|x_N\|_{\check{W}_x}^2. \tag{3.49}$$

Given the definition of $\check{W}_x$ in (3.42), the cost (3.49) is equivalent to an infinite horizon worst-case cost in which the maximization is performed subject to $w_i \in \mathcal{W}$ for $i = 0, \ldots, N-1$ and without constraints on w_i for $i \geq N$. With the definitions

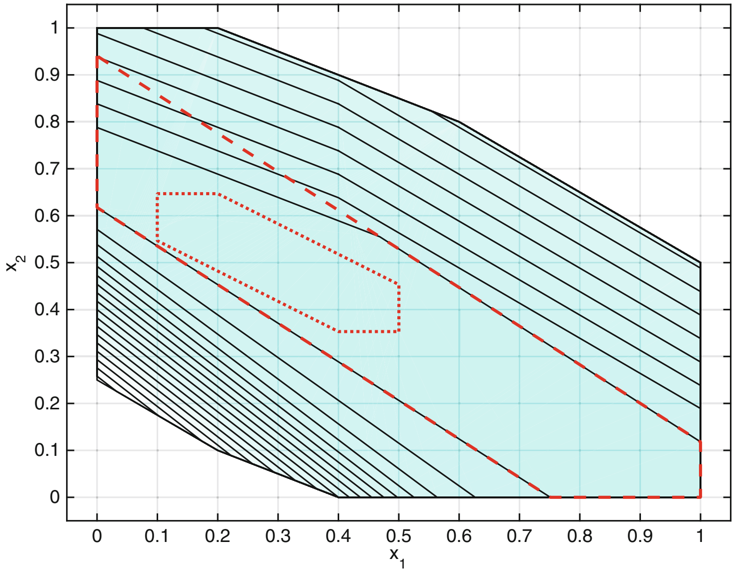

Fig. 3.7 The feasible initial condition sets $\mathcal{F}_N$, $N = 0, 1, \ldots, 18$, for Example 3.2; also shown are the sets $\{x : (F + GK)(x - x^0) \leq \mathbf{1}\}$ (*dashed line*) and $\hat{\mathcal{X}}^{\text{mRPI}} \oplus \{x^0\}$ (*dotted line*)

$$C_{xx} = \begin{bmatrix} I \\ \Phi \\ \vdots \\ \Phi^{N-1} \\ \Phi^N \end{bmatrix}, \; C_{xc} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ B & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \Phi^{N-2}B & \cdots & B & 0 \\ \Phi^{N-1}B & \cdots & \Phi B & B \end{bmatrix}, \; C_{xw} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ D & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \Phi^{N-2}D & \cdots & D & 0 \\ \Phi^{N-1}D & \cdots & \Phi D & D \end{bmatrix},$$

$$C_{ux} = \begin{bmatrix} K & & 0 \\ & \ddots & \vdots \\ & K & 0 \end{bmatrix}, \; \bar{Q} = \begin{bmatrix} Q & & \\ & \ddots & \\ & & Q \\ & & & \check{W}_x \end{bmatrix}, \; \bar{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}, \; \bar{V} = \begin{bmatrix} V & & \\ & \ddots & \\ & & V \end{bmatrix},$$

the sequences of predicted states $\mathbf{x}_k = (x_{0|k}, \ldots, x_{N|k})$ and inputs $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$ for the open-loop strategy (3.6) can be written explicitly in terms of x_k, $\mathbf{c}_k$ and the disturbance sequence $\mathbf{w}_k = (w_{0|k}, \ldots, w_{N-1|k})$ as

$$\begin{aligned} \mathbf{x}_k &= C_{xx}x_k + C_{xc}\mathbf{c}_k + C_{xw}\mathbf{w}_k, \\ \mathbf{u}_k &= C_{ux}\mathbf{x}_k + \mathbf{c}_k, \end{aligned}$$

and the cost (3.49) for the open-loop strategy (3.6) at time k can be expressed

$$\check{J}(x_k, \mathbf{c}_k) = \max_{\mathbf{w}_k \in \{\mathbf{w}:\, \bar{V}\mathbf{w} \le \mathbf{1}\}} \Big\{ \|C_{xx}x_k + C_{xc}\mathbf{c}_k + C_{xw}\mathbf{w}_k\|_{\bar{Q}}^2 \\ + \|C_{ux}(C_{xx}x_k + C_{xc}\mathbf{c}_k + C_{xw}\mathbf{w}_k) + \mathbf{c}\|_{\bar{R}}^2 - \gamma^2\|\mathbf{w}_k\|^2 \Big\}. \tag{3.50}$$

This maximization problem is a quadratic program, and it is convex if the matrix

$$\Delta \doteq \gamma^2 I - C_{xw}^T(\bar{Q} + C_{ux}^T \bar{R} C_{ux})C_{xw} \tag{3.51}$$

is positive-definite. Assuming that $\Delta \succ 0$, a more convenient but equivalent minimization can be derived from (3.50) using convex programming duality. This is based on the fact that, for $G \succ 0$, the optimal value of the QP:

$$\max_{x \in \{x:\, Ax \le b\}} \; g^T x - \tfrac{1}{2} x^T G x \tag{3.52}$$

is equal to the optimal value of the dual problem defined by the QP:

$$\min_{\lambda \in \{\lambda:\, \lambda \ge 0\}} \; b^T\lambda + \tfrac{1}{2}(g - A^T\lambda)^T G^{-1}(g - A^T\lambda), \tag{3.53}$$

(see e.g. [18] for a proof of this result).

Lemma 3.4 *For $\Delta \succ 0$, the worst-case cost (3.49) for the open-loop strategy (3.6) is equal to*

$$\check{J}(x_k, \mathbf{c}_k) = \min_{\mu \in \{\mu:\, \mu \ge 0\}} \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}^T \begin{bmatrix} \check{W}_x & 0 \\ 0 & \check{W}_c \end{bmatrix} \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} - 2\mu^T \check{W}_{\mu z} \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \\ + 2\mu^T \mathbf{1} + \mu^T \check{W}_{\mu\mu}\mu \tag{3.54}$$

where $\check{W}_x$ satisfies the Riccati equation (3.42), $\check{W}_c$ is defined in (3.45), and

$$\check{W}_{\mu z} = \bar{V}\Delta^{-1} C_{xw}^T \Big((\bar{Q} + C_{ux}^T \bar{R} C_{ux}) \begin{bmatrix} C_{xx} & C_{xc} \end{bmatrix} + C_{ux}^T \bar{R} \begin{bmatrix} 0 & I \end{bmatrix} \Big) \\ \check{W}_{\mu\mu} = \bar{V}\Delta^{-1}\bar{V}^T.$$

Proof This follows from (3.50) and the equivalence of (3.52) and (3.53). □

To use the worst-case cost (3.54) as the objective of the MPC optimization, we replace the optimization (3.47) in step (i) of Algorithm 3.2 by the following problem in $N(n_u + n_V)$ variables and $n_C(\nu + 1) + Nn_V$ constraints.

$$\begin{aligned} \underset{\mathbf{c}_k,\, \mu}{\text{minimize}} \quad & \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}^T \begin{bmatrix} \check{W}_x & 0 \\ 0 & \check{W}_c \end{bmatrix} \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} - 2\mu^T \check{W}_{\mu z} \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} + 2\mu^T \mathbf{1} + \mu^T \check{W}_{\mu\mu}\mu \\ \text{subject to} \quad & \bar{F}\Psi^i \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \le \mathbf{1} - h_i, \; i = 0, \ldots, \nu \\ & \mu \ge 0 \end{aligned} \tag{3.55}$$

With this modification the online MPC optimization remains a convex quadratic program; note however that it involves a larger number of variables and constraints than the online optimization (3.47).

The presence of disturbance constraints implies that (3.54) gives a tighter bound on the worst-case performance of the MPC algorithm in closed loop operation than the cost of (3.44); hence the optimization (3.55) is likely to result in improved worst-case performance of the MPC law. However, although the guarantee of recursive feasibility is not affected, the stability and convergence results in (b)–(d) of Corollary 3.1 no longer apply when (3.47) is replaced by (3.55) in Algorithm 3.2. This is to be expected of course, since the presence of the disturbance constraints (3.3) in the definition of the worst-case cost (3.49) implies that $u = Kx$ is not necessarily optimal for this cost, even when the constraints on x and u in (3.4) are inactive. Therefore, in the general case of persistent disturbances, the MPC law will not necessarily converge asymptotically to this linear feedback law.

On the other hand a bound on the disturbance l_2 gain similar to (3.48) does hold for the closed-loop system when (3.55) replaces the optimization in step (i) of Algorithm 3.2. This can be shown by an extension of the argument that was used in the proof of Theorem 3.3.

Theorem 3.4 *If $x_0 \in \mathcal{F}_N$ and $\Delta \succ 0$, Algorithm 3.2 with (3.55) in place of (3.47) satisfies, for all $n \geq 0$, the bound:*

$$\sum_{k=0}^{n} \bigl(\|x_k\|_Q^2 + \|u_k\|_R^2\bigr) \leq \check{J}(x_0, \mathbf{c}_0^*) + \gamma^2 \sum_{k=0}^{n} \|w_k\|^2. \tag{3.56}$$

Proof From Lemma 3.4 the optimal value of the objective in (3.55) is equal to

$$\check{J}(x_k, \mathbf{c}_k^*) = \max_{\substack{w_{i|k} \in \mathcal{W} \\ i=0,\ldots,N-1}} \sum_{i=0}^{N-1} \bigl(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - \gamma^2 \|w_{i|k}\|^2\bigr) + \|x_{N|k}\|_{\check{W}_x}^2,$$

but $\|x_{N|k}\|_{\check{W}_x}^2 = \max_{w_{N|k}} (\|x_{N|k}\|_Q^2 + \|u_{N|k}\|_R^2 - \gamma^2 \|w_{N|k}\|^2 + \|x_{N+1|k}\|_{\check{W}_x}^2)$ so that

$$\begin{aligned}
\check{J}(x_k, \mathbf{c}_k^*) &\geq \max_{\substack{w_{i|k} \in \mathcal{W} \\ i=0,\ldots,N}} \sum_{i=0}^{N} \bigl(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - \gamma^2 \|w_{i|k}\|^2\bigr) + \|x_{N+1|k}\|_{\check{W}_x}^2 \\
&\geq \|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2 \|w_k\|^2 \\
&\quad + \max_{\substack{w_{i|k} \in \mathcal{W} \\ i=1,\ldots,N}} \sum_{i=0}^{N-1} \bigl(\|x_{i|k+1}\|_Q^2 + \|u_{i|k+1}\|_R^2 - \gamma^2 \|w_{i|k+1}\|^2\bigr) + \|x_{N|k+1}\|_{\check{W}_x}^2
\end{aligned}$$

where $u_{i|k+1} = Kx_{i|k+1} + \mathbf{c}^*_{i+1|k}$ for all $i = 0, 1, \ldots$ It follows that

$$\begin{aligned}\check{J}(x_k, \mathbf{c}^*_k) &\geq \|x_k\|^2_Q + \|u_k\|^2_R - \gamma^2\|w_k\|^2 + \check{J}(x_{k+1}, M\mathbf{c}^*_k)\\ &\geq \|x_k\|^2_Q + \|u_k\|^2_R - \gamma^2\|w_k\|^2 + \check{J}(x_{k+1}, \mathbf{c}^*_{k+1}),\end{aligned}$$

and hence

$$\sum_{k=0}^{n}\bigl(\|x_k\|^2_Q + \|u_k\|^2_R\bigr) \leq \check{J}(x_0, \mathbf{c}^*_0) - \check{J}(x_{n+1}, \mathbf{c}^*_{n+1}) + \gamma^2\sum_{k=0}^{n}\|w_k\|^2.$$

which implies the bound (3.56) since $\check{J}(x, \mathbf{c}) \geq 0$ for all $(x, \mathbf{c})$. The conclusion here that $\check{J}(x, \mathbf{c})$ is nonnegative follows from the convexity of $\check{J}(x, \mathbf{c})$ and from the fact that the optimal unconstrained feedback laws given by (3.40) are feasible for sufficiently small x; thus $\check{J}(0, 0) = 0$ is the global minimum of $\check{J}(x, \mathbf{c})$. Note also that $\check{J}(x, \mathbf{c})$ is necessarily convex whenever $\Delta \succ 0$ because the expression maximized on the RHS of (3.50) is convex in $(x, \mathbf{c})$ for any given $\mathbf{w}$, and since the pointwise maximum of convex functions is convex [19]. □

A numerical example comparing the minmax MPC strategies defined by the alternative online optimizations of (3.47) and (3.55) is provided in Question 7 on p. 116.

3.5 Rigid and Homothetic Tubes

The focus of this chapter has so far been on robust MPC laws with constraints derived from the decomposition (3.8) with initial conditions $s_{0|k} = x_k$ and $e_{0|k} = 0$ for the nominal and uncertain components of the predicted state. In this section, we consider alternative definitions of the nominal state and uncertainty tube that can provide alternative, potentially stronger, stability guarantees. The stability properties of the nominal cost and game theoretic approaches of Sects. 3.3 and 3.4 were stated in terms of a finite l_2 gain from the disturbance input to the state and control input in closed-loop operation, as well as a guarantee of asymptotic convergence of the system state to the mRPI set for the unconstrained optimal feedback law. Instead, by relaxing the requirement that $s_{0|k} = x_k$ and $e_{0|k} = 0$, the tube MPC approaches of this section, which are based on [6, 20], ensure exponential stability of an outer approximation of the mRPI set.

The guarantee of exponential stability of a given limit set for the closed-loop system state comes at a price. This is because the initial condition $\mathcal{E}_{0|k} = \{0\}$ in the uncertainty tube dynamics (3.19) results in an uncertainty tube $\{\mathcal{E}_{0|k}, \mathcal{E}_{1|k}, \ldots\}$ which is minimal in the sense that $\mathcal{E}_{i|k}$ is the smallest set that contains the uncertain component of the predicted state given the disturbance bounds. Consequently, if $e_{0|k} \neq 0$, so that the initial set $\mathcal{E}_{0|k}$ contains more points than just the origin, then the amount by which the constraints on the nominal predicted trajectories must be tightened in order to ensure that constraints are satisfied for all realizations of uncertainty will be overestimated. This leads to smaller sets of feasible initial conditions than are obtained using the methods of Sects. 3.3 and 3.4.

However the flexibility afforded by allowing non-zero initial conditions for the uncertainty tube enables the algorithms of this section to reduce the potential conservativeness of their constraint handling. This possibility is developed here by combining different linear feedback laws in the definition of the predicted input trajectories. For a robust MPC algorithm employing the open-loop strategy (3.6), the size of the feasible set of states depends on the sizes of the MPI set for the nominal model and the mRPI set in the presence of disturbances, both of which depend on the feedback gain K. For a small mRPI set we require good disturbance rejection, whereas a large MPI set for the nominal dynamics requires good tracking performance in the presence of constraints. These can be conflicting requirements, thus motivating the consideration of closed-loop strategies for which the uncertainty tube $\{\mathcal{E}_{0|k}, \mathcal{E}_{1|k}, \ldots\}$ may depend explicitly on the future state and constraints. Within the framework of open-loop strategies however, this issue can be addressed by incorporating different linear feedback gains in the nominal and uncertain components of the dynamics (3.8).

To this end, let $x_{i|k} = s_{i|k} + e_{i|k}$, and

$$u_{i|k} = Ks_{i|k} + K_e e_{i|k} + c_{i|k}, \tag{3.57a}$$

$$s_{i+1|k} = \Phi s_{i|k} + Bc_{i|k}, \tag{3.57b}$$

$$e_{i+1|k} = \Phi_e e_{i|k} + Dw_{i|k}, \quad w_{i|k} \in \mathcal{W} \tag{3.57c}$$

where $\Phi = A + BK$ and $\Phi_e = A + BK_e$. The freedom to choose different gains K_e and K can allow for improved disturbance rejection without adversely affecting the MPI set for the nominal state. However, in this framework, the initial conditions $e_{0|k} = 0$ and $s_{0|k} = x_k$ cannot be assumed because there is no single value of $c_{0|k+1}$ that makes $u_{1|k} = Ks_{1|k} + K_e e_{1|k} + c_{1|k}$ equal to $u_{0|k+1} = Kx_{k+1} + c_{0|k+1}$ for all $w_k \in \mathcal{W}$. Therefore the method described in Sect. 3.2.1 for ensuring recursive feasibility of an open-loop strategy would fail with this parameterization. However, if $\mathcal{E}_{0|k}$ is permitted to contain more points than just the origin, then it is possible to construct a feasible but suboptimal trajectory at time $k + 1$ by choosing $s_{i|k+1} = s_{i+1|k}$, $e_{i|k+1} = e_{i+1|k}$ and $c_{i|k+1} = c_{i+1|k}$ for $i = 0, 1, \ldots$ This is the case for tube MPC algorithms considered in this section that allow non-singleton initial uncertainty sets. Note that in this context, and throughout the current section, the term "nominal state" does not have the usual meaning (namely the state of the disturbance-free model) because $s_{0|k}$ is not necessarily chosen to coincide with the current state $x_{0|k} = x_k$.

3.5.1 Rigid Tube MPC

The convenience of the decomposition (3.57) is that, if $s_{0|k}$ is chosen so that $e_{0|k} = x_{0|k} - s_{0|k}$ belongs to a set $\mathcal{S}$ that is RPI for (3.57c), namely if

$$\Phi_e \mathcal{S} \oplus D\mathcal{W} \subseteq \mathcal{S}, \tag{3.58}$$

then $e_{i|k}$ must also lie in $\mathcal{S}$ for all $i = 1, 2, \ldots$ It is assumed that $\mathcal{S}$ is compact, convex and polytopic, and is described by $n_{\mathcal{S}}$ linear inequalities:

$$\mathcal{S} = \{e : V_{\mathcal{S}} e \leq 1\}.$$

Under these circumstances the predicted trajectory $\{e_{0|k}, e_{1|k}, \ldots\}$ lies in a tube of fixed cross section $\mathcal{S}$. Such tubes were used in [6]; we refer to them as *rigid* tubes to distinguish them from *homothetic* tubes [20] which allow for variable scaling of the tube cross sections.

This strategy simplifies the problem of ensuring robust satisfaction of the constraints (3.4). Applying these constraints to the predicted trajectories of (3.57) requires that

$$\bar{F}\Psi^i z_{0|k} \leq \mathbf{1} - h_{\mathcal{S}}, \quad i = 0, 1, \ldots \tag{3.59}$$

where $z_{0|k} = (s_{0|k}, \mathbf{c}_k)$ is the initial state of the autonomous dynamics (3.9) and the vector $h_{\mathcal{S}}$ can be computed offline by solving a set of linear programs:

$$h_{\mathcal{S}} = \max_{e \in \mathcal{S}} (F + GK_e)e.$$

The RHS of each constraint in (3.59) is independent of the time index i. By Theorem 2.3 therefore, the infinite sequence of inequalities in (3.59) can be reduced to an equivalent constraint set described by a finite number of inequalities: $z_{0|k} \in \mathcal{Z}^{\mathrm{MPI}}$. Under the necessary assumption that $h_{\mathcal{S}} < \mathbf{1}$, $\mathcal{Z}^{\mathrm{MPI}}$ is defined by

$$\mathcal{Z}^{\mathrm{MPI}} \doteq \{z : \bar{F}\Psi^i z \leq \mathbf{1} - h_{\mathcal{S}}, \; i = 0, \ldots, \nu\},$$

where ν is the (necessarily finite) integer that satisfies the conditions of Theorem 2.3 with the RHS of (2.15) replaced by $\mathbf{1} - h_{\mathcal{S}}$.

Using rigid tubes to represent the evolution of the uncertain component of the predicted state of (3.1) causes the uncertainty in predicted trajectories to be overestimated, especially in the early part of the prediction horizon. This results in smaller feasible sets of initial conditions than the exact approach described in Sect. 3.2.1. Clearly $\mathcal{S}$ can be no smaller than the minimal RPI set for (3.57c), and to reduce the degree of conservatism it is therefore advantageous to define $\mathcal{S}$ as a close approximation of the minimal RPI. For example

$$\mathcal{S} \doteq \frac{1}{1-\rho} \bigoplus_{j=0}^{r-1} \Phi_e^j D\mathcal{W} \tag{3.60}$$

where r and $\rho \in [0, 1)$ satisfy $\Phi_e^r D\mathcal{W} \subseteq \rho D\mathcal{W}$. The following lemma shows that this choice of $\mathcal{S}$ meets the condition (3.58) for robust positive invariance.

Lemma 3.5 *The set $\mathcal{S}$ in (3.60) satisfies (3.58) and hence is RPI for (3.57c).*

Proof Under the assumption that $\Phi_e^r D\mathcal{W} \subseteq \rho D\mathcal{W}$, we obtain

$$\begin{aligned}\Phi_e \mathcal{S} \oplus D\mathcal{W} &= \frac{1}{1-\rho} \bigoplus_{j=1}^{r-1} \Phi_e^j D\mathcal{W} \oplus \frac{1}{1-\rho} \Phi_e^r D\mathcal{W} \oplus D\mathcal{W} \\ &\subseteq \frac{1}{1-\rho} \bigoplus_{j=1}^{r-1} \Phi_e^j D\mathcal{W} \oplus \frac{\rho}{1-\rho} D\mathcal{W} \oplus D\mathcal{W} \\ &= \frac{1}{1-\rho} \bigoplus_{j=0}^{r-1} \Phi_e^j D\mathcal{W} = \mathcal{S}.\end{aligned}$$

□

The final ingredient needed for the definition of a robust MPC algorithm is the choice of the cost function. This is taken to be the nominal cost of (3.29), with $s_{i|k}$ defined as the nominal predicted trajectory generated by (3.57b) and with $v_{i|k} = K s_{i|k} + c_{i|k}$ for all i. Therefore, we set $J(s_{0|k}, \mathbf{c}_k) \doteq \|z_{0|k}\|_W^2$ where W is the solution of the Lyapunov equation (2.34), and by Theorem 2.10 we obtain

$$J(s_{0|k}, \mathbf{c}_k) = \|s_{0|k}\|_{W_x}^2 + \|\mathbf{c}_k\|_{W_c}^2.$$

Unlike the nominal cost of Sect. 3.3, the initial condition, $s_{0|k}$, from which this cost is computed is not necessarily equal to the plant state x_k, but is instead treated as an optimization variable subject to the constraint that $e_{0|k} = x_k - s_{0|k} \in \mathcal{S}$. This choice of cost is justified by the argument that the primary objective of the MPC law is to steer the plant state into or close to the mRPI set associated with the linear feedback law $u = K_e x$ (which by assumption has been designed to provide good disturbance rejection), and that x_k necessarily converges to the mRPI approximation $\mathcal{S}$ if $s_{0|k}$ converges to zero. Since the aim is to enforce convergence of $s_{0|k}$, the cost weights Q and R in (3.29) are taken to be positive-definite matrices [6, 20].

The resulting online optimization problem is a quadratic program in $Nn_u + n_x$ variables and $n_C(\nu + 1) + n_{\mathcal{S}}$ constraints.

Algorithm 3.3 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization

$$\begin{aligned} \underset{s_{0|k}, \mathbf{c}_k}{\text{minimize}} \quad & \|s_{0|k}\|_{W_x}^2 + \|\mathbf{c}_k\|_{W_c}^2 \\ \text{subject to} \quad & \bar{F}\Psi^i \begin{bmatrix} s_{0|k} \\ \mathbf{c}_k \end{bmatrix} \leq \mathbf{1} - h_{\mathcal{S}}, \ i = 0, \ldots, \nu \\ & x_k - s_{0|k} \in \mathcal{S} \end{aligned} \tag{3.61}$$

where ν satisfies the conditions of Theorem 2.3 with the RHS of (2.15) replaced by $\mathbf{1} - h_{\mathcal{S}}$.

(ii) Apply the control law $u_k = Ks^*_{0|k} + K_e(x_k - s^*_{0|k}) + c^*_{0|k}$, where $(s^*_{0|k}, \mathbf{c}^*_k)$ is the optimizer of problem (3.61), and $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$. ◁

To determine the feasible set for the state x_k in (3.61), let $\mathcal{F}^s_N$ denote the feasible set for $s_{0|k}$ in the optimization (3.61):

$$\mathcal{F}^s_N \doteq \left\{ s : \exists \mathbf{c} \text{ such that } \bar{F}\Psi^i \begin{bmatrix} s \\ \mathbf{c} \end{bmatrix} \le \mathbf{1} - h_{\mathcal{S}},\ i = 0, \ldots, \nu \right\}.$$

Then, given that any feasible state x_k must satisfy $x_k = s_{0|k} + e_{0|k}$ for some $e_{0|k} \in \mathcal{S}$, the set of feasible initial conditions for Algorithm 3.3 can be expressed as

$$\mathcal{F}_N = \mathcal{F}^s_N \oplus \mathcal{S}.$$

Theorem 3.5 *For the system (3.1) with the control law of Algorithm 3.3, the feasible set $\mathcal{F}_N$ is RPI, and $\mathcal{S}$ is exponentially stable with region of attraction equal to $\mathcal{F}_N$ if $Q \succ 0$ and $R \succ 0$ in (3.29).*

Proof To demonstrate that $\mathcal{F}_N$ is RPI, suppose that $x_k \in \mathcal{F}_N$ so that $(s^*_{0|k}, \mathbf{c}^*_k) \in \mathcal{Z}^{\text{MPI}}$ and $x_k - s^*_{0|k} \in \mathcal{S}$. Let $s_{0|k+1} = \Phi s^*_{0|k} + BE\mathbf{c}^*_k$ and $\mathbf{c}_{k+1} = M\mathbf{c}^*_k$, then since $\mathcal{Z}^{\text{MPI}}$ is an invariant set for the autonomous dynamics (3.9) and $\mathcal{S}$ is RPI for (3.57) it follows that $(s_{0|k+1}, \mathbf{c}_{k+1}) \in \mathcal{Z}^{\text{MPI}}$ and $x_{k+1} - s_{0|k+1} = \Phi_e(x_k - s_{0|k}) + Dw_k \in \mathcal{S}$ for any disturbance $w_k \in \mathcal{W}$; therefore $(s_{0|k+1}, \mathbf{c}_{k+1})$ is feasible for (3.61), implying $x_{k+1} \in \mathcal{F}_N$.

The exponential stability of $\mathcal{S}$ follows from the definition (3.29) of the cost in (3.61), which implies the bound

$$J(s^*_{0|k+1}, \mathbf{c}^*_{k+1}) \le J(s^*_{0|k}, \mathbf{c}^*_k) - (\|s_{0|k}\|^2_Q + \|Ks_{0|k} + \mathbf{c}^*_k\|^2_R).$$

Therefore, by the same argument as was used the proof of Theorem 2.8, the closed-loop application of Algorithm 3.3 gives, for any $x_0 \in \mathcal{F}_N$ and all $k > 0$,

$$\|s^*_{0|k}\|^2 \le \frac{b}{a}\left|1 - \frac{\underline{\lambda}(Q)}{b}\right|^k \|s^*_{0|0}\|^2,$$

for some constants $a, b > 0$ satisfying $a\|s_{0|k}\|^2 \le J(s^*_{0|k}, \mathbf{c}^*_k) \le b\|s_{0|k}\|^2$ for any $x_k \in \mathcal{F}_N$, and where $\underline{\lambda}(Q)$ is the smallest eigenvalue of Q. The constraint $s^*_{0|k} = x_k - e$ for $e \in \mathcal{S}$ implies that $\min_{e \in \mathcal{S}} \|x_k - e\| \le \|s^*_{0|k}\|$; hence the distance of x_k from $\mathcal{S}$ is upper-bounded by an exponentially decaying function of time. □

Example 3.3 Considering again the supply chain model of Example 3.1, K and K_e are initially chosen to be equal, with $K = K_e = [-0.89\ -0.78]$, which is the unconstrained LQ-optimal feedback gain for the nominal cost with $Q = I, R = 0.01$.

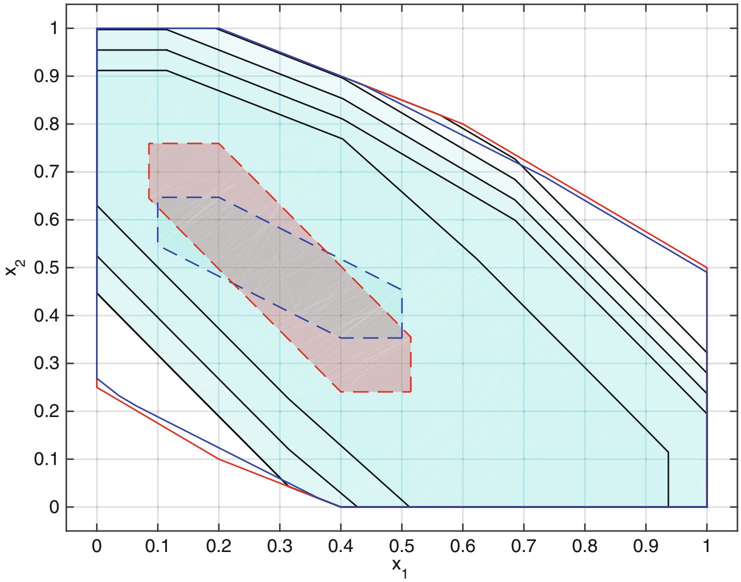

Fig. 3.8 The feasible initial condition sets $\mathcal{F}_N$, $N = 0, 1, 2, 3, 4$ for Algorithm 3.3 with $K_e = K$ chosen as the unconstrained LQ-optimal feedback gain (*black lines*). Also shown are the maximal feasible set for Algorithm 3.1 (*solid red line*) and the maximal feasible set for Algorithm 3.3 with K_e equal to the unconstrained min-max optimal feedback gain of Example 3.2 (*solid blue line*), and the mRPI sets for the unconstrained LQ-optimal and unconstrained min-max optimal feedback gains (*dotted red line* and *dotted blue line*, respectively)

The set $\mathcal{S}$ is taken to be the mRPI set approximation (3.60) for $r = 6$, $\rho = 0.127$, so that $h_{\mathcal{S}}$ is equal to the value of $\hat{h}_\infty$ in Example 3.1. The feasible initial condition sets, $\mathcal{F}_N$, $N = 0, 1, 2, 3, 4$, for Algorithm 3.3 are shown in Fig. 3.8; for these choices of K, K_e and $\mathcal{S}$, the largest possible feasible set for Algorithm 3.3 is obtained with $N = 4$. For comparison, the figure shows the maximal feasible set of states for Algorithm 3.1 (which is also obtained for $N = 4$). This contains and extends outside the feasible set for rigid tube MPC as a result of the conservative definition of the uncertainty tube in Algorithm 3.3. For this example, and for all values of N, Algorithm 3.1 has $6(N + 2)$ constraints whereas Algorithm 3.3 has $6(N + 2) + 8$ constraints.

Replacing $\mathcal{S}$ with a smaller mRPI set approximation reduces the degree of conservativeness of the constraints in Algorithm 3.3. In fact Fig. 3.8 shows that almost all of the discrepancy between the maximal feasible sets of Algorithms 3.1 and 3.3 disappears if $K_e = [-1.27\ \ -1.55]$ (which is the unconstrained min-max optimal feedback gain for $\gamma^2 = 10$) and, as before, $K = [-0.89\ \ -0.78]$. However, the maximal set is then obtained for a larger value of N; here $N = 20$ is needed to achieve the maximal feasible set. ♢

3.5.2 Homothetic Tube MPC

The rigid tube MPC described in Sect. 3.5.1 assumes that uncertainty in the error state e is uniform through the prediction horizon. This however is conservative given that e could initially be small (indeed $e_{0|k} = 0$ if $s_{0|k} = x_k$), and the set containing the uncertain component of the predicted future state only approaches the mRPI set asymptotically. Rather than tightening constraints on the nominal predicted trajectories by considering the worst case $e_{i|k} \in \mathcal{S}$ as is done in rigid tube MPC, it is therefore more reasonable to assume that

$$e_{i|k} \in \alpha_{i|k}\mathcal{S}^0 \tag{3.62}$$

for positive scalars $\alpha_{i|k}$, $i = 0, 1, \ldots$ that are taken to be variables in the online optimization, and where the set

$$\mathcal{S}^0 = \{e : V_{\mathcal{S}}^0 e \le \mathbf{1}\} \tag{3.63}$$

is determined offline. This replaces the rigid tube $\{\mathcal{S}, \mathcal{S}, \ldots\}$ that is used in rigid tube MPC to bound the uncertainty tube $\{\mathcal{E}_{0|k}, \mathcal{E}_{1|k}, \ldots\}$ with an uncertainty tube given by $\{\alpha_{0|k}\mathcal{S}^0, \alpha_{1|k}\mathcal{S}^0, \ldots\}$. The sets $\alpha_{i|k}\mathcal{S}^0$ in this expression are homothetic to $\mathcal{S}^0$, and hence, the approach is known as homothetic tube MPC [20].

The presence of the scalar variables $\alpha_{i|k}$ implies that $\mathcal{S}^0$ (unlike $\mathcal{S}$) need not be RPI for the error dynamics (3.57c). Instead it is assumed that $\mathcal{S}^0$ is compact and satisfies the invariance condition $\Phi_e\mathcal{S}^0 \subseteq \mathcal{S}^0$. A convenient way to invoke the inclusion condition (3.62) is through a recursion relating $\alpha_{i+1|k}$ to $\alpha_{i|k}$ so as to ensure that $e_{i+1|k} \in \alpha_{i+1|k}\mathcal{S}^0$ whenever $e_{i|k} \in \alpha_{i|k}\mathcal{S}^0$. The required condition[1] can be expressed as

$$\Phi_e\alpha_{i|k}\mathcal{S}^0 \oplus D\mathcal{W} \subseteq \alpha_{i+1|k}\mathcal{S}^0, \tag{3.64}$$

or equivalently, given the representation (3.63), as

$$\alpha_{i|k} \max_{e\in\mathcal{S}^0} V_{\mathcal{S}}^0\Phi_e e + \max_{w\in\mathcal{W}} V_{\mathcal{S}}^0 D w \le \alpha_{i+1|k}\mathbf{1}. \tag{3.65}$$

This condition is equivalent to

$$\alpha_{i|k}\bar{e} + \bar{w} \le \alpha_{i+1|k}\mathbf{1} \tag{3.66}$$

[1] This is a simplified version of the more general inclusion condition that is considered in [20]: $\{\Phi s_{i|k} + Bc_{i|k}\} \oplus \Phi_e\alpha_{i|k}\mathcal{S}^0 \oplus D\mathcal{W} \subseteq \{s_{i+1|k}\} \oplus \alpha_{i+1|k}\mathcal{S}^0$.

where the vectors $\bar{e}$ and $\bar{w}$ can be computed offline by solving a pair of linear programs:

$$\bar{e} \doteq \max_{e} V_{\mathcal{S}}^{0} \Phi_e e \text{ subject to } V_{\mathcal{S}}^{0} e \leq \mathbf{1}$$

$$\bar{w} \doteq \max_{w} V_{\mathcal{S}}^{0} D w \text{ subject to } V w \leq \mathbf{1}.$$

Given that the condition (3.66) ensures that $e_{i|k} \in \alpha_{i|k} \mathcal{S}^0$ throughout the prediction horizon, the constraints (3.4) can be invoked as

$$\bar{F} \Psi^i z_{0|k} + \alpha_{i|k} h_{\mathcal{S}}^{0} \leq \mathbf{1}, \quad i = 0, 1, \ldots \tag{3.67}$$

where $z_{0|k} = (s_{0|k}, \mathbf{c}_k)$ is the initial state of the autonomous dynamics (3.9) and where the vector $h_{\mathcal{S}}^{0}$ can be computed offline by solving a linear program:

$$h_{\mathcal{S}}^{0} \doteq \max_{e} (F + G K_e) e \text{ subject to } V_{\mathcal{S}}^{0} e \leq \mathbf{1}.$$

Hence, robust satisfaction of the constraints (3.4) by the predicted trajectories of (3.57) is ensured by conditions (3.66) and (3.67), which are linear in the variables $s_{0|k}, \mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ and $\alpha_{i|k}, i = 0, 1, \ldots$

To restrict the sequence $\{\alpha_{0|k},\ i = 0, 1, \ldots\}$ to a finite number of degrees of freedom, we invoke (3.66) for $i \geq N$ by the sufficient condition

$$\alpha_{i+1|k} = \lambda \alpha_{i|k} + \mu,\ i = N, N+1, \ldots \tag{3.68}$$

where $\lambda \doteq \|\bar{e}\|_\infty$ and $\mu \doteq \|\bar{w}\|_\infty$. For simplicity we assume here that the condition (3.68) is to be imposed after a horizon equal to N, but in general this could be replaced by any finite horizon. Under the necessary assumption that $\lambda < 1$ the dynamics of (3.68) are stable and converge to the limit

$$\bar{\alpha} = \frac{1}{1 - \lambda} \mu$$

Thus, for $i \geq N$, $\alpha_{i|k}$ is given in terms of $\alpha_{N|k}$ by

$$\alpha_{i|k} = \lambda^{i-N} (\alpha_{N|k} - \bar{\alpha}) + \bar{\alpha},$$

and (3.66), (3.67) therefore constitute an infinite set of linear constraints in a finite number of variables: $s_{0|k}$, $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ and $\{\alpha_{0|k}, \ldots, \alpha_{N-1|k}\}$. These constraints are equivalent to a finite number of linear conditions by the following result, the proof of which is similar to the proof of Theorem 2.3.

Corollary 3.2 *Let ν be the smallest integer greater than or equal to N such that*

$$\bar{F} \Psi^{\nu+1} z + \left(\lambda^{\nu+1-N} (\alpha_N - \bar{\alpha}) + \bar{\alpha}\right) h_{\mathcal{S}}^{0} \leq \mathbf{1}$$

for all z and $\{\alpha_0, \ldots, \alpha_N\}$ *satisfying*

$$\bar{F}\Psi^i z \le \begin{cases} \mathbf{1} - \alpha_i h^0_{\mathcal{S}}, & i = 0, \ldots, N-1 \\ \mathbf{1} - \big(\lambda^{i-N}(\alpha_N - \bar{\alpha}) + \bar{\alpha}\big) h^0_{\mathcal{S}}, & i = N, \ldots, \nu \end{cases}$$

$$\alpha_i \bar{e} + \bar{w} \le \alpha_{i+1}\mathbf{1}, \quad i = 0, \ldots, N-1$$

then (3.67) holds for all $i = 0, 1, \ldots$ *Furthermore* ν *is necessarily finite if* Ψ *is strictly stable and* $(\Psi, \bar{F})$ *is observable.*

The definition of an online predicted cost is needed before an algorithm can be stated. This is taken to be the same as the cost (3.29) employed by rigid tube MPC, but with the addition of terms that penalize the deviation of $\alpha_{i|k}$ from the asymptotic value $\bar{\alpha}$,

$$J(s_{0|k}, \mathbf{c}_k, \boldsymbol{\alpha}_k) = \|s_{0|k}\|^2_{W_x} + \|\mathbf{c}_k\|^2_{W_c} + \sum_{i=0}^{N-1} q_\alpha(\alpha_{i|k} - \bar{\alpha})^2 + p_\alpha(\alpha_{N|k} - \bar{\alpha})^2, \tag{3.69}$$

where $\boldsymbol{\alpha}_k = (\alpha_{0|k}, \ldots, \alpha_{N|k})$. In order to ensure the monotonic non-increasing property of the optimized cost, we assume that the weights $p_\alpha, q_\alpha > 0$ satisfy the condition

$$p_\alpha \ge (1 - \lambda^2)^{-1} q_\alpha. \tag{3.70}$$

This results in an online optimization consisting of a quadratic program in $Nn_u + n_x + N + 1$ variables and $n_C(\nu + 1) + N + n_{\mathcal{S}^0}$ constraints, where $n_{\mathcal{S}^0}$ is the number of rows of $V^0_{\mathcal{S}}$.

Algorithm 3.4 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization

$$\begin{aligned} \underset{s_{0|k}, \mathbf{c}_k, \boldsymbol{\alpha}_k}{\text{minimize}} \quad & \|s_{0|k}\|^2_{W_x} + \|\mathbf{c}_k\|^2_{W_c} + \sum_{i=0}^{N-1} q_\alpha(\alpha_{i|k} - \bar{\alpha})^2 + p_\alpha(\alpha_{N|k} - \bar{\alpha})^2 \\ \text{subject to} \quad & \bar{F}\Psi^i \begin{bmatrix} s_{0|k} \\ \mathbf{c}_k \end{bmatrix} \le \mathbf{1} - \alpha_{i|k} h^0_{\mathcal{S}}, \qquad i = 0, \ldots, \nu \\ & \alpha_{i|k}\bar{e} + \bar{w} \le \alpha_{i+1|k}\mathbf{1}, \qquad i = 0, \ldots, N-1 \\ & \alpha_{i|k} = \lambda^{i-N}(\alpha_{N|k} - \bar{\alpha}) + \bar{\alpha} \quad i = N, \ldots, \nu \\ & x_k - s_{0|k} \in \alpha_{0|k}\mathcal{S}^0 \end{aligned} \tag{3.71}$$

where ν satisfies the conditions of Corollary 3.2.

(ii) Apply the control law $u_k = Ks^*_{0|k} + K_e(x_k - s^*_{0|k}) + c^*_{0|k}$, where $(s^*_{0|k}, \mathbf{c}^*_k, \boldsymbol{\alpha}^*_k)$ is the optimizer of (3.71), and $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$. ◁

Theorem 3.6 *For the system (3.1) and control law of Algorithm 3.4, the set $\mathcal{F}_N$ of feasible states x_k for (3.71) is RPI, and $\bar{\alpha}\mathcal{S}^0$ is exponentially stable with region of attraction equal to $\mathcal{F}_N$ if $Q, R \succ 0$ and $q_\alpha > 0$ in (3.69).*

Proof The recursive feasibility of the optimization (3.71) is demonstrated by the argument that was used to show recursive feasibility in the proof of Theorem 3.5. The exponential stability of $\bar{\alpha}\mathcal{S}^0$ can be shown using the feasible but suboptimal values for the optimization variables in (3.71) at time $k+1$ that are given by

$$s_{0|k+1} = \Phi s^*_{0|k} + BE\mathbf{c}^*_{0|k}, \quad \mathbf{c}_{k+1} = M\mathbf{c}^*_k,$$
$$\boldsymbol{\alpha}_{k+1} = \big(\alpha^*_{1|k}, \ldots, \alpha^*_{N|k}, \lambda(\alpha^*_{N|k} - \bar{\alpha}) + \bar{\alpha}\big).$$

These allow the optimal value of the cost in (3.71) at time $k+1$, denoted $J^*_{k+1} \doteq J(s^*_{0|k+1}, \mathbf{c}^*_{k+1}, \boldsymbol{\alpha}^*_{k+1})$, to be bounded as follows,

$$\begin{aligned}
J^*_{k+1} &\le \|\Phi s^*_{0|k} + BE\mathbf{c}^*_{0|k}\|^2_{W_x} + \|M\mathbf{c}^*_k\|^2_{W_c} + \sum_{i=1}^{N} q_\alpha(\alpha^*_{i|k} - \bar{\alpha})^2 + p_\alpha\lambda^2(\alpha^*_{N|k} - \bar{\alpha})^2 \\
&\le \|\Phi s^*_{0|k} + BE\mathbf{c}^*_{0|k}\|^2_{W_x} + \|M\mathbf{c}^*_k\|^2_{W_c} + \sum_{i=1}^{N-1} q_\alpha(\alpha^*_{i|k} - \bar{\alpha})^2 + p_\alpha(\alpha^*_{N|k} - \bar{\alpha})^2 \\
&\le J^*_k - (\|s^*_{0|k}\|^2_Q + \|Ks^*_{0|k} + \mathbf{c}^*_k\|^2_R) - q_\alpha(\alpha^*_{0|k} - \bar{\alpha})^2,
\end{aligned}$$

where (3.70) has been used. By the argument of the proof of Theorem 2.8 therefore, for any initial condition x_0 in the feasible set $\mathcal{F}_N$ for (3.71), we obtain, for all $k > 0$,

$$\|s^*_{0|k}\|^2 + |\alpha^*_{0|k} - \bar{\alpha}|^2 \le \frac{b}{a}\left|1 - \frac{\min\{\underline{\lambda}(Q), q_\alpha\}}{b}\right|^k \big(\|s^*_{0|0}\|^2 + |\alpha^*_{0|0} - \bar{\alpha}|^2\big), \quad (3.72)$$

where $a, b > 0$ are constants such that, for all $x_k \in \mathcal{F}_N$,

$$a\big(\|s_{0|k}\|^2 + |\alpha_{0|k} - \bar{\alpha}|^2\big) \le J(s^*_{0|k}, \mathbf{c}^*_k) \le b\big(\|s_{0|k}\|^2 + |\alpha_{0|k} - \bar{\alpha}|^2\big).$$

Since $x_k - e^*_{0|k} = s^*_{0|k}$ and $e^*_{0|k} \in \alpha^*_{0|k}\mathcal{S}^0$, the minimum Euclidean distance from x_k to any point in $\bar{\alpha}\mathcal{S}^0$ is bounded by

$$\begin{aligned}
\min_{e\in\bar{\alpha}\mathcal{S}^0} \|x_k - e\| \le \left\|x_k - \frac{\bar{\alpha}}{\alpha^*_{0|k}} e^*_{0|k}\right\| &\le \|s^*_{0|k}\| + \frac{|\alpha^*_{0|k} - \bar{\alpha}|}{\alpha^*_{0|k}} \max_{e\in\alpha^*_{0|k}\mathcal{S}^0} \|e\| \\
&= \|s^*_{0|k}\| + \beta|\alpha^*_{0|k} - \bar{\alpha}| \qquad (3.73)
\end{aligned}$$

where $\beta = \max_{e\in\bar{\alpha}\mathcal{S}^0} \|e\|/\bar{\alpha}$ is a constant, and it follows from (3.72) and (3.73) that the distance of x_k from $\bar{\alpha}\mathcal{S}^0$ is upper-bounded by an exponential decay. □

The cost and constraints of the HTMPC optimization (3.71) can be simplified (as discussed in Question 9 on p. 117) if the set $\mathcal{S}^0$ is robustly invariant for the error dynamics (3.57c). We note also that it is possible to relax the constraints of this approach using the equi-normalization technique described in [21]. This is achieved through exact scaling of the set $\mathcal{S}^0$, allowing for an expansion of the region attraction of Algorithm 3.4. Further improvements in the size of the feasible initial condition set can be achieved by formulating the degrees of freedom $\alpha_{i|k}$ as vectors rather than the scalars that, in Algorithm 3.4, scale the set $\mathcal{S}^0$ equally in all directions. This is possible through an appropriate use of Farkas' Lemma, and is discussed in detail in Chap. 5.

3.6 Early Robust MPC for Additive Uncertainty

To conclude this chapter we describe two of the main precursors of the robust MPC techniques described in Sects. 3.2, 3.3 and 3.5. The first of these is concerned with a robust extension of SGPC for systems with additive disturbances [22]. This approach imposes tightened constraints on nominal predicted trajectories to ensure robustness, and it also provides conditions for recursive feasibility analogous to those of Sect. 3.2.1, but in the context of input-output discrete time models and equality terminal constraints. We then discuss the tube MPC algorithms of [4, 23]. These use low-complexity polytopes to bound predicted trajectories, treating the parameters defining these sets as variables in the online MPC optimization. Similarly to the homothetic tubes considered in Sect. 3.5.2, the condition that these tubes should contain the predicted trajectories of the uncertain plant model is invoked through a recursive sequence of constraints.

3.6.1 Constraint Tightening

This section describes a formulation of robust MPC for the case of additive disturbances which is based on the SGPC algorithm described in Sect. 2.10. As in Sect. 3.3, a nominal cost is assumed. This is computed under the assumption that the nominal disturbance input is zero, and is therefore equal to the predicted cost when there are no future disturbances. In the constant setpoint problem considered in Sect. 2.10, SGPC with integral action steers the state asymptotically to a reference state (which for simplicity is taken to be zero in this section) whenever the future disturbance reaches a steady state. However, this presupposes recursive feasibility which is achieved in [22] through constraint tightening. The tightened constraints are derived in two stages, the first of which achieves an a posteriori feasibility that ensures the feasibility of predictions, and the second invokes feasibility a priori so that feasibility is retained recursively. The terms a posteriori and a priori are used in the sense that the former involves conditions based on past information whereas the latter anticipates the future in an attempt to ensure recursive feasibility.

To simplify presentation we consider here the case of a single-input single-output system. The convenience of the SGPC approach is that, for the disturbance-free case, it develops prediction dynamics which involve transfer functions described by finite impulse response (FIR) filters. Hence, a terminal (equality) stability constraint can be imposed on predicted trajectories implicitly, without the need to invoke any terminal constraints. This convenience can be preserved for the case when an additive disturbance is introduced into the system model (2.68):

$$
\begin{aligned}
y_k &= \frac{z^{-1}b(z^{-1})}{a(z^{-1})}u_k + \frac{1}{a(z^{-1})}\zeta_k \\
&= \frac{z^{-1}b(z^{-1})}{\alpha(z^{-1})}\Delta u_k + \frac{1}{\alpha(z^{-1})}\xi_k
\end{aligned}
\tag{3.74}
$$

where $\alpha(z^{-1}) = \Delta(z^{-1})a(z^{-1})$, $\Delta(z^{-1}) = 1 - z^{-1}$ and $\zeta_k = \frac{1}{\Delta(z^{-1})}\xi_k$ with ξ_k denoting a zero mean white noise process, and where the polynomial matrices $A(z^{-1})$, $B(z^{-1})$ in (2.68) are replaced by polynomials $a(z^{-1})$, $b(z^{-1})$ for the single-input single-output case considered here. As explained in Chap. 2, consideration is given to the control increments, Δu_k (rather than the values of the control input, u_k), as a means of introducing integral action into the feedback loop.

Similarly to the decomposition of predicted trajectories in Sect. 3.2, we decompose the z-transforms, $y(z^{-1})$, $u(z^{-1})$, of the predicted output and control input sequences according to

$$
y(z^{-1}) = y^{(1)}(z^{-1}) + y^{(2)}(z^{-1}), \quad u(z^{-1}) = u^{(1)}(z^{-1}) + u^{(2)}(z^{-1}).
$$

Here, $y^{(1)}, u^{(1)}$ denote nominal predicted output and input sequences that correspond to the disturbance-free case, while $y^{(2)}, u^{(2)}$ model the effects of the additive disturbance in (3.74). Following the development of Sect. 2.10, the control law $\Delta u^{(1)}(z^{-1}) = \big(c(z^{-1}) - z^{-1}N(z^{-1})y^{(1)}(z^{-1})\big)/M(z^{-1})$ for some polynomials $N(z^{-1})$ and $M(z^{-1})$, results in predictions

$$
y^{(1)}(z^{-1}) = z^{-1}b(z^{-1})c(z^{-1}) + y_f(z) \tag{3.75a}
$$

$$
\Delta u^{(1)}(z^{-1}) = \alpha(z^{-1})c(z^{-1}) + \Delta u_f(z) \tag{3.75b}
$$

provided that $N(z^{-1})$ and $M(z^{-1})$ satisfy the Bezout identity

$$
\alpha(z^{-1})M(z^{-1}) + z^{-1}b(z^{-1})N(z^{-1}) = 1. \tag{3.76}
$$

Note that $y_f(z)$ and $\Delta u_f(z)$ in (3.75) are polynomials in positive powers of z that relate to past values of outputs and control increments, thereby taking account of non-zero initial conditions.

In the absence of disturbances, a terminal equality constraint requiring the nominal predicted outputs and inputs to be identically zero after a finite initial prediction horizon is imposed implicitly by setting $\Delta u(z^{-1}) = \Delta u^{(1)}(z^{-1})$. The case of non-zero disturbances can also be handled through use of the Bezout identity (3.76). From (3.74), $\Delta u^{(2)}(z^{-1})$ and $y^{(2)}(z^{-1})$ are related by

$$\alpha(z^{-1})y^{(2)}(z^{-1}) - z^{-1}b(z^{-1})\Delta u^{(2)}(z^{-1}) = \xi(z^{-1}),$$

and (3.76) therefore implies that the transfer functions from $\xi(z^{-1})$ to $y^{(2)}(z^{-1})$ and $\Delta u^{(2)}(z^{-1})$ have the form of FIR filters if the predicted control increments are defined by $\Delta u(z^{-1}) = \big(c(z^{-1}) - z^{-1}N(z^{-1})y(z^{-1})\big)/M(z^{-1})$ since from (3.75b) we then obtain

$$y^{(2)}(z^{-1}) = M(z^{-1})\xi(z^{-1}) \tag{3.77a}$$

$$\Delta u^{(2)}(z^{-1}) = -z^{-1}N(z^{-1})\xi(z^{-1}). \tag{3.77b}$$

The fixed order polynomials $M(z^{-1}), z^{-1}N(z^{-1})$ appearing in these expressions enable the worst-case values of the predicted values for $y^{(2)}$, $\Delta u^{(2)}$ to be computed conveniently and thus allow constraints to be applied robustly, for all allowable disturbance sequences $\xi(z^{-1})$.

To illustrate this point, consider the case where the system is subject to rate constraints only:

$$|\Delta u_{i|k}| \le R, \quad i = 0, 1, \ldots$$

Then the implied constraints on the nominal control sequence must be tightened to give

$$|\Delta u^{(1)}_{i|k}| \le R - R^{\#}_i, \quad i = 0, 1, \ldots \tag{3.78}$$

where $R^{\#}_i$ denotes the prediction i steps ahead of the worst-case absolute value of $\Delta u^{(2)}$ in (3.77b). To determine this, some assumption as to the size of uncertainty has to be made, and (analogously to the disturbance set (3.3)) a bound can be imposed through $\zeta_{i|k}$, which would in practice be limited as

$$|\zeta_{i|k}| \le d, \quad i = 0, 1, \ldots.$$

Given this limit it is straightforward to compute $R^{\#}_i$, by writing (3.77b) in terms of $\zeta(z^{-1})$ as

$$\Delta u^{(2)}(z^{-1}) = -z^{-1}N(z^{-1})[\Delta(z^{-1})\zeta(z^{-1}) - \zeta_k],$$

and then extracting the worst-case value of the coefficient of z^{-i} in $\Delta u^{(2)}(z^{-1})$ over the allowable range of values of coefficients of $\zeta(z^{-1})$. Given that $M(z^{-1})$ and $\Delta u^{(1)}_{i|k}(z^{-1})$ are both finite-degree polynomials in z^{-1}, it is clear that only a finite number of terms: $R^{\#}_i$, $i = 0, 1, \ldots, \nu$, for finite ν, need to be evaluated in order to invoke the constraints (3.78) over an infinite prediction horizon.

Condition (3.78), imposed on the degrees of freedom in (3.75b), namely the coefficients of $c(z^{-1})$, defines the a posteriori conditions that ensure the feasibility of the control input increments, given the current measurements. However, this is not enough to guarantee recursive feasibility, as can be seen by considering the tail at time $k+1$ of a trajectory that was feasible at time k. This tail is generated by replacing the $c(z^{-1})$ polynomial in (3.75) with $z\big(c(z^{-1}) - c_{0|k}\big)$. In the absence of disturbances, such a tail would necessarily be feasible, but through the initial conditions term, Δu_f, of (3.75b), the effect on the sequence of predictions for $\Delta u^{(1)}$ at time $k+1$ due to non-zero ζ will be the addition of a term which can be shown to be

$$f_{\Delta u}(\zeta_k, \zeta_{k+1}) = -N(z^{-1})(\zeta_{k+1} - \zeta_k),$$

This term must be accommodated by the tightened constraints when they are applied at the next time instant, $k+1$. Thus application of (3.78) at time $k+1$ must ensure that for each prediction step i, $R^{\#}_i - R^{\#}_{i+1}$ is at least as large as the modulus of the corresponding element of $f_{\Delta u}(\zeta_k, \zeta_{k+1})$. Detailed calculation shows that this is indeed the case if the tightening parameters are

$$R^{\#}_1 = 0,\ \ R^{\#}_{i+1} = R^{\#}_i + 2d|N_{i-1}|, \quad i = 1, \ldots, \mu - 1 \tag{3.79}$$

where μ denotes the sum of the degrees of $\alpha(z^{-1})$ and $c(z^{-1})$ and N_i is the coefficient of z^{-i} in $N(z^{-1})$. The constraint (3.78) with (3.79) defines the a priori feasibility conditions which also satisfy the a posteriori conditions and are in fact the least conservative constraint tightening bounds that provide the guarantees of recursive feasibility.

The development presented in this section has obvious extensions to absolute input (rather than rate) constraints as well as mixed input/output constraints. Indeed, as shown in Sects. 3.3 and 3.5, a similar constraint tightening approach can also be extended to state space models and to algorithms other than SGPC. What makes the approach described here convenient is the fact that, by using the Bezout identity (3.76), the uncertainty tube converges to the mRPI set in a finite number of steps. This ensures that the sequence of constraint tightening parameters $R^{\#}_1, R^{\#}_2, \ldots$ converges to a limit in a finite number of steps, without the need for the more general, but computationally more demanding, theoretical framework of Sects. 3.2.1 and 3.2.2.

3.6.2 Early Tube MPC

Uncertainty tubes that are parameterized explicitly in terms of optimization variables were proposed in the robust MPC algorithms of [4, 24]. Although similar in this respect to the homothetic tubes of [20], the tubes of [4, 24] allow more variation between the shapes of the sets defining the tube cross section than simple translation and scaling of a given set. To avoid the need for large numbers of optimization variables, attention is restricted to tube cross sections defined as low-complexity polytopes. The approach is explained in this section within the context of additive model uncertainty; the application of low-complexity polytopes to the case of multiplicative model uncertainty (which was proposed in [4, 23]) is discussed in Chap. 5.

A low-complexity polytope is a linearly transformed hypercube. Let $\mathcal{S}_{i|k}$ denote the low-complexity polytope defined by

$$\mathcal{S}_{i|k} \doteq \{x : \underline{\alpha}_{i|k} \leq V_{\mathcal{S}} x \leq \overline{\alpha}_{i|k}\} \tag{3.80}$$

where $V_{\mathcal{S}} \in \mathbb{R}^{n_x \times n_x}$ is non-singular and $\underline{\alpha}_{i|k}, \overline{\alpha}_{i|k} \in \mathbb{R}^{n_x}$. Then the condition that the state $x \in \mathbb{R}^{n_x}$ of (3.1) belongs to $\mathcal{S}_{i|k}$ can be expressed as $2n_x$ linear inequalities. It is convenient to define a transformed state vector as

$$\xi = V_{\mathcal{S}} x.$$

In terms of this transformed state, the open-loop strategy of (3.6) and the corresponding prediction dynamics (3.7) can be re-written as

$$u_{i|k} = \tilde{K} \xi_{i|k} + c_{i|k}, \tag{3.81a}$$

$$\xi_{i|k} = \tilde{\Phi} \xi_{i|k} + \tilde{B} c_{i|k} + \tilde{D} w_{i|k}, \tag{3.81b}$$

where $\tilde{\Phi} = V_{\mathcal{S}} \Phi V_{\mathcal{S}}^{-1}$, $\tilde{B} = V_{\mathcal{S}} B$, $\tilde{D} = V_{\mathcal{S}} D$ and $\tilde{K} = K V_{\mathcal{S}}^{-1}$. As before, we assume that $c_{i|k} = 0$ for all $i \geq N$, where N is the mode 1 prediction horizon, and K is assumed to be the unconstrained LQ-optimal feedback gain.

Low-complexity polytopic tubes provide a compact and efficient means of bounding the predicted state as a function of the degrees of freedom $(c_{0|k}, \ldots, c_{N-1|k})$ in (3.81b), the initial state $\xi_{0|k} = V_{\mathcal{S}} x_k$, and the disturbance set $\mathcal{W}$. This can be achieved by propagating the tube cross sections recursively using the following result.

Lemma 3.6 *For $A \in \mathbb{R}^{n_A \times n_y}$, let $A^+ \doteq \max\{A, 0\}$ and $A^- \doteq \max\{-A, 0\}$, then $\underline{y} \leq y \leq \bar{y}$ implies*

$$A^+ \underline{y} - A^- \bar{y} \leq Ay \leq A^+ \bar{y} - A^- \underline{y}, \tag{3.82}$$

and, for each of the $2n_A$ elementwise bounds in (3.82), there exists y satisfying $\underline{y} \leq y \leq \bar{y}$ such that the bound holds with equality.

Consider the conditions on $\underline{\alpha}_{i|k}, \overline{\alpha}_{i|k}, c_{i|k}$ and $\mathcal{W}$ in order that $x_{i|k} \in \mathcal{S}_{i|k}$ implies that $x_{i+1|k} \in \mathcal{S}_{i+1|k}$. If we define

$$\underline{w}_D \doteq \min_{w \in \mathcal{W}} \tilde{D} w, \quad \bar{w}_D \doteq \max_{w \in \mathcal{W}} \tilde{D} w,$$

then, given that $x_{i|k} \in \mathcal{S}_{i|k}$, the elements of $\xi_{i+1|k}$ are bounded according to $\underline{\alpha}_{i+1|k} \le \xi_{i+1|k} \le \overline{\alpha}_{i+1|k}$ where

$$\underline{\alpha}_{i+1|k} \le \tilde{\Phi}^{+}\underline{\alpha}_{i|k} - \tilde{\Phi}^{-}\overline{\alpha}_{i|k} + \tilde{B}c_{i|k} + \underline{w}_D \tag{3.83a}$$

$$\tilde{\Phi}^{+}\overline{\alpha}_{i|k} - \tilde{\Phi}^{-}\underline{\alpha}_{i|k} + \tilde{B}c_{i|k} + \bar{w}_D \le \overline{\alpha}_{i|k}. \tag{3.83b}$$

Therefore the tube $\{\mathcal{S}_{0|k}, \ldots, \mathcal{S}_{N|k}\}$ contains the predicted state trajectories of (3.1) if and only if these conditions hold for $i = 0, \ldots, N-1$, starting from $\underline{\alpha}_{0|k} = \overline{\alpha}_{0|k} = V_{\mathcal{S}} x_k$.

Under the control law of (3.81a), the constraints of (3.4), expressed in terms of $\tilde{F} = F V_{\mathcal{S}}^{-1}$ and $\tilde{K}$, can be written as

$$(\tilde{F} + G\tilde{K})\xi_{i|k} + Gc_{i|k} \le 1. \tag{3.84}$$

These constraints are satisfied for all $\xi_{i|k} \in \mathcal{S}_{i|k}$ if and only if

$$(\tilde{F} + G\tilde{K})^{+}\overline{\alpha}_{i|k} - (\tilde{F} + G\tilde{K})^{-}\underline{\alpha}_{i|k} + Gc_{i|k} \le 1 \tag{3.85}$$

This takes care of constraints for $i = 0, \ldots, N-1$, whereas the constraints for $i \ge N$ are accounted for by imposing the condition that the terminal tube cross section should lie in a terminal set, $\mathcal{S}_T$, which is RPI. To allow for a guarantee of recursive feasibility, [4, 24] proposed a low-complexity polytopic terminal set:

$$\{x : |V_{\mathcal{S}} x| \le \alpha_T\} \tag{3.86}$$

where the absolute value and inequality sign in (3.86) apply on an element by element basis. This terminal set is invariant under the dynamics of (3.81b) and the constraints of (3.84) for $c_{i|k} = 0$ if and only if

$$|\tilde{\Phi}|\alpha_T + \max\{\bar{w}_D, -\underline{w}_D\} \le \alpha_T, \tag{3.87}$$

and

$$|\tilde{F} + G\tilde{K}|\alpha_T \le 1 \tag{3.88}$$

The conditions (3.87) and (3.88) follow from the fact that, for $|\xi| \le \alpha_T$, the achievable maximum (elementwise) values of $\tilde{\Phi}\xi$ and $(\tilde{F} + G\tilde{K})\xi$ are $|\tilde{\Phi}|\alpha_T$ and $|\tilde{F} + G\tilde{K}|\alpha_T$ respectively.

Combining the robust constraint handling of this section with the nominal cost of Sect. 3.3 gives the following online optimization problem which is a quadratic program in $N(n_u + 2n_x)$ variables and $2(N+1)n_x + Nn_C$ constraints.

$$\begin{aligned} \underset{\mathbf{c}_k,\underline{\alpha}_{i|k},\overline{\alpha}_{i|k},i=0,\dots,N}{\text{minimize}} \quad & \|\mathbf{c}_k\|^2_{W_c} \\ \text{subject to} \quad & (3.83), (3.85) \text{ for } i = 0,\dots,N-1, \\ & \underline{\alpha}_{N|k} \geq -\alpha_T, \;\; \overline{\alpha}_{N|k} \leq \alpha_T \\ & \underline{\alpha}_{0|k} = \overline{\alpha}_{0|k} = V_{\mathcal{S}} x_k \end{aligned} \tag{3.89}$$

The optimal value of $\mathbf{c}_k$ for this problem is clearly also optimal for the problem of minimizing, subject to the constraints of (3.89), the nominal cost $J(x_k, \mathbf{c}_k) = \|x_k\|^2_{W_x} + \|\mathbf{c}_k\|^2_{W_c}$ defined in (3.29). Hence the stability analysis of Sect. 3.3 applies to the MPC law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k},\dots,c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ for problem (3.89). In particular, recursive feasibility is implied by the feasibility of the following values for the variables in (3.89) at time $k+1$:

$$\begin{gathered} \mathbf{c}_{k+1} = M\mathbf{c}^*_k \\ \underline{\alpha}_{i|k+1} = \underline{\alpha}^*_{i+1|k}, \quad \overline{\alpha}_{i|k+1} = \overline{\alpha}^*_{i+1|k}, \quad i = 0,\dots,N-1 \\ \underline{\alpha}_{N|k+1} = \xi_T, \quad \overline{\alpha}_{N|k+1} = -\xi_T. \end{gathered}$$

where $\mathbf{c}^*_k$ and $\underline{\alpha}^*_{i|k}, \overline{\alpha}^*_{i|k}, i = 0,\dots,N$ are optimal for (3.89) at time k. The constraints of (3.89) must therefore be feasible at times $k = 1, 2, \dots$ if the initial condition x_0 lies in the feasible set for (3.89) at $k = 0$. In addition, the quadratic bound (3.38) holds for the closed-loop system, and asymptotic convergence of x_k to the minimal RPI set $\mathcal{X}^{\text{mRPI}}$ defined in (3.23) follow from Lemmas 3.1, 3.2 and the bound (3.37).

We close this section by noting that the offline computation that is required so that the MPC optimization (3.89) can be performed online concerns the selection of $V_{\mathcal{S}}$ and α_T defining the terminal set in (3.86). Since this terminal set must be invariant, a convenient choice for $V_{\mathcal{S}}$ is provided by the eigenvector matrix of Φ with the columns of $V_{\mathcal{S}}$ that correspond to complex conjugate eigenvalues of Φ replaced by the real and imaginary parts of the corresponding eigenvectors. With this choice of $V_{\mathcal{S}}$, a necessary and sufficient condition for existence of α_T satisfying (3.87) is that the eigenvalues of Φ should lie inside the box in the complex plane with corners at ± 1 and $\pm j$ (Fig. 3.9). Having defined $V_{\mathcal{S}}$, the elements of α can be determined so as to maximize the volume of the terminal set subject to (3.88) through the solution of a convex optimization problem.

Despite their computational convenience, the relatively inflexible geometry of low-complexity polytopes makes them rather restrictive. For this reason Chap. 5 replaces low-complexity polytopes with general polytopic sets through an appropriate use of Farkas' Lemma (see [25] or [26]).

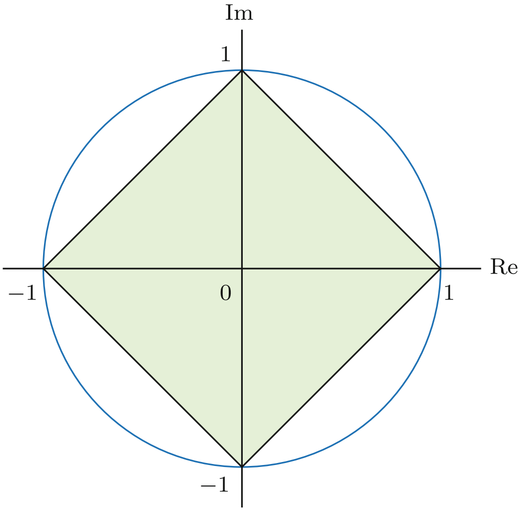

Fig. 3.9 The box in the complex plane with vertices at ±1 and $\pm j$ (*shaded region*)

3.7 Exercises

1 A production planning problem requires the quantity u_k of product made in week k, for $k = 0, 1, \ldots$ to be optimized. The quantity w_k of product that is sold in week k is unknown in advance but lies in the interval $0 \le w_k \le W$ and has a nominal value of $\hat{w}$. The quantity x_{k+1} remaining unsold at the start of week $k+1$ is governed by

$$x_{k+1} = x_k + u_k - w_k, \quad k = 0, 1, \ldots$$

Limits on storage and manufacturing capacities imply that x and u can only take values in the intervals

$$0 \le x_k \le X, \quad 0 \le u_k \le U.$$

The desired level of x in storage is x^0, and the planned values $u_{0|k}, u_{1|k}, \ldots$ are to be optimized at the beginning of week k given a measurement of x_k.

(a) What are the advantages of using a receding horizon control strategy that is recomputed at $k = 0, 1, \ldots$ in this application instead of an open-loop control sequence computed at $k = 0$?
Let the planned production at time k be $u_{i|k} = \hat{w} - (x_{i|k} - x^0) + c_{i|k}$ for $i = 0, 1, \ldots$, where $(c_{0|k}, \ldots, c_{N-1|k}) = \mathbf{c}_k$ is a vector of optimization variables at time k, $c_{i|k} = 0$ for $i \ge N$, and $x_{i|k}$ is a prediction of x_{k+i}.

(b) Let $s_{i|k}$ be the nominal value of $x_{i|k} - x^0$ (namely the value that would be obtained if $w_{i|k} = \hat{w}$ for all $i \ge 0$, with $s_{0|k} = x_k - x^0$). Show that the nominal cost $J(x_k, \mathbf{c}_k) \doteq \sum_{i=0}^{\infty} s_{i|k}^2$ is given by

$$J(x_k, \mathbf{c}_k) = (x_k - x^0)^2 + \|\mathbf{c}_k\|^2.$$

(c) If $s_{i|k} + e_{i|k} = x_{i|k} - x^0$, verify that $e_{i|k} \in [\hat{w} - W, \hat{w}]$ for all $i \geq 1$. Hence show that $u_{i|k} \in [0, U]$ and $x_{i|k} \in [0, X]$ for all $w_{i|k} \in [0, W]$ and all $i \geq 0$ if and only if the following conditions hold,

$$\begin{aligned} c_{i|k} + \hat{w} + x^0 &\in [W, X] & i &= 0, \ldots, N-1 \\ c_{i|k} - c_{i-1|k} &\in [0, U - W] & i &= 1, \ldots, N-1 \\ c_{0|k} + \hat{w} + x^0 - x_k &\in [0, U - W] \\ c_{N-1|k} &\in [-(U - W), 0] \end{aligned}$$

and state the conditions on X, U, W that are required in order that the feasible set for x_0 is non-empty.

(d) Let $\mathbf{c}_k^*$ be the minimizer of the cost in (b) subject to the constraints in (c) at time k, and define the MPC law as $u_k = \hat{w} - (x_k - x^0) + c_{0|k}^*$, where $c_{0|k}^*$ is the first element of $\mathbf{c}_k^*$. Show that the constraints of (c) are recursively feasible and that $c_{0|k}^*$ converges to zero as $k \to \infty$.

(e) Explain why $x_0 \leq \hat{w} + x^0$ is needed for feasibility of the constraints of (c) at $k = 0$. How might this condition be relaxed?

2 If Φ is a nilpotent matrix, with $\Phi^n = 0$ for some integer $n > 0$, and Ψ is defined by

$$\Psi = \begin{bmatrix} \Phi & \Gamma \\ 0 & M \end{bmatrix}, \quad M = \begin{bmatrix} 0 & I_{n_u} & 0 & \cdots & 0 \\ 0 & 0 & I_{n_u} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I_{n_u} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{Nn_u \times Nn_u}$$

for a given matrix Γ, prove that Ψ is nilpotent with $\Psi^m = 0$ for $m = n + N$.

Use this property to write down expressions for:

(a) the minimal RPI set for the dynamics $e_{k+1} = \Phi e_k + D w_k$,

(b) the set of inequalities defining the maximal RPI set for the system $z_{k+1} = \Psi z_k + \bar{D} w_k$ and constraints $\bar{F} z_k \leq \mathbf{1}$,

(c) the conditions under which the set of feasible initial states for the system $z_{k+1} = \Psi z_k + \bar{D} w_k$ and constraints $\bar{F} z_k \leq \mathbf{1}$ is non-empty,

where w_k lies in a compact polytopic set $\mathcal{W}$ for all k and $\bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}$.

3 A system has dynamics $x_{k+1} = Ax_k + Bu_k + w_k$ and state constraints $Fx \le \mathbf{1}$, with

$$A = \begin{bmatrix} -1 & 0.2 \\ -0.25 & 0.65 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \quad F = \begin{bmatrix} I \\ -I \end{bmatrix}$$

where the disturbance input w_k is unknown at time k and satisfies, for all $k \ge 0$

$$w_k \in \sigma\mathcal{W}_0, \quad \mathcal{W}_0 \doteq \left\{ w : \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

for some constant scalar parameter $\sigma > 0$.

Verify that $\Phi = A + BK$ is nilpotent if $K = \begin{bmatrix} 0.5 & 0.3 \end{bmatrix}$. Hence determine the maximal RPI set, $\mathcal{Z}^{\text{MRPI}}(\sigma)$, for the system $z_{k+1} = \Psi z_k + \bar{D} w_k$, $w_k \in \sigma\mathcal{W}_0$, where Ψ is defined as in Question 2, with $N = 2$ and $\Gamma = B \begin{bmatrix} 1 & 0 \end{bmatrix}$. Show that $\mathcal{Z}^{\text{MRPI}}(\sigma)$ is non-empty if and only if $\sigma \le \frac{2}{3}$.

4 For the system considered in Question 3 an MPC law is defined at each time instant $k = 0, 1, \ldots$ as $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, c^*_{1|k})$ is the solution of the QP:

$$\mathbf{c}^*_k = \arg\min_{\mathbf{c}_k} \ \|\mathbf{c}_k\|^2 \ \text{ subject to } \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{Z}^{\text{MRPI}}(\sigma)$$

and where K and $\mathcal{Z}^{\text{MRPI}}(\sigma)$ are as defined in Question 3. The controller is designed to operate with any σ in the interval $[0, \frac{2}{3})$, and σ can be assumed to be known and constant when the controller is in operation.

(a) Show that the closed-loop system is stable if the MPC optimization is feasible at time $k = 0$. What limit set will the closed-loop state converge to?

(b) Comment on the suggestion that better performance would be obtained with respect to the cost $\sum_{k=0}^{\infty}(\|x_k\|^2_Q + u_k^2)$, for a given matrix $Q \succeq 0$, if the MPC optimization at time k was defined

$$\mathbf{c}^*_k = \arg\min_{\mathbf{c}_k} \left\| \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \right\|^2_W \ \text{ subject to } \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{Z}^{\text{MRPI}}(0.5),$$

where W is the solution of the Lyapunov equation

$$W - \Psi^T W \Psi = \hat{Q}, \quad \hat{Q} = \begin{bmatrix} Q + K^T K & K^T & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

5 (a) A matrix Φ and a compact convex polytopic set $\mathcal{W}$ satisfy the inclusion condition

$$\Phi^r \mathcal{W} \subseteq \rho \mathcal{W}$$

for some integer r and $\rho \in [0, 1)$. Show that $h_\infty \leq \hat{h}_\infty$, where

$$h_\infty \doteq \sum_{j=0}^{\infty} \max_{w_j \in \mathcal{W}} F\Phi^j w_j, \quad \hat{h}_\infty \doteq \frac{1}{1-\rho} \sum_{j=0}^{r-1} \max_{w_j \in \mathcal{W}} F\Phi^j w_j$$

and prove that the fractional error, $(\hat{h}_\infty - h_\infty)/h_\infty$, in this bound is no greater than $\rho/(1-\rho)$.

(b) For (A, B) and $\mathcal{W}_0$ as defined in Question 3 and $K = \begin{bmatrix}0.479 & 0.108\end{bmatrix}$, use the bounds in (a) to determine h_∞ to an accuracy of 1 % when $\Phi = A + BK$ and $\mathcal{W} \doteq 0.1\mathcal{W}_0$.

(c) Suggest an over-bounding approximation of the minimal RPI set for the system $e_{k+1} = (A + BK)e_k + w_k$, $w_k \in \mathcal{W}$ which is based on the inclusion condition in (a). What can be said about the accuracy of this approximation?

6 A robust MPC law is to be designed for the system $x_{k+1} = Ax_k + Bu_k + w_k$ with the state constraints $Fx \leq \mathbf{1}$ and disturbance bounds $w_k \in 0.5\mathcal{W}$, where (A, B), F and $\mathcal{W}_0$ are as defined in Question 3. The predicted control sequence is parameterized as $u_{i|k} = Kx_{i|k} + c_{i|k}$ with $K = \begin{bmatrix}0.479 & 0.108\end{bmatrix}$, where $c_{i|k}$ for $i < N$ are optimization variables and $c_{i|k} = 0$ for all $i \geq N$.

(a) For $N = 1$, construct matrices Ψ, $\bar{D}$ and $\bar{F}$ such that $Fx_{i|k} = \bar{F}z_{i|k}$ for all $i \geq 0$, where $z_{i+1|k} = \Psi z_{i|k} + \bar{D}w_{k+i}$ and hence determine the constraint set for $z_k = (x_k, \mathbf{c}_k)$ that gives the largest possible feasible set for x_k for this prediction system.

(b) Determine the matrix W_z defining the nominal cost

$$\sum_{i=0}^{\infty} (\|s_{i|k}\|^2 + v_{i|k}^2) = z_k^T W_z z_k$$

for the nominal predictions $s_{i+1|k} = (A + BK)s_{i|k} + Bc_{i|k}$ and $v_{i|k} = Ks_{i|k} + c_{i|k}$, with $s_{0|k} = x_k$. Verify that for $x_0 = (0, 1)$ the minimal value of this cost subject to the constraints computed in (a) is 1.707.

(c) Starting from the initial condition $x_0 = (0, 1)$, simulate the closed-loop system under the MPC law $u_k = Kx_k + c^*_{0|k}$, assuming that the disturbance sequence (which is unknown to the controller) is given by

$$\{w_0, w_1, w_2, w_3, \ldots\} = \left\{ \begin{bmatrix}0.5\\0\end{bmatrix}, \begin{bmatrix}-0.5\\0\end{bmatrix}, \begin{bmatrix}0\\0.5\end{bmatrix}, \begin{bmatrix}0\\-0.5\end{bmatrix} \right\}$$

and verify that the closed-loop state and control sequences satisfy

$$\sum_{k=0}^{3}\left(\|x_k\|^2 + u_k^2\right) = 4.49.$$

7 The worst case predicted cost:

$$\check{J}(x_k, \mathbf{c}_k) = \max_{\{w_k, w_{k+1}, \dots\}} \sum_{i=0}^{\infty}\left(\|x_{i|k}\|^2 + u_{i|k}^2 - \gamma^2\|w_{k+i}\|^2\right)$$

is to be used to define a robust MPC law for the system with the model, disturbance bounds and state constraints of Question 6, with $N = 1$ and $\gamma^2 = 3.3$. For this value of γ^2 the unconstrained optimal feedback gain is $K = \begin{bmatrix}0.540 & 0.249\end{bmatrix}$ and the corresponding Riccati equation has the solution

$$\check{W}_x = \begin{bmatrix} 2.336 & -0.904 \\ -0.904 & 2.103 \end{bmatrix}.$$

(a) Determine W_x and W_c so that $\check{J}(x_k, \mathbf{c}_k) = \|x_k\|^2_{W_x} + \|\mathbf{c}_k\|^2_{W_c}$.

(b) Find the smallest integer ν such that $\bar{F}\Psi^{\nu+1}z \leq \mathbf{1} - h_{\nu+1}$ for all z satisfying $\bar{F}\Psi^i z \leq \mathbf{1} - h_i$, for $i = 0, \dots, \nu$. Hence, solve the MPC optimization (3.47) at $k = 0$ for $x_0 = (0, 1)$, and verify that the optimal solution is $\mathbf{c}_0^* = 0.051$ and that $\check{J}^*(x_0) = 2.294$.

(c) Consider the alternative worst-case cost defined by

$$\check{J}(x_k, \mathbf{c}_k) = \max_{\substack{w_{k+i}\in\mathcal{W} \\ i=0,\dots,N-1}} \sum_{i=0}^{N-1}\left(\|x_{i|k}\|^2 + u_{i|k}^2 - \gamma^2\|w_{k+i}\|^2\right) + \|x_{N|k}\|^2_{\check{W}_x}.$$

and determine the matrices $W_{\mu z}$, $W_{\mu\mu}$ in the online MPC optimization of (3.55). Hence verify that the optimum $\mathbf{c}_0^*$ is unchanged but $\check{J}^*(x_0) = 2.222$.

(d) Why is the predicted cost smaller in (c) than (b)? What are the advantages of (c) relative to (b), and what are the possible disadvantages?

8 In this problem the rigid tube MPC strategy (Sect. 3.5.1) is applied to the system with model $x_{k+1} = Ax_k + Bu_k + w_k$, state constraints $Fx_k \leq \mathbf{1}$ and disturbance bounds $w_k \in \mathcal{W}$, $\mathcal{W} \doteq 0.4\mathcal{W}_0$, with A, B, F and $\mathcal{W}_0$ as given in Question 3, and feedback gain $K = K_e = \begin{bmatrix}0.479 & 0.108\end{bmatrix}$, which is optimal for the nominal cost $\sum_{k=0}^{\infty}(\|x_k\|^2 + u_k^2)$ in the absence of constraints.

(a) For $r = 2$ and $\Phi = A + BK$ find the smallest scalar ρ such that $\Phi^r\mathcal{W} \subseteq \rho\mathcal{W}$ and compute $h_{\mathcal{S}} \doteq \max_{e\in\mathcal{S}} Fe$ where

$$\mathcal{S} = \frac{1}{1-\rho}(\mathcal{W} \oplus \Phi\mathcal{W}).$$

(b) For predictions $u_{i|k} = Kx_{i|k} + c_{i|k}$, with $c_{i|k} = 0$ for $i \geq N$ and $N = 1$, verify that the maximal invariant set for the dynamics $z_{k+1} = \Psi z_k$ and constraints $\bar{F}z_k \leq \mathbf{1} - h_{\mathcal{S}}$, $k = 0, 1, \ldots$ is

$$\mathcal{Z}^{\text{MPI}} = \{z : \bar{F}\Psi^i z \leq \mathbf{1} - h_{\mathcal{S}},\ i = 0, 1, 2\}$$

where $\bar{F} = \begin{bmatrix} F & 0 \end{bmatrix}$ and $\Psi = \begin{bmatrix} \Phi & BE \\ 0 & M \end{bmatrix}$.

(c) The online MPC optimization is performed subject to the constraints $(s_{0|k}, \mathbf{c}_k) \in \mathcal{Z}^{\text{MPI}}$ and $x_k - s_{0|k} \in \mathcal{S}$. Explain the function of the optimization variable $s_{0|k}$ and the reason for including the constraint $x_k - s_{0|k} \in \mathcal{S}$. How can this constraint be expressed in terms of linear conditions on the optimization variables?

(d) Determine the matrices W_x, W_c that define the nominal predicted cost

$$\sum_{i=0}^{\infty}\bigl(\|s_{i|k}\|^2 + v_{i|k}^2\bigr) = \|s_{0|k}\|_{W_x}^2 + \|\mathbf{c}_k\|_{W_c}^2$$

where $s_{i+1|k} = As_{i|k} + Bv_{i|k}$, $v_{i|k} = Ks_{i|k} + c_{i|k}$. Solve the MPC optimization for $x_0 = (0, 1)$ and verify that the optimal value of the objective function is $\|s_{0|0}^*\|_{W_x}^2 + \|\mathbf{c}_0^*\|_{W_c}^2 = 0.122$.

(e) What is the advantage of using a different feedback gain K_e in the definition of $\mathcal{S}$ and implementing the controller as $u_k = Ks_{0|k}^* + K_e(x_k - s_{0|k}^*) + c_{0|k}^*$?

9 The homothetic tube MPC strategy of Sect. 3.5.2 does not require the set $\mathcal{S}^0$ to be robustly invariant for the dynamics, $e_{k+1} = \Phi e_k + w_k$, $w_k \in \mathcal{W}$, of the uncertain component of the predicted state. This question concerns a simplification of the online HTMPC optimization that becomes possible when $\mathcal{S}^0$ is replaced by a set, $\mathcal{S} = \{s : V_{\mathcal{S}}s \leq \mathbf{1}\}$, which is robustly invariant for these dynamics.

(a) Show that, if $\Phi\mathcal{S} \oplus \mathcal{W} \subseteq \mathcal{S}$, then $\bar{e} + \bar{w} \leq \mathbf{1}$ where

$$\bar{e} = \max_{e \in \mathcal{S}} V_{\mathcal{S}}\Phi e \qquad \bar{w} = \max_{w \in \mathcal{W}} V_{\mathcal{S}}w.$$

(b) Show that, if $\nu \geq N - 1$ is an integer such that $\bar{F}\Psi^{\nu+1}z \leq \mathbf{1} - h_{\mathcal{S}}$ (where $h_{\mathcal{S}} = \max_{e \in \mathcal{S}} Fe$, and Ψ, $\bar{F}$ are as defined in Question 8(b)) for all z and $(\alpha_0, \ldots, \alpha_{N-1})$ satisfying

$$\begin{aligned}
&\bar{F}\Psi^i z \leq \mathbf{1} - \alpha_i h_{\mathcal{S}}, && i = 0, \ldots, \nu \\
&\alpha_i \bar{e} + \bar{w} \leq \alpha_{i+1}\mathbf{1}, && i = 0, \ldots, N-1 \\
&\alpha_i = 1, && i \geq N,
\end{aligned}$$

then, for the system $x_{k+1} = Ax_k + Bu_k + w_k$, $w_k \in \mathcal{W}$ and control law $u_k = Kx_k + c_{0|k}$, there exists $(s_{0|k}, \mathbf{c}_k, \boldsymbol{\alpha}_k)$ satisfying the following constraints for all $k > 0$ if they are feasible at $k = 0$:

$$\begin{aligned} &\bar{F}\Psi^i z_k \leq \mathbf{1} - \alpha_{i|k} h_{\mathcal{S}}, && i = 0, \ldots, \nu \\ &\alpha_{i|k}\bar{e} + \bar{w} \leq \alpha_{i+1|k}\mathbf{1}, && i = 0, \ldots, N-1 \\ &\alpha_{i|k} = 1, && i \geq N \\ &x_k - s_{0|k} \in \alpha_{0|k}\mathcal{S} \end{aligned}$$

with $z_k = (s_{0|k}, \mathbf{c}_k)$, $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$, $\boldsymbol{\alpha}_k = (\alpha_{0|k}, \ldots, \alpha_{N-1|k})$.

(c) Let $\mathbf{c}_k^* = (c_{0|k}^*, \ldots, c_{N-1|k}^*)$ be optimal for the problem of minimizing the predicted cost at time k:

$$J(s_{0|k}, \mathbf{c}_k, \boldsymbol{\alpha}_k) = \|s_{0|k}\|_{W_x}^2 + \|\mathbf{c}_k\|_{W_c}^2 + \sum_{i=0}^{N-1} q_\alpha(\alpha_{i|k} - 1)^2,$$

over $(s_{0|k}, \mathbf{c}_k, \boldsymbol{\alpha}_k)$ subject to the recursively feasible constraints of part (b), where W_x satisfies the Riccati equation (2.9) for $Q, R \succ 0$, $W_c = \text{diag}\{B^T W_x B + R, \ldots, B^T W_x B + R\}$ and q_α is any nonnegative scalar.

Show that, if this minimization is feasible at $k = 0$, then for any disturbance sequence with $w_k \in \mathcal{W}$ for all $k \geq 0$, the closed-loop system $x_{k+1} = Ax_k + Bu_k + w_k$ under the MPC law $u_k = Kx_k + c_{0|k}^*$ satisfies the constraints $Fx_k \leq \mathbf{1}$ for all $k \geq 0$, and its state converges asymptotically to the set $\mathcal{S}$.

10 This question considers the design and implementation of the homothetic tube MPC strategy of Question 9 for the system model, disturbance bounds and constraints of Question 8.

(a) Using the set $\mathcal{S}$ determined in Question 8, namely

$$\mathcal{S} = \frac{1}{1-\rho}(\mathcal{W} \oplus \Phi\mathcal{W})$$

where ρ is the smallest scalar such that $\Phi^2\mathcal{W} \subseteq \rho\mathcal{W}$, determine $\bar{e}$ and $\bar{w}$ defined in Question 9(a). Verify that, for $N = 1$, the smallest integer ν satisfying the conditions of Question 9(b) is $\nu = 2$.

(b) Taking $q_\alpha = 1$ solve the MPC optimization defined in Question 9(c) for $x_0 = (0, 1)$ and $N = 1$. Why is the optimal solution for $\mathbf{c}_0$ in this problem equal to the optimum for rigid tube MPC computed for the same x_0 in Question 8(d)?

(c) Compare HTMPC in terms of its closed-loop performance and the size of its feasible initial condition set with: (i) the robust MPC algorithm of Question 6 and (ii) the rigid tube MPC algorithm of Question 8.

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Chapter 4
Closed-Loop Optimization Strategies for Additive Uncertainty

The performance and constraint handling capabilities of a robust predictive control law are limited by the amount of information on future model uncertainty that is made available to the controller. However, the manner in which the controller uses this information is equally important. Although the realization of future model uncertainty is by definition unknown when a predicted future control trajectory is optimized, this information may be available to the controller at the future instant of time when the control law is implemented. For example, the prediction, $u_{i|k}$, at time k of the control input i steps into the future should ideally depend on the predicted model state i steps ahead, but this is unknown at time k because of the unknown model uncertainty at times $k, \ldots, k+i-1$, even though the state x_{k+i} is, by assumption, available to the controller at time $k+i$. In order to fully exploit the potential benefits of this information, a closed-loop optimization strategy is needed that allows the degrees of freedom over which the predicted trajectories are optimized to depend on the realization of future model uncertainty.

This chapter considers again the control problem defined in terms of a linear system model in Sect. 3.1, which is restated here for convenience:

$$x_{k+1} = Ax_k + Bu_k + Dw_k. \tag{4.1}$$

The matrices A, B, D are known, the state x_k is known when the control input u_k is chosen, and w_k is an unknown additive disturbance input at time k. As in Chap. 3, w is assumed to lie in a compact convex polytopic set $\mathcal{W}$ containing the origin, described either by its vertices:

$$\mathcal{W} = \text{Co}\{w^{(j)},\ j = 1, \ldots, m\}, \tag{4.2}$$

or by the intersection of half-spaces:

$$\mathcal{W} = \{w : Vw \leq \mathbf{1}\}, \tag{4.3}$$

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and the state and control input of (4.1) are subject to linear constraints:

$$Fx_k + Gu_k \leq \mathbf{1}, \quad k = 0, 1, \ldots \tag{4.4}$$

for given matrices $F \in \mathbb{R}^{n_c \times n_x}$ and $G \in \mathbb{R}^{n_c \times n_u}$.

The control objective is to steer the system state into a prescribed target set under any possible realization of model uncertainty, while minimizing an appropriately defined performance cost. Consideration is given to general class of control laws, and hence we remove the restriction to open-loop strategies that was imposed in Chap. 3.

A predictive control law that employs a closed-loop optimization strategy optimizes predicted performance over parameters that define predicted future feedback policies. In its most general setting, the problem can be solved using dynamic programming, and the first part of this chapter gives a brief overview of this technique and its application to robust MPC for systems with additive disturbances. In practice, the computation required by dynamic programming is often prohibitive. To render computation tractable, it may therefore be necessary to restrict the class of control policies considered in the MPC optimization, and the later sections of this chapter consider approaches that optimize over restricted classes of feedback policies in order to reduce computation. We discuss policies with affine dependence on disturbances, and then consider more general piecewise affine parameterizations. As in Chap. 3, the emphasis is on the tradeoff that can be achieved between computational tractability and performance while ensuring satisfaction of constraints for all uncertainty realizations.

4.1 General Feedback Strategies

Robust optimal control laws for constrained systems subject to unknown disturbances were proposed in [1–4]. The essence of these approaches is to construct a robust controller as a feedback solution to the problem of minimizing a worst-case (min-max) performance objective over all admissible realizations of model uncertainty. The description of model uncertainty in terms of sets of allowable parameter values leads naturally to set theoretic methods for robust constraint handling (e.g. [3, 4]). This is the starting point for this chapter's discussion of closed-loop optimization strategies for the system (4.1) with disturbance sets and constraints of the form of (4.3) and (4.4).

Multistage min-max control problems can be solved using Dynamic Programming (DP) techniques [5, 6]. In principle, the approach is able to determine the optimal feedback strategy in a very general setting, i.e. given only the performance objective, the system dynamics, the constraints on states and control and disturbance inputs, and knowledge of the information that is available to the controller at each time step. Consider for example the problem of robustly steering the state of (4.1) from an initial condition x_0, which belongs to an allowable set of initial conditions that is to be made as large as possible, to a pre-specified target set, $\mathcal{X}_T$, over N time steps. This problem can be formulated (see e.g. [4]) as an optimal control problem involving a terminal cost $\mathcal{I}(x_N)$ defined by

$$\mathcal{I}(x_N) \doteq \begin{cases} 0 & \text{if } x_N \in \mathcal{X}_T \\ 1 & \text{otherwise} \end{cases}$$

From the model (4.1) and the assumptions on the information that is available to the controller, it follows that the maximizing disturbance at time i depends on the control input u_i, whereas the minimizing control input must be computed without knowledge of the realization of the disturbance w_i at time i. Therefore the optimal sequence of feedback laws $u_i^*(x)$, $i = 0, \ldots, N-1$ can be obtained by solving the sequential min-max problem:

$$\mathcal{I}_N^*(x_0) \doteq \min_{\substack{u_0 \\ Fx_0+Gu_0 \leq \mathbf{1}}} \max_{\substack{w_0 \\ w_0 \in \mathcal{W}}} \cdots \min_{\substack{u_{N-1} \\ Fx_{N-1}+Gu_{N-1} \leq \mathbf{1}}} \max_{\substack{w_{N-1} \\ w_{N-1} \in \mathcal{W}}} \mathcal{I}(x_N). \tag{4.5}$$

Dynamic programming solves this problem by recursively determining the optimal costs and control laws for successive stages. Thus let $m = N - i$, where i is the time index and m denotes the number of stages to go until the target set is reached. Then set $\mathcal{I}_0^*(x) \doteq \mathcal{I}(x)$ and, for $m = 1, \ldots, N$, solve:

$$w_i^*(x, u) = \arg \max_{\substack{w \\ w \in \mathcal{W}}} \mathcal{I}_{m-1}^*(Ax + Bu + Dw) \tag{4.6a}$$

$$u_i^*(x) = \arg \min_{\substack{u \\ Fx+Gu \leq \mathbf{1}}} \mathcal{I}_{m-1}^*\big(Ax + Bu + Dw_i^*(x, u)\big) \tag{4.6b}$$

with

$$\mathcal{I}_m^*(x) \doteq \mathcal{I}_{m-1}^*\Big(Ax + Bu_i^*(x) + Dw_i^*\big(x, u_i^*(x)\big)\Big) \tag{4.6c}$$

The sequence of feedback laws $u_i^*(x_i)$, $i = 0, \ldots, N-1$ satisfying (4.6b) necessarily steers the state of (4.1) into $\mathcal{X}_T$ over N steps, whenever this is possible for the given constraints, disturbance set and initial condition. The set of initial conditions from which $\mathcal{X}_T$ can be reached in N steps is referred to as the N-step controllable set to $\mathcal{X}_T$.

Definition 4.1 (*Controllable set*) The N-step controllable set to $\mathcal{X}_T$ is the set of all states x_0 of (4.1) for which there exists a sequence of feedback laws $u_i^*(x)$, $i = 0, \ldots, N-1$ such that, for any admissible disturbance sequence $\{w_0, \ldots, w_{N-1}\} \in \mathcal{W} \times \cdots \times \mathcal{W}$, we obtain $Fx_i + Gu_i \leq \mathbf{1}$, $i = 0, \ldots, N-1$ and $x_N \in \mathcal{X}_T$ along trajectories of (4.1) under $u_i = u_i^*(x_i)$.

In (4.6), the problem of computing the N-step controllable set is split into N subproblems, each of which requires the computation of a 1-step controllable set. The conceptual description of the procedure given by (4.6) has the following geometric interpretation [4]. Let $\mathcal{X}^{(m)}$ denote the m-step controllable set to $\mathcal{X}_T$. Then clearly $\mathcal{X}^{(1)}$ is the set of states x such that there exists u satisfying $Fx + Gu \leq \mathbf{1}$ and $Ax + Bu + Dw \in \mathcal{X}_T$ for all $w \in \mathcal{W}$, i.e.

$$\mathcal{X}^{(1)} = \{x : \exists u \text{ such that } Fx + Gu \leq \mathbf{1}, \text{ and } Ax + Bu \in \mathcal{X}_T \ominus D\mathcal{W}\}.$$

By (4.6), the m-step controllable set $\mathcal{X}^{(m)}$ is defined in terms of $\mathcal{X}^{(m-1)}$ for $m = 1, 2, \ldots, N$ by

$$\mathcal{X}^{(m)} = \{x : \exists u \text{ such that } Fx + Gu \leq \mathbf{1} \text{ and } Ax + Bu \in \mathcal{X}^{(m-1)} \ominus D\mathcal{W}\}$$

with $\mathcal{X}^{(0)} = \mathcal{X}_T$. Given the polytopic uncertainty description (4.2) and the linearity of the dynamics (4.1) and constraints (4.4), $\mathcal{X}^{(m)}$ is therefore convex and polytopic whenever it is non-empty. Hence, given the representation $\mathcal{X}^{(m-1)} = \{x : H^{(m-1)}x \leq \mathbf{1}\}$, $\mathcal{X}^{(m)}$ can be determined for $m = 1, 2, \ldots, N$ using the following procedure.

Algorithm 4.1 (*Controllable sets*) Set $\mathcal{X}^{(0)} = \mathcal{X}_T$. For $m = 1, 2, \ldots, N$:

(i) Compute $\hat{\mathcal{X}}^{(m-1)} \doteq \mathcal{X}^{(m-1)} \ominus D\mathcal{W}$. Since $\mathcal{X}^{(m-1)} = \{x : H^{(m-1)}x \leq \mathbf{1}\}$, this is given by $\hat{\mathcal{X}}^{(m-1)} = \{x : H^{(m-1)}x \leq \mathbf{1} - h_{m-1}\}$ where each element of h_{m-1} is the solution of a linear program, namely

$$h_{m-1} = \max_{w \in \mathcal{W}} H^{(m-1)}w.$$

(ii) Define $\mathcal{Y}^{(m-1)} \subseteq \mathbb{R}^{n_x+n_u}$ as the set

$$\mathcal{Y}^{(m-1)} \doteq \left\{(x, u) : \begin{bmatrix} H^{(m-1)}A & H^{(m-1)}B \\ F & G \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \mathbf{1} - \begin{bmatrix} h_{m-1} \\ 0 \end{bmatrix}\right\}.$$

(iii) Compute $\mathcal{X}^{(m)}$ as the projection of $\mathcal{Y}^{(m-1)}$ onto the x-subspace:

$$\mathcal{X}^{(m)} = \{x : \exists u \text{ such that } (x, u) \in \mathcal{Y}^{(m-1)}\}.$$

and determine $H^{(m)}$ such that $\{x : H^{(m)}x \leq \mathbf{1}\} = \mathcal{X}^{(m)}$. ◁

In applications of dynamic programming to receding horizon control problems, the target set $\mathcal{X}_T$ is usually chosen as a robustly controlled positively invariant (RCPI) set, i.e. $\mathcal{X}_T$ is robustly positively invariant for (4.1) and (4.4) under some feedback law. In this case, it is easy to show that the controllable sets $\mathcal{X}^{(m)}$, $m = 1, \ldots, N$ are themselves RCPI and nested:

$$\mathcal{X}^{(N)} \supseteq \mathcal{X}^{(N-1)} \supseteq \cdots \supseteq \mathcal{X}^{(1)} \supseteq \mathcal{X}_T.$$

This nested property necessarily holds because, under the assumption that $\mathcal{X}_T$ is RCPI, $\mathcal{X}_T$ must lie in m-step controllable set to $\mathcal{X}_T$ for $m = 1, 2, \ldots$ It follows that $\mathcal{X}_T \subseteq \mathcal{X}^{(1)}$ and hence $\mathcal{X}^{(1)}$ is itself RCPI. By the same argument therefore, $\mathcal{X}^{(m)} \subseteq \mathcal{X}^{(m+1)}$, and hence $\mathcal{X}^{(m)}$ is RCPI for $m = 1, 2, \ldots$.

To illustrate the procedure of Algorithm 4.1, we next consider its application to a first-order system. For this simple example, the controllable set can be determined by straightforward algebra, providing some additional insight into each step of Algorithm 4.1. We also compare the controllable set for a general closed-loop strategy with the sets of initial conditions from which a given terminal set can be reached under either a fixed feedback law or an open-loop strategy.

Example 4.1 A first-order system with control and disturbance inputs is described by the dynamics

$$x_{k+1} = x_k + u_k - w_k.$$

The state x, control input u and disturbance input w are scalars that are constrained to lie in intervals:

$$x \in [0, X], \quad u \in [0, U], \quad w \in [0, W].$$

The control objective is to steer the state into a target set $\mathcal{X}_T = [\underline{x}^{(0)}, \bar{x}^{(0)}]$.

In this example, $\mathcal{X}^{(m)}$, the m-step controllable set to $\mathcal{X}_T$, is equal to an interval on the real line, i.e. for each $m = 1, \ldots, N$ we have

$$\mathcal{X}^{(m)} = [\underline{x}^{(m)}, \bar{x}^{(m)}].$$

It is therefore straightforward to determine the recursion relating the $(m-1)$-step controllable set to the m-step set without using Algorithm 4.1. Consider first the conditions defining the upper limit of $\mathcal{X}^{(m)}$ in terms of $\bar{x}^{(m-1)}$. By linearity, $\bar{x}^{(m-1)}$ must be the 1-step ahead state from $\bar{x}^{(m)}$ for some control input u and disturbance w. Since $u \geq 0$ and $w \geq 0$, the maximum over u of the minimum over w of $\bar{x}^{(m)} = \bar{x}^{(m-1)} - u + w$ is obtained with $u = w = 0$. Similarly, for the lower limit of $\mathcal{X}^{(m)}$, the worst case value of w (in the sense of maximizing $\underline{x}^{(m)} = \underline{x}^{(m-1)} - u + w$) is W, while the minimizing value of u is U. Taking into account the constraint that $0 \leq \underline{x}^{(m)} \leq \bar{x}^{(m)} \leq X$, we therefore obtain

$$\underline{x}^{(m)} = \max\{0,\ \underline{x}^{(m-1)} + W - U\}, \quad \bar{x}^{(m)} = \min\{X,\ \bar{x}^{(m-1)}\}. \tag{4.7}$$

We can verify this result using Algorithm 4.1: $\hat{\mathcal{X}}^{(m-1)} = \mathcal{X}^{(m-1)} \ominus [-W, 0]$ in step (i) yields

$$\hat{\mathcal{X}}^{(m-1)} = [\underline{x}^{(m-1)} + W,\ \bar{x}^{(m-1)}],$$

so the conditions defining $\mathcal{Y}^{(m-1)}$ in step (ii) are

$$0 \leq x \leq X, \quad \underline{x}^{(m-1)} + W \leq x + u \leq \bar{x}^{(m-1)}, \quad 0 \leq u \leq U,$$

and, performing step (iii) by eliminating u using

$$x + u \geq \underline{x}^{(m-1)} + W \quad \forall u \in [0, U] \quad \text{if and only if} \quad x \geq \underline{x}^{(m-1)} + W - U$$
$$x + u \leq \bar{x}^{(m-1)} \quad \forall u \in [0, U] \quad \text{if and only if} \quad x \leq \bar{x}^{(m-1)}$$

we therefore obtain $\mathcal{X}^{(m)} = [\underline{x}^{(m)}, \bar{x}^{(m)}]$ as defined in (4.7).

To demonstrate the improvement in the size of controllable set that is achievable using a general feedback law rather than a particular linear feedback law or open-loop control, consider a specific example with the constraints

$$x \in [0, 5], \quad u \in [0, 1], \quad w \in [0, \tfrac{1}{2}],$$

and target set

$$\mathcal{X}_T = [\underline{x}^{(0)}, \bar{x}^{(0)}] = [3, 4].$$

In this case the control input can exert a greater influence on the state than the disturbance input since $U > W$, and it is therefore to be expected that the m-step controllable set converges for finite m to the maximal controllable set for any horizon. Figure 4.1 shows that $\mathcal{X}^{(m)}$ for $m \geq 6$ is equal to the maximal set $\mathcal{X}^{(\infty)} = [0, 4]$. (Note that the upper limit of $\mathcal{X}^{(m)}$ cannot exceed the upper limit $\bar{x}^{(0)} = 4$ of the target interval because u is constrained to be non-negative.)

With an open-loop control law such as

$$u = \hat{w}, \quad \hat{w} = W/2 = \tfrac{1}{4},$$

we get $\mathcal{X}^{(m)} = [\underline{x}^{(m-1)} + W - \hat{w}, \bar{x}^{(m-1)} - \hat{w}] = [\underline{x}^{(m-1)} + \frac{1}{4}, \bar{x}^{(m-1)} - \frac{1}{4}]$. Consequently the set of states from which $\mathcal{X}_T$ can be reached in m steps necessarily shrinks as m increases and is in fact empty for $m > 2$ (Fig. 4.1).

Consider next the maximal controllable sets that can be obtained with a linear feedback law. A solution of (4.6b) for the optimal feedback is given by

$$u = \min\big\{U, \ \max\{0, \ K(x - \bar{x}^{(0)})\}\big\}$$

with $K = -1$ for all m, and it can moreover be shown that any control law capable of generating the m-step controllable set for $m > 2$ must depend nonlinearly on x. However it is possible to approximate the m-step controllable set using a linear feedback law. Thus, for example

$$u = K(x - \bar{x}^{(0)})$$

ensures $u = 0$ for $x = \bar{x}^{(0)} = 4$, which maximizes the upper limit of the set of initial conditions that can be driven in m steps to $\mathcal{X}_T$ under linear feedback. The gain must satisfy $K < -0.5$ in order that the lower limit of the m-step set to $\mathcal{X}_T$ extends below

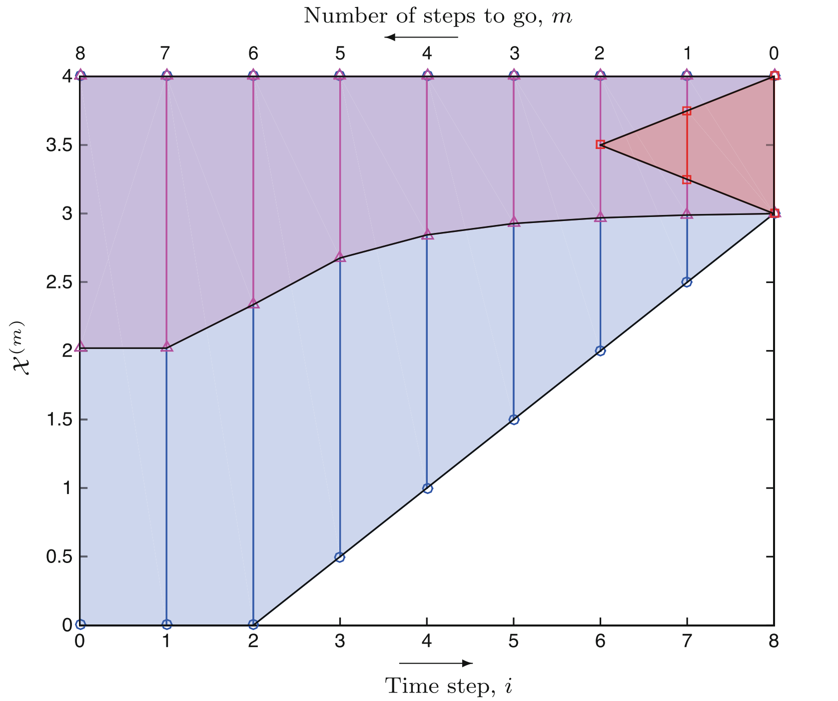

Fig. 4.1 The m-step controllable sets for Example 4.1 with $m \leq 8$ for general feedback laws (*marked with* ◯). Also shown are the sets of states that can be steered to $\mathcal{X}_T = [3, 4]$ in m steps under the linear feedback law $u = -0.51(x - 4)$ (*marked* △), and open-loop control $u = \hat{w}$ (*marked* □)

$\underline{x}^{(0)} = 3$, and for this range of gains, the asymptotic size of the m-step set for large m is limited by the constraint $u \leq 1$. Hence the asymptotic m-step set under this linear feedback law necessarily increases as $|K|$ decreases, while the number of steps needed for convergence to this set increases as $|K|$ decreases. For a horizon $N = 8$, the 8-step set is maximized with $K = -0.51$ (to 2 decimal places); this is shown in Fig. 4.1. ♢

In addition to its uses in computing robust controllable sets, dynamic programming can also be used to solve problems involving the minimization of performance indices consisting of sums of stage costs over a horizon. For example, the optimal value of the quadratic min-max cost with horizon N considered in Sect. 3.4:

$$\check{J}_N^*(x_0) \doteq \min_{\substack{u_0 \\ Fx_0+Gu_0\leq \mathbf{1}}} \max_{\substack{w_0 \\ w_0\in\mathcal{W}}} \cdots \min_{\substack{u_{N-1} \\ Fx_{N-1}+Gu_{N-1}\leq \mathbf{1} \\ x_N\in\mathcal{X}_T}} \max_{\substack{w_{N-1} \\ w_{N-1}\in\mathcal{W}}} \sum_{i=0}^{N-1}(\|x_i\|_Q^2 + \|u_i\|_R^2 - \gamma^2\|w_i\|^2) + \|x_N\|_{\check{W}_x}^2 \tag{4.8}$$

can be rewritten, using the fact that the optimal values of u_i and w_i depend, respectively, on x_i and (x_i, u_i), as

$$\check{J}_N^*(x_0) = \min_{\substack{u_0 \\ Fx_0+Gu_0\le \mathbf{1}}} \max_{\substack{w_0 \\ w_0\in\mathcal{W}}} \Bigg\{ \|x_0\|_Q^2 + \|u_0\|_R^2 - \gamma^2\|w_0\|^2 + \cdots + \min_{\substack{u_{N-1} \\ Fx_{N-1}+Gu_{N-1}\le \mathbf{1} \\ x_N\in\mathcal{X}_T}} \max_{\substack{w_{N-1} \\ w_{N-1}\in\mathcal{W}}} \Big\{ \|x_{N-1}\|_Q^2 + \|u_{N-1}\|_R^2 - \gamma^2\|w_{N-1}\|^2 + \|x_N\|_{\check{W}_x}^2 \Big\}\Bigg\}.$$

This exposes the structure of the optimal control problem as a sequence of subproblems. Each subproblem involves the optimization, over the control and disturbance inputs at a given time step, of a single stage of the cost plus the remaining cost-to-go. Therefore, analogously to (4.6a–4.6c), the optimal closed-loop strategy at time i is given by the solution of

$$\big(u_i^*(x), w_i^*(x,u)\big) = \arg \min_{\substack{u \\ Fx+Gu\le\mathbf{1}}} \max_{\substack{w \\ w\in\mathcal{W}}} \check{J}_{N-i}(x,u,w) \quad \text{subject to } Ax+Bu \in \mathcal{X}^{(N-i-1)} \ominus D\mathcal{W}, \tag{4.9a}$$

where $\mathcal{X}^{(m)}$ is the m-step controllable set to a given target set $\mathcal{X}_T$, and where $m = N-i$ is the number of time steps until the end of the N-step horizon. The cost with m steps to go, $\check{J}_m$, is defined for $m = 1, 2, \ldots, N$ by the dynamic programming recursion

$$\check{J}_m(x,u,w) = \|x\|_Q^2 + \|u\|_R^2 - \gamma^2\|w\|^2 + \check{J}_{m-1}^*(Ax+Bu+Dw) \tag{4.9b}$$

with

$$\check{J}_{N-i}^*(x) = \check{J}_{N-i}\Big(x, u_i^*(x), w_i^*\big(x, u_i^*(x)\big)\Big) \tag{4.9c}$$

and the terminal conditions:

$$\check{J}_0^*(x) = \|x\|_{\check{W}_x}^2, \tag{4.9d}$$

$$\mathcal{X}^{(0)} = \mathcal{X}_T. \tag{4.9e}$$

The corresponding receding horizon control law for a prediction horizon of N time steps is given by $u = u_N^*(x)$.

The decomposition of (4.8) into a sequence of single-stage min-max problems in (4.9a–4.9e) enables the optimal feedback law to be determined without imposing a suboptimal controller parameterization on the problem. However, (4.9) shows an obvious difficulty with this approach, namely that at each stage $m = 1, 2, \ldots, N$ the function $\check{J}_{m-1}^*(x)$ giving the optimal cost for $m-1$ stages must be known in order to be able to determine the optimal cost and control law for the m-stage problem. Conventional implementations of dynamic programming for this problem therefore require the optimal control problem to be solved globally, for all admissible states. This makes the method computationally intensive and, crucially, it results in poor scalability of the approach because the computational demand grows exponentially with the dimensions of the state and input variables.

Predictive controllers aim to avoid such computational difficulties by optimizing the predicted trajectories that emanate from a particular model state rather than globally, and this is the focus of the remainder of this chapter. We next describe two general feedback strategies before discussing parameterized feedback strategies that optimize over restricted classes of feedback law. The remainder of this chapter uses the following example problem to illustrate and compare robust control laws.

Example 4.2 A triple integrator with control and disturbance inputs is given by (4.1) with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The disturbance input w is constrained to lie in the set $\mathcal{W}$ defined by

$$\mathcal{W} = \left\{ w : \begin{bmatrix} -0.25 \\ -0.25 \\ -0.25 \end{bmatrix} \leq w \leq \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} \right\}$$

and the state x and control input u are subject to constraints

$$-500 \leq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \leq 5, \quad -4 \leq u \leq 4$$

so that the constraints take the form of (4.4) with

$$F = \begin{bmatrix} 0.2 & 0 & 0 \\ -0.002 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0.25 \\ -0.25 \end{bmatrix}.$$

The cost is defined by (4.8) with the weights

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = 0.1, \quad \gamma^2 = 50,$$

and the terminal weighting matrix $\check{W}_x$ is defined as the solution of the Riccati equation (3.42). The terminal set $\mathcal{X}_T$ is defined as the maximal RPI set for (4.1) and (4.4) under $u = Kx$, where $K = [-0.77 \; -2.40 \; -2.59]$ is the optimal feedback gain for the infinite horizon cost (3.39).

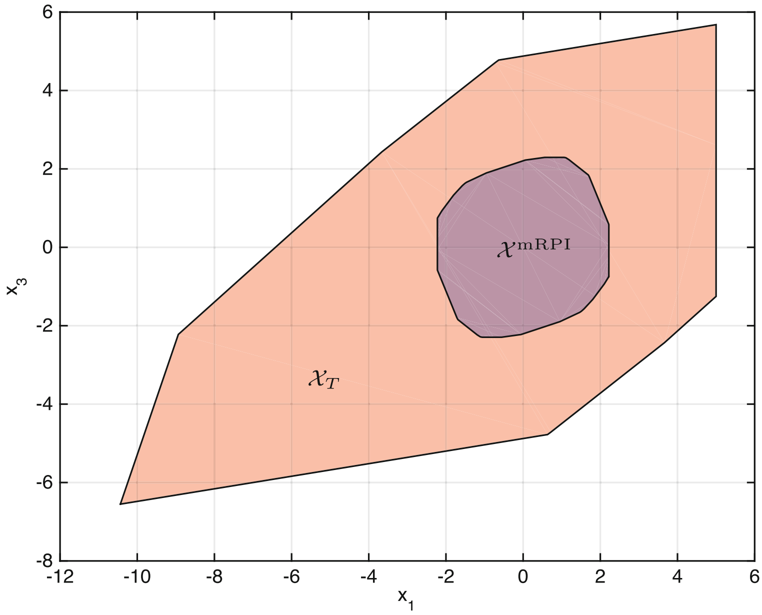

Fig. 4.2 The projections onto the plane $\{x : [0\ 1\ 0]x = 0\}$ of the terminal set $\mathcal{X}_T$ and the mRPI set $\mathcal{X}^{\mathrm{mRPI}}$ under $u = Kx$

Figure 4.2 shows the projection of the terminal set $\mathcal{X}_T$ onto the plane on which $[0\ 1\ 0]x = 0$. For comparison, the projection of the minimal RPI set under $u = Kx$ is also shown in this figure. The projections of the N-step controllable sets to $\mathcal{X}_T$ onto the plane $[0\ 1\ 0]x = 0$, for $4 \leq N \leq 10$, are shown in Fig. 4.3. ◊

4.1.1 Active Set Dynamic Programming for Min-Max Receding Horizon Control

At each stage of the min-max problem defined in (4.9a–4.9e), the optimal control and worst case disturbance inputs are piecewise affine functions of the model state at that stage. This is a consequence of the quadratic nature of the cost and the linearity of the constraints, and it implies that the optimal control law for the full N-stage problem (4.8) is a piecewise affine function of the initial state x_0. Each constituent affine feedback law of this function depends on which of the inequality constraints are active (namely which constraints hold with equality) at the solution, and it follows that the regions of state space within which the optimal control law is given by a particular affine state feedback law are convex and polytopic.

A possible solution method consists of computing offline all of these polytopic regions and their associated affine feedback laws, in a similar manner to the multi-

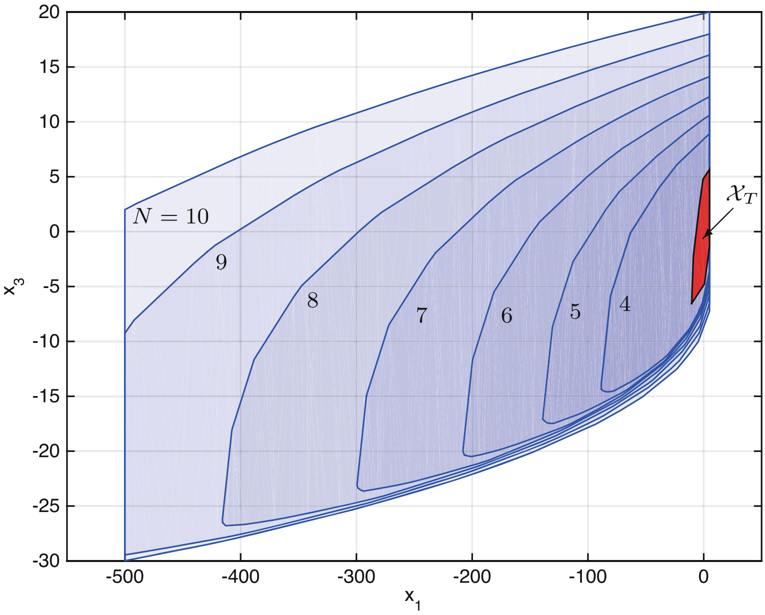

Fig. 4.3 The projections onto the plane $\{x : [0\ 1\ 0]x = 0\}$ of the N-step controllable sets to $\mathcal{X}_T$ for $N = 4, 5, \ldots, 10$ and the terminal set $\mathcal{X}_T$

parametric approach considered in Sect. 2.8. The optimal feedback law could then be determined online by identifying which region of state space contains the current state. This approach is described in the context of linear systems with unknown disturbances and piecewise linear cost indices in [7]. However, as in the case of no model uncertainty, the approach suffers from poor scalability of its computational requirements with the dimension of the state and the horizon length. An illustration of this is given in Example 4.3, where, for a horizon of $N = 4$ the number of affine functions and polytopic regions defining the optimal control law for (4.9a–4.9e) is around 100 (Fig. 4.4). For $N = 10$, the number of regions increases to around 10,000 (Fig. 4.5), implying a large increase in the online computation that is required to determine which region contains the current state, as well as large increases in the offline computation and storage requirements of the controller.

To avoid computing the optimal feedback law at all points in state space, it is possible instead to use knowledge of the solution of (4.9a–4.9e) at a particular point in state space to determine the optimal control law at the current plant state. This is the motivation behind the homotopy-based methods for constrained receding horizon control proposed for uncertainty-free control problems in [8, 9] and developed subsequently for min-max robust control problems with bounded additive disturbances in [10, 11].

Algorithms based on homotopy track the changes in the optimal control and worst case disturbance inputs as the system state varies in state space. For the problem

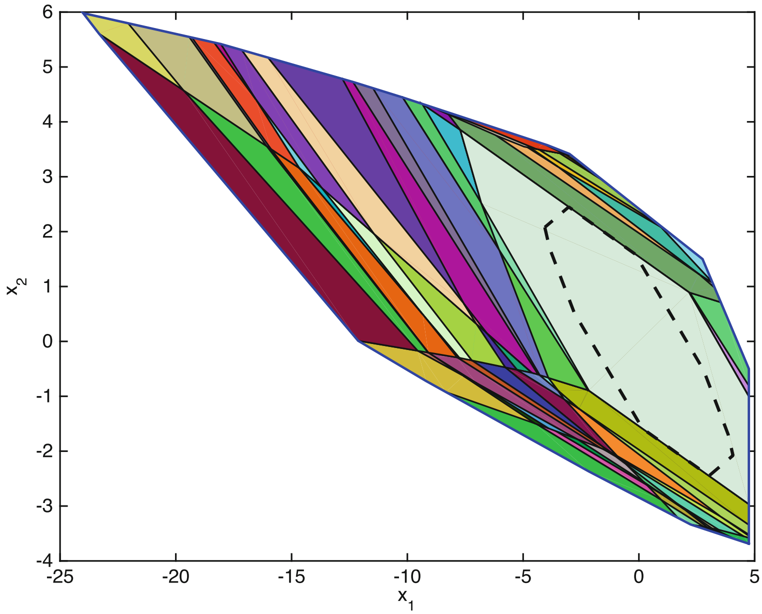

Fig. 4.4 Active set regions in the plane $\{x : [0\ 1\ 0]x = 0\}$ for $N = 4$ (74 regions). The *dashed lines* show the intersection of the terminal set $\mathcal{X}_T$ with this plane

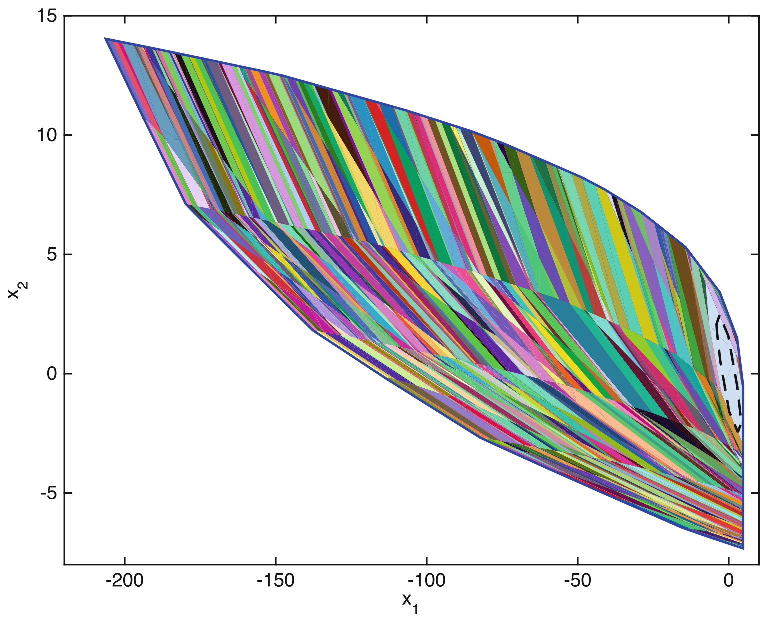

Fig. 4.5 Active set regions in the plane $\{x : [0\ 1\ 0]x = 0\}$ for $N = 10$ (6352 regions). The *dashed lines* show the intersection of the terminal set $\mathcal{X}_T$ with this plane

(4.9a–4.9e), this is done by determining, for a given set of active constraints, the optimal feedback laws at each stage of the problem and hence the optimal state, control and worst case disturbance trajectories as functions of the initial model state. As the initial model state x_0 moves along a search path in state space towards the location of the current plant state, a change in the active set is detected by determining the point at which any inactive constraint becomes active, or any active constraint becomes inactive. This information allows the active set to be updated, and a new piecewise affine control law to be determined, thus enabling the initial model state to move into a new region of state space while continuing to track the optimal solution.

With the search path for the initial model state x_0 defined as a straight line in state space, the detection of an active set change becomes a univariate feasibility problem with linear constraints. Furthermore the computational requirement of updating the optimal solution after an active set change scales linearly with the horizon length and polynomially with state and input dimensions. Clearly this is not the whole story because the overall computation also depends on the number of active set changes, and, as is usual for an active set solver for a constrained optimization problem, upper bounds on this grow exponentially with the problem size. In practice, however, the number of active set changes is likely to be small, and, when used in a receding horizon control setting, the algorithm can be initialized using knowledge of the active set corresponding to the optimal solution at a previous sampling instant.

A summary of the method is as follows. Suppose the optimal cost for the sub-problem (4.9a) with $N-i-1$ stages to go is given by

$$\check{J}^*_{N-i-1}(x) = x^T P_i x + 2x_i^T q_i + r_i \tag{4.10}$$

for some matrix P_i, vector q_i and scalar r_i, and let the active constraints in problem (4.9a) be given by

$$E_i(Ax + Bu) = \mathbf{1} \tag{4.11a}$$

$$F_i x + G_i u = \mathbf{1} \tag{4.11b}$$

$$V_i w = \mathbf{1} \tag{4.11c}$$

(where $E_i(Ax + Bu) = \mathbf{1}$ represents the active constraints in the condition $Ax + Bu \in \mathcal{X}^{N-i-1} \ominus D\mathcal{W}$). The first-order optimality conditions defining the maximizing function $w_i^*(x, u)$ are given by

$$\begin{bmatrix} \gamma^2 I - D^T P_i D & V_i^T \\ V_i & 0 \end{bmatrix} \begin{bmatrix} w_i \\ \eta_i \end{bmatrix} = \begin{bmatrix} D^T P_i \\ 0 \end{bmatrix} (Ax + Bu) + \begin{bmatrix} D^T q_i \\ \mathbf{1} \end{bmatrix},$$

where η_i is a vector of Lagrange multipliers for the constraints (4.11c). Under the assumption that γ is sufficiently large that the maximization subproblem in (4.9a) is concave, namely that

$$V_{i,\perp}^T \big(\gamma^2 I - D^T P_i D\big) V_{i,\perp} \succ 0,$$

where the columns of $V_{i,\perp}$ span the kernel of V_i, the maximizer $w_i^*(x,u)$ is unique and is given by

$$\begin{bmatrix} w_i^*(x,u) \\ \eta_i^*(x,u) \end{bmatrix} = M_i(Ax+Bu)+m_i$$

for some matrix M_i and vector m_i. Similarly, for the minimization subproblem in (4.9a), let

$$\begin{bmatrix} \hat{P}_i & \hat{q}_i \end{bmatrix} = \begin{bmatrix} P_i & q_i \end{bmatrix} + \begin{bmatrix} P_i D & 0 \end{bmatrix}\begin{bmatrix} M_i & m_i \end{bmatrix}.$$

Then the first-order optimality conditions defining the minimimizing function $u_i^*(x)$ are given by

$$\begin{bmatrix} R+B^T\hat{P}_iB & B^TE_i^T & G_i^T \\ E_iB & 0 & 0 \\ G_i & 0 & 0 \end{bmatrix}\begin{bmatrix} u_i \\ \nu_i \\ \mu_i \end{bmatrix} = -\begin{bmatrix} B^T\hat{P}_iA \\ E_iA \\ F_i \end{bmatrix}x_k + \begin{bmatrix} -B^T\hat{q}_k \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix},$$

where ν_i, η_i are, respectively, Lagrange multipliers for the constraints (4.11a), (4.11b). Assuming convexity of the minimization in (4.9a), or equivalently

$$\begin{bmatrix} B^TE_i^T & G_i^T \end{bmatrix}_\perp (R+B^T\hat{P}_iB)\begin{bmatrix} E_iB \\ G_i \end{bmatrix}_\perp \succ 0$$

where the columns of $\begin{bmatrix} E_iB \\ G_i \end{bmatrix}_\perp$ span the kernel of $\begin{bmatrix} E_iB \\ G_i \end{bmatrix}$, then

$$\begin{bmatrix} u_i^*(x) \\ \nu_i^*(x) \\ \mu_i^*(x) \end{bmatrix} = L_ix+l_i,$$

for some matrix L_i and vector l_i.

The affine forms of $u_i^*(x)$ and $w_i^*(x,u)$ imply that the optimal cost $\check{J}_{N-i}^*(x)$ for $N-i$ stages to go is quadratic, justifying by induction the quadratic form assumed in (4.10). Furthermore, having determined M_i, m_i, L_i, l_i for each $i=0,\ldots,N-1$ for a given active set, it is possible to express as functions of x_0 the sequences of minimizing control inputs and maximizing disturbance inputs

$$\mathbf{u}(x_0) = \{u_0^*,\ldots,u_{N-1}^*\},\quad \mathbf{w}(x_0) = \{w_0^*,\ldots,w_{N-1}^*\},$$

and the corresponding multiplier sequences

$$\boldsymbol{\eta}(x_0) = \{\eta_0^*,\ldots,\eta_{N-1}^*\},\quad \boldsymbol{\nu}(x_0) = \{\nu_0^*,\ldots,\nu_{N-1}^*\},\quad \boldsymbol{\mu}(x_0) = \{\mu_0^*,\ldots,\mu_{N-1}^*\}.$$

If $\mathbf{u}(x_0)$ and $\mathbf{w}(x_0)$ satisfy the constraints of (4.9a–4.9e) and if the multipliers satisfy $\boldsymbol{\eta}(x_0)\geq 0$, $\boldsymbol{\nu}(x_0)\geq 0$, $\boldsymbol{\mu}(x_0)\geq 0$, then these sequences are optimal for the N-stage problem (4.8) provided the constraints in (4.9a–4.9e) are linearly dependent. The affine form of each of these functions implies that x_0 must lie in a convex polytopic set, which for a given active constraint set, denoted by $\mathcal{A}$, we denote as $\mathcal{R}(\mathcal{A})$.

Algorithm 4.2 (*Online active set DP*)
Offline: compute the controllable sets $\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \ldots, \mathcal{X}^{(N)}$ to $\mathcal{X}_T$.
Online, at each time instant $k = 0, 1, \ldots$:

(i) Set $x = x_k$ and initialize the solver with $x_0^{(0)}$ and an active set $\mathcal{A}^{(0)}$ such that $x_0^{(0)} \in \mathcal{R}(\mathcal{A}^{(0)})$. At each iteration $j = 0, 1, \ldots$:

  (a) Determine M_i, m_i, L_i, l_i for $i = N-1, N-2, \ldots, 0$, and hence determine $\mathbf{u}(x_0), \mathbf{w}(x_0), \boldsymbol{\eta}(x_0), \boldsymbol{\nu}(x_0), \boldsymbol{\mu}(x_0)$.

  (b) Perform the line search:

$$\alpha^{(j)} = \max_{\alpha \le 1} \ \alpha \ \text{ subject to } \ x_0^{(j)} + \alpha(x - x_0^{(j)}) \in \mathcal{R}(\mathcal{A}^{(0)}).$$

  (c) If $\alpha^{(j)} < 1$, set $x_0^{(j+1)} = x_0^{(j)} + \alpha^{(j)}(x - x_0^{(j)})$ and use the active boundary of $\mathcal{R}(\mathcal{A}^{(j)})$ to determine $\mathcal{A}^{(j+1)}$ from $\mathcal{A}^{(j)}$.
  Otherwise set $\mathcal{A}_k^* = \mathcal{A}^{(j)}$ and proceed to (ii).

(ii) Apply the control law $u_k = u_0^*(x)$. $\triangleleft$

If the constraints of (4.9a) are linearly dependent, the optimal multiplier sequences may be non-unique [12]. This situation can be handled by introducing into the problem additional equality constraints that enforce compatibility of the linearly dependent constraints. The first-order optimality conditions can then be used to relate the multipliers of these additional constraints to the free variables appearing in the solutions for the multipliers of linearly dependent constraints. Furthermore the degrees of freedom in the optimal multiplier sequences can be chosen so as to ensure that the multipliers are continuous when the active set changes. With this approach, the sequences of primal and dual variables are again determined uniquely as functions of x_0, and the continuity of these sequences at the boundaries of $\mathcal{R}(\mathcal{A})$ in x_0-space is preserved (see [11] for details).

The optimization in step (i) of Algorithm 4.2 can be initialized with a cold start by setting $x_0^{(0)} = 0$ and $\mathcal{A}^{(0)} = \emptyset$, since by assumption the origin lies inside the minimal RPI set for the unconstrained optimal feedback law $u = Kx$, and hence all constraints are inactive for this choice of x_0. Alternatively, if the previous optimal solution is known, then it can be warm started by setting $x_0^{(0)}$ at time k equal to the plant state x_{k-1} at the previous time step and setting $\mathcal{A}^{(0)}$ equal to the corresponding optimal active set $\mathcal{A}_{k-1}^*$. Convergence of the optimization in step (i) in a finite number of iterations is guaranteed since the active set $\mathcal{A}^{(j+1)}$ is uniquely defined at each iteration and since there are a finite number of possible active sets.

Finally it can be shown that the control law of Algorithm 4.2 is recursively feasible and robustly stabilizing for the system (4.1), for all initial conditions in the N-step controllability set, $\mathcal{X}^{(N)}$, to $\mathcal{X}_T$. Recursive feasibility follows from the constraints of (4.9a–4.9e) since these ensure that, at each time k, x_{k+1} will necessarily be steered into $\mathcal{X}^{(N-1)}$. Closed-loop stability for all initial conditions $x_0 \in \mathcal{X}^{(N)}$ can be demonstrated using an identical argument to the proof of Theorem 3.4 to show that the bound:

$$\sum_{k=0}^{n} \bigl(\|x_k\|_Q^2 + \|u_k\|_R^2\bigr) \le \check{J}^*(x_0) + \gamma^2 \sum_{k=0}^{n} \|w_k\|^2 \tag{4.12}$$

holds for all $n \ge 0$ along closed-loop trajectories under Algorithm 4.2.

Example 4.3 For the triple integrator of Example 4.2, Figs. 4.4 and 4.5 show regions of state space within which the optimal control law for problem (4.8) is given by a single affine feedback law, for horizons of $N = 4$ and $N = 10$, respectively. Since this is a third-order system, figures show the intersection of these regions with the plane $\{x : [0\ 1\ 0]x = 0\}$. As expected, for any given active set $\mathcal{A}$, the region $\mathcal{R}(\mathcal{A})$ is a convex polytope, and the union of all regions covers the N-step controllable set. The terminal set $\mathcal{X}_T$ is contained in the region $\mathcal{R}(\emptyset)$, within which the solution of (4.8) coincides with the unconstrained optimal feedback law, $u = Kx$. Note that $\mathcal{R}(\emptyset)$ is not necessarily invariant under $u = Kx$, and hence $\mathcal{R}(\emptyset)$ extends beyond the boundaries of $\mathcal{X}_T$ even though $\mathcal{X}_T$ is the maximal RPI set for $u = Kx$. Comparing Figs. 4.4 and 4.5, it can be seen that the number of active set regions increases rapidly with N.

To illustrate how the online computational requirement of Algorithm 4.2 varies with horizon length for this system, the optimization in step (i) was solved for 50 values of the model state randomly selected on the boundary of $\mathcal{X}^{(N)}$ for $N = 1, 2, \ldots, 20$. Each optimization was cold started (initialized with $x_0^{(0)} = 0$ and $\mathcal{A}^{(0)} = \emptyset$). Figure 4.6 shows that the average, maximum and minimum execution times per iteration[1] for the chosen set of initial conditions depend approximately linearly on N. This is in agreement with the expectation that the computation per iteration should depend approximately linearly on horizon length. Despite exponential growth of the total number of active set regions with N, the number of iterations required is a polynomial function of N for this example. This is illustrated by Fig. 4.7, which shows that the average number of iterations grows approximately as $O(0.25N^2)$. For $N = 20$, the average total execution time was 0.58 s and the maximum execution time 2.23 s. $\diamondsuit$

[1]The execution times in Fig. 4.6 provide an indication of how computation scales with horizon length—Algorithm 4.2 was implemented in Matlab without being fully optimized for speed.

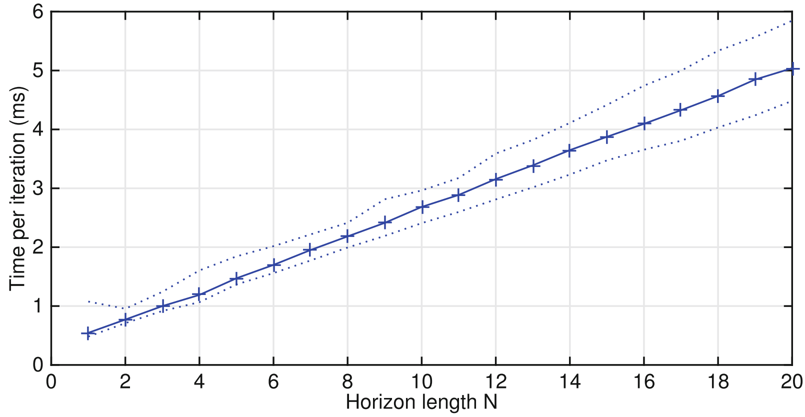

Fig. 4.6 Computation per iteration of the optimization in step (i) of Algorithm 4.2 for Example 4.3 with varying horizon N; average execution time (*solid line*), and minimum and maximum execution times (*dotted lines*)

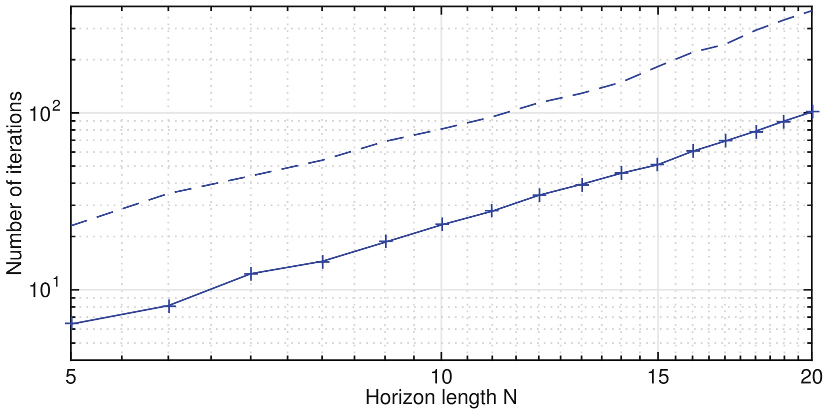

Fig. 4.7 Number of iterations of the optimization in step (i) of Algorithm 4.2 for Example 4.3 with varying horizon N; average (*solid line*) and maximum (*dashed line*)

4.1.2 MPC with General Feedback Laws

As stated previously, dynamic programming methods that provide global solutions are often computationally prohibitive, particularly for problems in which the optimal control is only required locally in a particular operating region of the state space. Although the approach of Sect. 4.1.1 reduces computation using active sets that apply locally within regions of state space, the approach requires prior knowledge of the controllable sets. However, even when performed offline, the computation of these controllable sets can be challenging, and this is a particular concern for problems with long horizons and for systems with many states and input variables.

A major advantage of MPC, and perhaps its defining feature, is that it determines an optimal control input locally, usually for a particular value of the system state, by propagating input and state sequences forwards in time. Since the constraints (4.4) are linear and the disturbance set $\mathcal{W}$ is polytopic, it should not be surprising that each vertex of the controllable set to a convex polytopic target set is determined by a particular maximizing sequence of vertices of $\mathcal{W}$ in (4.5). Therefore, to allow for the full generality of future feedback laws that is needed to achieve the largest possible set of feasible initial conditions, a predictive control strategy must assign a control sequence to each possible sequence of future disturbance inputs [13]. The approach, which is described in this section, leads to an optimization problem with computation that grows exponentially with horizon length. As a result it is computationally intractable for the vast majority of control applications and is mainly of interest from a conceptual point of view. However the approach provides important motivation for the computationally viable MPC strategies that are discussed in Sect. 4.2.

Consider a sequence of disturbance inputs, each of which is equal in value to one of the vertices $w^{(j)}$, $j = 1, \ldots, m$, of the disturbance set $\mathcal{W}$ in (4.3):

$$\{w^{(j_1)}, w^{(j_2)}, \ldots\}.$$

As before we assume that the plant state x_i is known to the controller at the ith time step and a causal control law is assumed; thus u_i depends on x_i but cannot depend on $x_{i+1}, x_{i+2}, \ldots$ If $\{w_0, \ldots, w_{i-1}\} = \{w^{(j_1)}, \ldots, w^{(j_i)}\}$, then it follows that u_i is a function of x_0 and $(j_1, \ldots, j_i)$. Hence we denote the control sequence as $\{u_0, u^{(j_1)}, u^{(j_1,j_2)}, \ldots\}$ and denote $\{x_0, x^{(j_1)}, x^{(j_1,j_2)}, \ldots\}$ as the corresponding sequence of states, which evolve according to the model (4.1):

$$x^{(j_1)} = Ax_0 + Bu_0 + Dw^{(j_1)} \tag{4.13a}$$

and for $i = 1, 2, \ldots$,

$$x^{(j_1,\ldots,j_i,j_{i+1})} = Ax^{(j_1,\ldots,j_i)} + Bu^{(j_1,\ldots,j_i)} + Dw^{(j_{i+1})}. \tag{4.13b}$$

In this prediction scheme, the m^i distinct disturbance sequences $\{w^{(j_1)}, \ldots w^{(j_i)}\}$, in which $j_r \in \{1, \ldots, m\}$ for $r = 1, \ldots, i$, generate m^i state and control sequences with the tree structure shown in Fig. 4.8.

The linearity of the dynamics (4.1) implies that the convex hull of $x^{(j_1,\ldots,j_{i+1})}$ for $j_r \in \{1, \ldots, m\}$, $r = 1, \ldots, i+1$ contains the model state x_{i+1} under any convex combination of control sequences $\{u_0, u^{(j_1)}, \ldots, u^{(j_1,\ldots,j_i)}\}$. To show this, let $X_0 = \{x_0\}$, $U_0 = \{u_0\}$, and for $i = 1, 2, \ldots$ define the sets

$$X_i = \mathrm{Co}\{x^{(j_1,\ldots,j_i)},\ (j_1, \ldots, j_i) \in \mathcal{L}_i\}, \tag{4.14a}$$

$$U_i = \mathrm{Co}\{u^{(j_1,\ldots,j_i)},\ (j_1, \ldots, j_i) \in \mathcal{L}_i\}, \tag{4.14b}$$

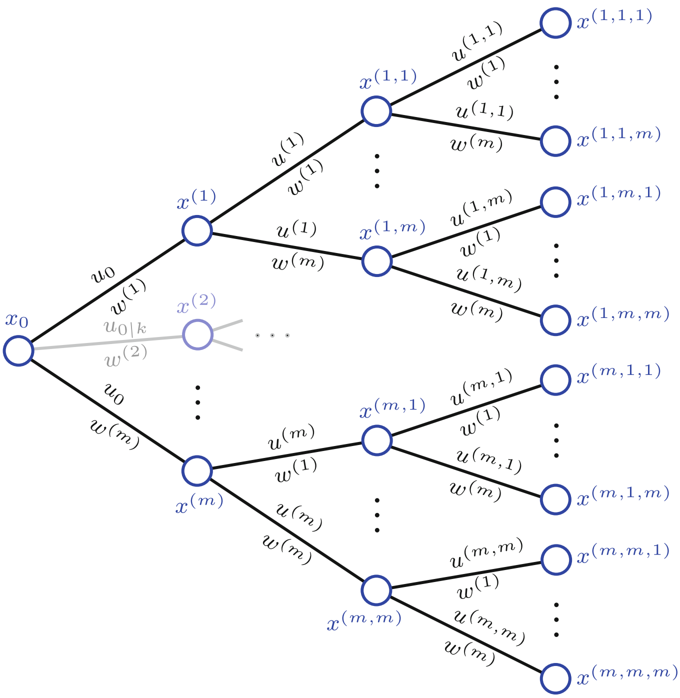

Fig. 4.8 The tree structure of predicted input and state sequences when each element of the disturbance sequence is equal to one of the m vertices of the disturbance set $\mathcal{W}$, shown here for a horizon of $N = 3$

where $\mathcal{L}_i$ denotes the set of all possible values of the sequence $\{j_1, \ldots, j_i\}$:

$$\mathcal{L}_i = \{(j_1, \ldots, j_i) : j_r \in \{1, \ldots, m\},\ r = 1, \ldots, i\},$$

for any positive integer i. Then, for any given $x_i \in X_i$, there necessarily exists $u_i \in U_i$ such that $x_{i+1} \in X_{i+1}$ for all $w_i \in \mathcal{W}$. In particular $x_i \in X_i$ implies $x_i = \sum_{(j_1,\ldots,j_i)\in\mathcal{L}_i} \lambda_{(j_1,\ldots,j_i)} x^{(j_1,\ldots,j_i)}$, where $\lambda_{(j_1,\ldots,j_i)}$ are non-negative scalars satisfying $\sum_{(j_1,\ldots,j_i)\in\mathcal{L}_i} \lambda_{(j_1,\ldots,j_i)} = 1$. Therefore setting

$$u_i = \sum_{(j_1,\ldots,j_i)\in\mathcal{L}_i} \lambda_{(j_1,\ldots,j_i)} u^{(j_1,\ldots,j_i)}$$

gives

$$Ax_i + Bu_i \in \mathrm{Co}\{Ax^{(j_1,\ldots,j_i)} + Bu^{(j_1,\ldots,j_i)},\ (j_1, \ldots, j_i) \in \mathcal{L}_i\},$$

but from (4.13a, 4.13b) and from the definition of X_i in (4.14a) we have

$$\mathrm{Co}\{Ax^{(j_1,\ldots,j_i)} + Bu^{(j_1,\ldots,j_i)},\ (j_1, \ldots, j_i) \in \mathcal{L}_i\} \oplus \mathcal{W} = X_{i+1}, \tag{4.15}$$

and it follows that the tubes $\{X_0, X_1, \ldots\}$ and $\{U_0, U_1, \ldots\}$ necessarily contain the state and control trajectories of (4.1) for any disturbance sequence $\{w_0, w_1, \ldots\}$ with $w_i \in \mathcal{W}$ for all $i = 0, 1, \ldots$.

Consider now a set of input and state trajectories predicted at time k in response to disturbance sequences, $\{w_{0|k}, \ldots, w_{i-1|k}\} = \{w^{(j_1)}, \ldots, w^{(j_i)}\}$, consisting entirely of vertices of $\mathcal{W}$. In order that the state and control trajectories $\{x_{0|k}, x_k^{(j_1)}, x_k^{(j_1,j_2)}, \ldots\}$ and $\{u_{0|k}, u_k^{(j_1)}, u_k^{(j_1,j_2)}, \ldots\}$ generated by the model:

$$x_k^{(j_1)} = Ax_k + Bu_{0|k} + Dw^{(j_1)} \tag{4.16a}$$

$$x_k^{(j_1,\ldots,j_k,j_{i+1})} = Ax_k^{(j_1,\ldots,j_i)} + Bu_k^{(j_1,\ldots,j_i)} + Dw^{(j_{i+1})}, \quad i = 1, 2, \ldots \tag{4.16b}$$

satisfy the constraints (4.4) for all $(j_1, \ldots, j_i) \in \mathcal{L}_i$, we require that

$$Fx_k + Gu_{0|k} \le \mathbf{1} \tag{4.17a}$$

and, for $i = 1, 2, \ldots$,

$$Fx_k^{(j_1,\ldots,j_i)} + Gu_k^{(j_1,\ldots,j_i)} \le \mathbf{1}, \text{ for all } (j_1, \ldots, j_i) \in \mathcal{L}_i. \tag{4.17b}$$

These constraints are imposed over an infinite future horizon if (4.17a, 4.17b) are invoked for $i = 1, \ldots, N-1$ together with a terminal constraint of the form

$$x_k^{(j_1,\ldots,j_N)} \in \mathcal{X}_T, \text{ for all } (j_1, \ldots, j_N) \in \mathcal{L}_N, \tag{4.18}$$

where $\mathcal{X}_T$ is a robustly positively invariant set for (4.1) and (4.4) under a particular feedback law. Assuming for convenience that this control law is linear, e.g. $u = Kx$, $\mathcal{X}_T$ can be defined as the maximal RPI set and determined using the method of Theorem 3.1.

Let $\mathcal{F}_N$ denote the set of feasible initial conditions x_k for (4.17a, 4.17b) and (4.18), i.e.

$$\mathcal{F}_N = \Big\{x_k : \exists\{u_{0|k}, u_k^{(j_1)}, \ldots, u_k^{(j_1,\ldots,j_{N-1})}\} \text{ for } (j_1, \ldots, j_{N-1}) \in \mathcal{L}_{N-1} \text{ such that } \{x_k, x_k^{(j_1)}, \ldots, x_k^{(j_1,\ldots,j_N)}\} \text{ satisfies (4.16a, b), (4.17a, b) for } i = 1, \ldots, N-1, \text{ and (4.18)}\Big\}. \tag{4.19}$$

Theorem 4.1 *$\mathcal{F}_N$ is identical to $\mathcal{X}^{(N)}$, the N-step controllable set to $\mathcal{X}_T$ for the system (4.1) subject to the constraints (4.4), defined in Definition 4.1.*

Proof From (4.14a, 4.14b) and (4.15), constraints (4.17a, 4.17b) and (4.18) ensure that every point $x_0 \in \mathcal{F}_N$ belongs to $\mathcal{X}^{(N)}$. To show that $\mathcal{F}_N$ is in fact equal to $\mathcal{X}^{(N)}$, note that, if $\mathcal{X}_T = \{x : V_T x \le \mathbf{1}\}$, then $\mathcal{X}^{(N)}$ can be expressed as $\{x : f_T(x) \le \mathbf{1}\}$

where $f_T(\cdot)$ is the solution of a sequential min-max problem:

$$f_T(x_0) \doteq \min_{\substack{u_0 \\ Fx_0+Gu_0\le \mathbf{1}}} \max_{\substack{w_0 \\ w_0\in\mathcal{W}}} \cdots \min_{\substack{u_{N-1} \\ Fx_{N-1}+Gu_{N-1}\le \mathbf{1}}} \max_{\substack{w_{N-1} \\ w_{N-1}\in\mathcal{W}}} \max_{r\in\{1,\ldots,n_V\}} V_{T,r}\, x_N, \tag{4.20}$$

in which $V_{T,r}$ is the rth row of V_T. Each stage of this problem can be expressed as a linear program, and hence the sequence of maximizing disturbance inputs is $\{w^{(j_1)}, \ldots, w^{(j_N)}\}$ for some sequence $(j_1, \ldots, j_N)$ of vertices of $\mathcal{W}$. Therefore $x_0 \in \mathcal{X}^{(N)}$ if and only if the optimal control sequence for this problem is $\{u_0, u^{(j_1)}, \ldots, u^{(j_1,\ldots,j_{N-1})}\}$ such that $V_T x^{(j_1,\ldots,j_N)} \le \mathbf{1}$. Since $x_0 \in \mathcal{F}_N$ if $V_T x^{(j_1,\ldots,j_N)} \le \mathbf{1}$ for some sequence $\{u_0, u^{(j_1)}, \ldots, u^{(j_1,\ldots,j_{N-1})}\}$, for each $(j_1, \ldots, j_N) \in \mathcal{L}_N$, it follows that every point $x_0 \in \mathcal{X}^{(N)}$ also belongs to $\mathcal{F}_N$. □

The following lemma shows that $\mathcal{F}_N$ is robustly positively invariant under the control law $u_k = u_{0|k}$.

Lemma 4.1 *For any $N > 0$, $\mathcal{F}_N$ is RPI for the dynamics (4.1), disturbance set (4.2) and constraints (4.4) if $u_k = u_{0|k}$.*

Proof Since $w_k \in \mathcal{W}$, there exist non-negative scalars λ_j, $j = 1, \ldots, m$, such that $\sum_{j=1}^m \lambda_j = 1$ and

$$w_k = \sum_{j=1}^m \lambda_j w^{(j)}.$$

Therefore, for any given $\{x_k, x_k^{(j_1)}, x_k^{(j_1,j_2)}, \ldots\}$ and $\{u_{0|k}, u_k^{(j_1)}, u_k^{(j_1,j_2)}, \ldots\}$ satisfying (4.16a, 4.16b), (4.17a, 4.17b) and (4.18), the trajectories defined by

$$u_{0|k+1} = \sum_{j=1}^m \lambda_j u_k^{(j)}, \quad u_{k+1}^{(j_1,\ldots,j_i)} = \sum_{j=1}^m \lambda_j u_k^{(j,j_1,\ldots,j_i)}, \; i = 1, \ldots, N-2,$$

$$x_{k+1}^{(j_1,\ldots,j_i)} = \sum_{j=1}^m \lambda_j x_k^{(j,j_1,\ldots,j_i)}, \; i = 1, \ldots, N-1,$$

for all $(j_1, \ldots, j_i) \in \mathcal{L}_i$, and

$$u_{k+1}^{(j_1,\ldots,j_{N-1})} = K x_{k+1}^{(j_1,\ldots,j_{N-1})}, \qquad \text{for all } (j_1, \ldots, j_{N-1}) \in \mathcal{L}_{N-1}$$

$$x_{k+1}^{(j_1,\ldots,j_{N-1},j_N)} = \Phi x_{k+1}^{(j_1,\ldots,j_{N-1})} + D w^{(j_N)}, \quad \text{for all } (j_1, \ldots, j_N) \in \mathcal{L}_N$$

satisfy, at time $k+1$, the constraints of (4.16a, 4.16b) (by linearity), (4.17a, 4.17b) (by convexity), and (4.18) since $x_{k+1}^{(j_1,\ldots,j_{N-1})} \in \mathcal{X}_T$ and $\mathcal{X}_T$ is RPI for (4.1) and (4.4) if $u = Kx$. □

Theorem 4.1 shows that x_0 lies in the N-step controllable set to a given target set if and only if a set of linear constraints is satisfied in the variables $\{u_0, u^{(j_1)}, \ldots, u^{(j_1,\ldots,j_{N-1})}\}$, $(j_1, \ldots, j_{N-1}) \in \mathcal{L}_{N-1}$. This feasibility problem could be combined with any chosen performance index to define an optimal control law with a set of feasible initial conditions equal to the controllable set for the given target set and horion. Furthermore, by Lemma 4.1, a receding horizon implementation of any such strategy would necessarily be recursively feasible, and hence could form basis of a robust MPC law. However, the feasible set $\mathcal{F}_N$ is defined in (4.19) in terms of the vertices of the predicted tubes for states and control inputs. In general, the optimal predictions with respect to a quadratic cost will be convex combinations of these vertices, and hence additional optimization variables would be needed in the MPC optimization if a quadratic performance index were used. With this in mind, a linear min-max cost is employed in [13], the cost index being defined as the sum of stage costs that depend linearly on the future state and control input. This choice of cost has the convenient property that the optimal control sequence is given by $\{u_{0|k}, u_k^{j_1}, \ldots, u_k^{j_1,\ldots,j_{N-1}}\}$ for some $(j_1, \ldots, j_{N-1}) \in \mathcal{L}_{N-1}$.

Linear or piecewise linear stage costs have the disadvantage that the unconstrained optimal is not straightforward to determine, and in [13] the terminal control law is restricted to one that enforces finite time convergence of the state (i.e. "deadbeat" control) in order that the cost is finite over an infinite prediction horizon. Instead we illustrate the MPC strategy here using a performance index similar to the nominal cost employed in Sect. 3.3. In this setting, we first reparameterize the predicted control input trajectories in terms of optimization variables $\mathbf{c}_k^{(l)} = \{c_{0|k}, c_k^{(j_1)}, \ldots, c_k^{(j_1,\ldots,j_{N-1})}\}$ for $l = (j_1, \ldots, j_{N-1}) \in \mathcal{L}_{N-1}$, so that

$$
\begin{aligned}
u_{0|k} &= Kx_k + c_{0|k} \\
u_k^{(j_1,\ldots,j_i)} &= \begin{cases} Kx_k^{(j_1,\ldots,j_i)} + c_k^{(j_1,\ldots,j_i)}, & i = 1, \ldots, N-1 \\ Kx_k^{(j_1,\ldots,j_i)}, & i = N, N+1, \ldots \end{cases}
\end{aligned}
\tag{4.21}
$$

Next define a set of nominal state and control trajectories, one for each sequence $l = (j_1, \ldots, j_{N-1}) \in \mathcal{L}_{N-1}$, as follows

$$
\begin{aligned}
v_{i|k}^{(l)} &= Ks_{1|k}^{(l)} + c_{i|k}^{(l)}, & s_{i+1|k}^{(l)} &= \Phi s_{i|k}^{(l)} + Bc_{i|k}^{(l)}, & i &= 0, 1, \ldots, N-1 \\
v_{i|k}^{(l)} &= Ks_{1|k}^{(l)}, & s_{i+1|k}^{(l)} &= \Phi s_{i|k}^{(l)}, & i &= N, N+1, \ldots
\end{aligned}
$$

where $c_{i|k}^{(l)}$ is the ith element of the sequence $\mathbf{c}_k^{(l)}$ for each $l \in \mathcal{L}_{N-1}$, namely $c_{i|k}^{(l)} = c_{0|k}$ for $i = 0$ and $c_{i|k}^{(l)} = c_k^{(j_1,\ldots,j_i)}$ for $i = 1, \ldots, N-1$. Finally, we define the worst case quadratic cost over these nominal predicted sequences as

$$
J\big(s_{0|k}, \{\mathbf{c}_k^{(l)},\ l \in \mathcal{L}_{N-1}\}\big) = \max_{l \in \mathcal{L}_{N-1}} \sum_{i=0}^{\infty} \big(\|s_{i|k}^{(l)}\|_Q^2 + \|v_{i|k}^{(l)}\|_R^2\big) \tag{4.22}
$$

Assuming for convenience that K is the unconstrained LQ-optimal feedback gain, Theorem 2.10 gives

$$J\big(s_{0|k}, \{\mathbf{c}_k^{(l)},\ l \in \mathcal{L}_{N-1}\}\big) = \|s_{0|k}\|^2_{W_x} + \max_{l \in \mathcal{L}_{N-1}} \|\mathbf{c}_k^{(l)}\|^2_{W_c},$$

where W_x is the solution of the Riccati equation (2.9), and where W_c is block diagonal, $W_c = \operatorname{diag}\{B^T W_x B + R, \dots, B^T W_x B + R\}$. Hence the problem of minimizing $J(s_{0|k}, \{\mathbf{c}_k^{(l)}\})$ is equivalent to the minimization of $\|s_{0|k}\|^2_{W_x} + \alpha^2$ subject to $\alpha^2 \geq \|\mathbf{c}_k^{(l)}\|^2_{W_c}$ for all $l \in \mathcal{L}_{N-1}$. This is a convex optimization problem that can be formulated as a second-order cone program (SOCP) [14].

By combining the cost of (4.22) with the linear constraints defining $\mathcal{F}_N$, we obtain the following MPC algorithm.

Algorithm 4.3 (*General feedback MPC*) At each time $k = 0, 1, \dots$:

(i) Perform the optimization

$$\begin{aligned} \underset{\{\mathbf{c}_k^{(l)},\, l \in \mathcal{L}_{N-1}\}}{\text{minimize}} \quad & \max_{l \in \mathcal{L}_{N-1}} \|\mathbf{c}_k^{(l)}\|^2_{W_c} \\ \text{subject to} \quad & \text{(4.16a, b), (4.17a, b)}, i = 1, \dots, N-1, \\ & \text{(4.18) and (4.21).} \end{aligned} \tag{4.23}$$

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\|\mathbf{c}_k^{l*}\|^2_{W_c}$ is the optimal value of the objective in (4.23) and $\mathbf{c}_k^{(l)*} = \{c^*_{0|k}, \dots, c_k^{(j_1,\dots,j_{N-1})*}\}$). ◁

The online optimization in step (i) can be formulated as a SOCP in $n_u(m^N - 1)/(m-1)$ variables, $n_c(m^N - 1)/(m-1) + n_T m^N$ linear inequality constraints and $m^N + 1$ second-order cone constraints, where n_T is the number of constraints defining $\mathcal{X}_T$. Furthermore Algorithm 4.3 is recursively feasible and enforces convergence to the minimal RPI set associated with $u = Kx$, as we now discuss.

The feasibility of (4.23) at all times given initial feasibility (i.e. $x_0 \in \mathcal{F}_N$) is implied by Lemma 4.1. Therefore, defining $c_{0|k+1}$ and $c_{k+1}^{(j_1,\dots,j_i)}$ for $i = 1, \dots, N-1$ analogously to the definitions of $u_{0|k+1}$ and $u_{k+1}^{(j_1,\dots,j_i)}$ in the proof of Lemma 4.1 gives

$$\begin{aligned} c_{0|k+1} &= \sum_{j=1}^{m} \lambda_j c_k^{(j)*} \\ c_{k+1}^{(j_1,\dots,j_i)} &= \sum_{j=1}^{m} \lambda_j c_k^{(j,j_1,\dots,j_i)*}, \quad i = 1, \dots, N-2 \\ c_{k+1}^{(j_1,\dots,j_{N-1})} &= 0 \end{aligned}$$

for each $l = (j_1, \ldots, j_{N-1}) \in \mathcal{L}_{N-1}$, and hence by convexity we have $\|\mathbf{c}_{k+1}^{(l)}\|_{W_c}^2 \le \|\mathbf{c}_k^{l*}\|_{W_c}^2 - \|c_{0|k}^*\|_{R+B^T W_x B}^2$ for all $l = (j_1, \ldots, j_{N-1}) \in \mathcal{L}_{N-1}$. The optimization of (4.23) at time $k+1$ therefore gives

$$\|\mathbf{c}_k^{(l)*}\|_{W_c}^2 - \|\mathbf{c}_{k+1}^{(l)*}\|_{W_c}^2 \ge \|c_{0|k}^*\|_{R+B^T W_x B}^2$$

which implies (by Lemma 3.1) that $c_{0|k}^* \to 0$ as $k \to \infty$. From Theorem 3.2, it follows that the l_2 gain from the disturbance input w to the state x is upper bounded by the l_2 bound for the unconstrained system under $u = Kx$, and furthermore $x_k \to \mathcal{X}^{\text{mRPI}}$ as $k \to \infty$, where $\mathcal{X}^{\text{mRPI}}$ is the minimal RPI for (4.1) under $u = Kx$.

Finally note that the control law of Algorithm 4.3 can also guarantee exponential convergence to an outer approximation, $\mathcal{S}$, of $\mathcal{X}^{\text{mRPI}}$ if $s_{0|k}$ is retained as an optimization variable in (4.23). Specifically, replacing the objective of (4.23) with $J(s_{0|k}, \{\mathbf{c}_k^{(l)}, l \in \mathcal{L}_{N-1}\})$ and including $x_k - s_{0|k} \in \mathcal{S}$ as an additional constraint in (4.23) ensures, by an argument similar to the proof of Theorem 3.5, that the optimal value $J(s_{0|k}^*, \mathbf{c}_k^{(l)*})$ converges exponentially to zero and hence that $\mathcal{S}$ is exponentially stable with region of attraction equal to $\mathcal{F}_N$. This argument relies on the existence of constants a, b satisfying $a\|s_{0|k}^*\|^2 \le J(s_{0|k}^*, \mathbf{c}_k^{(l)*}) \le b\|s_{0|k}^*\|^2$, which is ensured by the continuity [14] of the optimal objective of (4.23) and by the fact that $J(s_{0|k}^*, \mathbf{c}_k^{(l)*}) \ge 0$, whereas $J(s_{0|k}^*, \mathbf{c}_k^{(l)*}) = 0$ if and only if $x_{0|k}$ lies in $\mathcal{S}$.

Example 4.4 Although the general feedback MPC law of Algorithm 4.3 and the active set DP approach of Algorithm 4.2 are both feasible for all initial conditions in the N-step controllable set $\mathcal{X}^{(N)}$, their computational requirements are very different. The exponential growth in the numbers of variables and constraints of the optimization in step (i) of Algorithm 4.3 implies that its computation grows very rapidly with N. For the system of Example 4.2, we have $n_u = 1$, $n_c = 4$, $m = 8$,

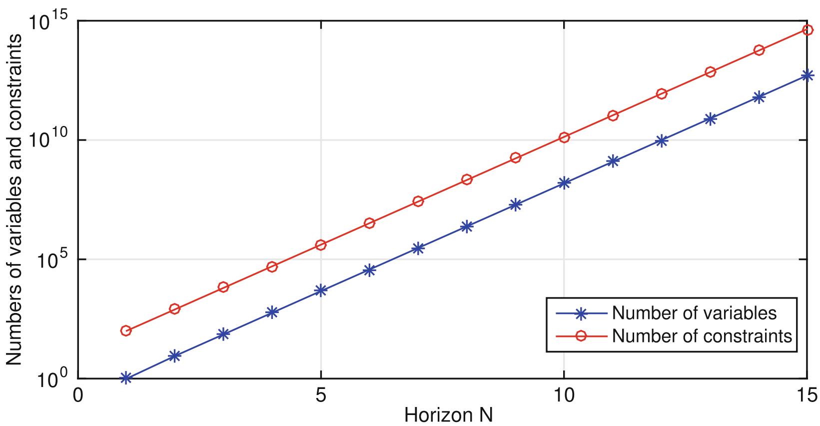

Fig. 4.9 Numbers of variables and linear inequality constraints for Algorithm 4.3 in Example 4.4

$n_T = 11$, and the number of optimization variables and linear inequality constraints grow with N as shown in Fig. 4.9. Even this simple third-order problem is limited to short horizons in order that the optimization (4.23) remains manageable; for example, if the number of optimization variables is required to be less than 1000, then N must be no greater than 4. $\diamondsuit$

4.2 Parameterized Feedback Strategies

Dynamic programming and MPC laws based on general feedback strategies have the definite advantages that they can provide optimal performance and the maximum achievable region of attraction. However, as discussed in Sect. 4.1.2 for the case of min-max robust control (and as also discussed in [15, 16] for stochastic problems), the computation of optimal control laws for these approaches often suffers from poor scalability with the problem size and the length of horizon. It is therefore perhaps inevitable that closed-loop optimization strategies with computational demands that grow less rapidly with problem size should be sought for robust MPC. These approaches reduce computation by restricting the class of closed-loop policies over which optimization is performed.

One such restriction is to the class of time-varying linear feedback plus feedforward control laws, where the linear feedback gains are parameters that are to be computed online. However, the dependence of the future state and input trajectories on these feedback gains is non-linear, and the optimization of predicted performance is non-convex. A way around this is offered by the Youla parameterization introduced into MPC in [17]. On account of the Bezout identity of (2.67), the transfer function matrices that map additive disturbances at the plant input to the plant output have an affine dependence on the Youla parameter. This property is exploited in [18] to devise a lower-triangular prediction structure in the degrees of freedom, leading to a convex online optimization. Later developments in this area, known as disturbance-affine MPC (DAMPC), are reported in [19, 20]. These proposals lead to an online optimization in a number of variables that grows quadratically with the length of the prediction horizon. This can be reduced to a linear growth if the lower-triangular structure is computed offline, but of course the resulting MPC algorithm is then no longer based on optimization of a closed-loop strategy.

An alternative triangular prediction structure to that of [18–20] was proposed in [21], which, like the approach of Sect. 4.1.2, parameterized predicted future feedback laws in terms of the vertices of the disturbance set. In this setting, the input at each prediction instant is known to lie in the convex hull of a linear combination of polytopic cross sections defined by the inputs associated with the disturbance set vertices. The approach thus implicitly employs a parameterization that defines tubes in which the predicted inputs and states will lie, and for this reason it is known as parameterized tube MPC (PTMPC). The disturbance-affine MPC strategy can be shown to be a special case of PTMPC (albeit with a restricted set of optimization variables) and both approaches have a number of optimization variables that grows quadratically with the prediction horizon. This can be reduced to a linear dependence if, instead of the triangular structure of PTMPC, a striped structure is employed [22]. This section discusses these three parameterized feedback MPC strategies.

4.2.1 Disturbance-Affine Robust MPC

A simple but highly restrictive feedback parameterization replaces the fixed linear feedback gain K in the open-loop strategy (3.6) with a linear time-varying state feedback law to give $u_{i|k} = K_k^{(i)} x_{i|k} + c_{i|k}$, where $K_k^{(i)}$ and $c_{i|k}$ for $i = 0, \ldots, N-1$ are to be optimized online at each time k. A less restrictive class of control laws is obtained by allowing the state dependence to be dynamic, in which case the time-varying state feedback is replaced by a convolutional sum, i.e. $u_{i|k} = \sum_{j=0}^{i} K_k^{(i-j)} x_{j|k} + c_{i|k}$. This class can be further widened if the state dependence of the control law is allowed to be both dynamic and time varying over the prediction horizon:

$$u_{i|k} = \sum_{j=0}^{i} K_{i|k}^{(j)} x_{j|k} + c_{i|k} \tag{4.24}$$

leading to prediction equations with a lower-triangular structure [18]

$$\mathbf{u}_k = \bar{K}_k \mathbf{x}_k + \mathbf{c}_k$$
$$\bar{K}_k = \begin{bmatrix} K_{0|k}^{(0)} & 0 & \cdots & 0 \\ K_{1|k}^{(0)} & K_{1|k}^{(1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{N-1|k}^{(0)} & K_{N-1|k}^{(1)} & \cdots & K_{N-1|k}^{(N-1)} \end{bmatrix} \tag{4.25}$$

Here $\mathbf{x}_k = (x_{0|k}, \ldots, x_{N|k})$ and $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$ are vectors of predicted states and inputs for the model of (4.1), with $x_{0|k} = x_k$. Also $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ is a vector of feedforward parameters, and the subscript k is used as a reminder that $\bar{K}_k$ and $\mathbf{c}_k$ are to be computed online at each time k.

Using (4.1) the state predictions $\mathbf{x}_k$ can be written in terms of the future control and disturbance inputs, $\mathbf{u}_k$ and $\mathbf{w}_k = (w_{0|k}, \ldots, w_{N-1|k})$, as

$$\mathbf{x}_k = C_{xx} x_k + C_{xu} \mathbf{u}_k + C_{xw} \mathbf{w}_k \tag{4.26}$$

where C_{xx}, C_{xu} and C_{xw}, respectively, denote the convolution matrices from x, $\mathbf{u}$ and $\mathbf{w}$ to $\mathbf{x}$ given by

$$C_{xx} = \begin{bmatrix} I \\ A \\ \vdots \\ A^{N-1} \\ A^N \end{bmatrix}, \; C_{xu} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ B & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A^{N-2}B & \cdots & B & 0 \\ A^{N-1}B & \cdots & AB & B \end{bmatrix}, \; C_{xw} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ D & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A^{N-2}D & \cdots & D & 0 \\ A^{N-1}D & \cdots & AD & D \end{bmatrix}. \tag{4.27}$$

Substituting (4.26) into (4.25) and solving for $\mathbf{u}_k$ yields

$$\begin{aligned} \mathbf{u}_k &= (I - \bar{K}_k C_{xu})^{-1} \bar{K}_k C_{xx} x_k + (I - \bar{K}_k C_{xu})^{-1} \mathbf{c}_k \\ &\quad + (I - \bar{K}_k C_{xu})^{-1} \bar{K}_k C_{xw} \mathbf{w}_k \end{aligned} \tag{4.28a}$$

$$\begin{aligned} \mathbf{x}_k &= \big[C_{xx} + C_{xu}(I - \bar{K}_k C_{xu})^{-1} \bar{K}_k C_{xx}\big] x_k + C_{xu}(I - \bar{K}_k C_{xu})^{-1} \mathbf{c}_k \\ &\quad + \big[C_{xw} + C_{xu}(I - \bar{K}_k C_{xu})^{-1} \bar{K}_k C_{xw}\big] \mathbf{w}_k \end{aligned} \tag{4.28b}$$

where $(I - \bar{K}_k C_{xu})$ is necessarily invertible since it is lower block diagonal with all its diagonal blocks equal to the identity. Thus, for given $\bar{K}_k$, the predicted state and control trajectories are given as affine functions of the current state and the vector of future disturbances. For any given value of $\bar{K}_k$, the worst case disturbance with respect to the constraints (4.4) could be determined by solving a sequence of linear programs similarly to the robust constraint handling approach of Sect. 3.2. However, for the closed-loop optimization strategy considered here, $\bar{K}_k$ is an optimization variable, and Eqs. (4.28a, 4.28b) depend nonlinearly on this variable. As a result, the implied optimization is non-convex and does not therefore lend itself to online implementation.

As mentioned in Sect. 2.10, the state and input predictions can be transformed into linear functions of the optimization variables through the use of a Youla parameter, and this is the route followed by [18]. Thus (4.28a) can be written equivalently as

$$\mathbf{u}_k = \bar{L}_k \mathbf{w}_k + \mathbf{v}_k, \tag{4.29}$$

with

$$\mathbf{v}_k = (I - \bar{K}_k C_{xu})^{-1} \bar{K}_k C_{xx} x_k + (I - \bar{K}_k C_{xu})^{-1} \mathbf{c}_k \tag{4.30a}$$

$$\bar{L}_k = (I - \bar{K}_k C_{xu})^{-1} \bar{K}_k C_{xw} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ L^{(1)}_{1|k} & 0 & \cdots & 0 & 0 \\ L^{(1)}_{2|k} & L^{(2)}_{2|k} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L^{(1)}_{N-1|k} & L^{(2)}_{N-1|k} & \cdots & L^{(N-1)}_{N-1|k} & 0 \end{bmatrix}. \tag{4.30b}$$

The transformation of (4.30a) is bijective, so for each $\mathbf{c}_k$ there exists a unique $\mathbf{v}_k$ and vice versa. Likewise $\bar{L}_k$ is uniquely defined by (4.30b) for arbitrary $\bar{K}_k$. On account of the lower-triangular structure of $\bar{L}_k$, the control policy of (4.29) is necessarily causal and realizable in that $u_{i|k}$ depends on disturbances $w_{0|k}, \ldots, w_{i-1|k}$ that will be known to the controller at the ith prediction time step. The corresponding vector of predicted states assumes the form

$$\mathbf{x}_k = C_{xx} x_k + C_{xu} \mathbf{v}_k + (C_{xw} + C_{xu} \bar{L}_k) \mathbf{w}_k. \tag{4.31}$$

The parameterization of (4.29) implies a disturbance-affine feedback law (e.g. [19, 23]) and is the basis of the disturbance-affine MPC laws proposed for example in [18, 20].

The predictions of (4.29) and (4.31) can be used to impose the constraints of (4.4) over an infinite future horizon by invoking

$$F x_{i|k} + G u_{i|k} \leq \mathbf{1}, \quad i = 0, \ldots, N-1, \tag{4.32a}$$

together with the terminal constraint, $x_{N|k} \in \mathcal{X}_T = \{x : V_T x \leq \mathbf{1}\}$:

$$V_T x_{N|k} \leq \mathbf{1}, \tag{4.32b}$$

where $\mathcal{X}_T$ is RPI for (4.1) and (4.4) under a feedback law $u = Kx$ with a fixed gain K. Here the linear dependence of the constraints (4.32a, 4.32b) on $x_{i|k}$ and $u_{i|k}$ means that they can be expressed in the form

$$\bar{F} \mathbf{x}_k + \bar{G} \mathbf{u}_k \leq \mathbf{1},$$

where $\bar{F}$ and $\bar{G}$ are block diagonal matrices:

$$\bar{F} = \begin{bmatrix} F & & & \\ & \ddots & & \\ & & F & \\ & & & V_T \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} G & & \\ & \ddots & \\ & & G \\ 0 & \cdots & 0 \end{bmatrix}.$$

Using (4.29) and (4.31), these constraints are equivalent to

$$\bar{F}_u \mathbf{v}_k + \max_{\mathbf{w} \in \mathcal{W} \times \cdots \times \mathcal{W}} (\bar{F}_w + \bar{F}_u \bar{L}_k) \mathbf{w} \leq \mathbf{1} - \bar{F}_x x_k,$$

where $\bar{F}_x = \bar{F} C_{xx}$, $\bar{F}_u = \bar{F} C_{xu} + \bar{G}$, $\bar{F}_w = \bar{F} C_{xw}$, and the maximization is performed element wise.

The vertex representation (4.2) of $\mathcal{W}$ allows these constraints to be expressed as a set of linear inequalities in $\mathbf{v}_k$. Thus, in the notation of Sect. 4.1.2, with $\mathbf{w}^{(l)}$ denoting the vector $(w^{(j_1)}, \ldots, w^{(j_N)})$ of vertices of $\mathcal{W}$ for each $l = (j_1, \ldots, j_N) \in \mathcal{L}_N$, the constraints (4.32a, 4.32b) are equivalent to linear constraints in $\mathbf{v}_k$ and $\bar{L}_k$:

$$\bar{F}_u \mathbf{v}_k + (\bar{F}_w + \bar{F}_u \bar{L}_k) \mathbf{w}^{(l)} \leq \mathbf{1} - \bar{F}_x x_k \ \text{ for all } l \in \mathcal{L}_N. \tag{4.33}$$

An alternative constraint formulation that preserves linearity while avoiding the exponential growth in the number of constraints with N that is implied by (4.33) uses convex programming duality to write these constraints equivalently as [20]:

$$H_k \mathbf{1} \le \mathbf{1} - \bar{F}_x x_k - \bar{F}_u \mathbf{v}_k, \quad H_k \ge 0, \quad H_k \bar{V} = \bar{F}_w + \bar{F}_u \bar{L}_k. \tag{4.34}$$

Here $H_k \in \mathbb{R}^{(Nn_c+n_T)\times n_V}$ is a matrix of additional variables in the MPC optimization performed online at times $k = 0, 1, \ldots$ and $\bar{V}$ is a block diagonal matrix containing N diagonal blocks, each of which is equal to V. Therefore, (4.34) constitutes a set of linear constraints in decision variables $\mathbf{v}_k$, $\bar{L}_k$ and H_k, and the total number of these constraints grows linearly with N. The technique used to derive (4.34) from (4.33) and the equivalence of these sets of constraints is discussed in Chap. 5 (see Lemma 5.6).

We next consider the definition of the predicted cost that forms the objective of an MPC strategy employing the constraints (4.34). One possible choice is a nominal cost that assumes all future disturbance inputs to be zero, i.e. $w_{i|k} = 0, i = 0, 1, \ldots$. By combining a quadratic nominal cost with the constraints (4.34), the online MPC optimization can be formulated conveniently as a quadratic program. It can be shown that the resulting MPC law ensures a finite l_2 gain from the disturbance input to the state and control input (see [20] for details), but the implied l_2 gain could be arbitrarily large because the nominal cost contains no information on the feedback gain matrix $\bar{L}_k$. By including in the cost an additional quadratic penalty on elements of $\bar{L}_k$, it is possible to derive stronger stability results [24], in particular the state of the closed-loop system can be shown to converge asymptotically to the minimal RPI set under a specific known linear feedback law. However, this approach relies on using sufficiently large weights in the penalty on $\bar{L}_k$, and we therefore consider here a conceptually simpler min-max approach proposed in [25], which uses the worst case cost considered in Sect. 3.4.

The predicted cost is therefore defined as the maximum, with respect to disturbances $w \in \mathcal{W}$, of a quadratic cost over a horizon of N steps:

$$\check{J}(x_0, \{u_0, u_1, \ldots\}) = \max_{\substack{w_i \in \mathcal{W} \\ i=0,\ldots,N-1}} \sum_{i=0}^{N-1} \left(\|x_i\|_Q^2 + \|u_i\|_R^2 - \gamma^2 \|w_i\|^2\right) + \|x_N\|_{\check{W}_x}^2. \tag{4.35}$$

Here $\check{W}_x$ is the solution of the Riccati equation (3.42), and hence the cost (4.35) is equivalent to the maximum of an infinite horizon cost over $w_i \in \mathcal{W}$ for $i = 0, \ldots, N-1$ and over $w_i \in \mathbb{R}^{n_w}$ for $i \ge N$. We denote the predicted cost evaluated at time k along the trajectories of (4.1) with the feedback strategy of (4.29) as $\check{J}(x_k, \mathbf{v}_k, \bar{L}_k)$:

$$\begin{aligned}\check{J}(x_k, \mathbf{v}_k, \bar{L}_k) = \max_{\mathbf{w}_k \in \{\mathbf{w}: \bar{V}\mathbf{w} \le \mathbf{1}\}} \Big\{ &\|C_{xx} x_k + C_{xu} \mathbf{v}_k + (C_{xw} + C_{xu} \bar{L}_k)\mathbf{w}_k\|_{\bar{Q}}^2 \\ &+ \|\bar{L}_k \mathbf{w}_k + \mathbf{v}_k\|_{\bar{R}}^2 - \gamma^2 \|\mathbf{w}_k\|^2 \Big\}\end{aligned} \tag{4.36}$$

where

$$\bar{Q} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & \check{W}_x \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}.$$

Following [25] and using the approach of Sect. 3.4, the cost (4.36) can be expressed in terms of conditions that are convex in the variables $\mathbf{v}_k$ and $\bar{L}_k$, provided γ is sufficiently large so that $\check{J}(x_k, \mathbf{v}_k, \bar{L}_k)$ is concave in $\mathbf{w}_k$. This concavity condition is equivalent to the requirement that $\Delta \succ 0$, where

$$\Delta \doteq \gamma^2 I - \Big((C_{xw} + C_{xu}\bar{L}_k)^T \bar{Q}(C_{xw} + C_{xu}\bar{L}_k) + \bar{L}_k^T \bar{R}\bar{L}_k\Big).$$

Lemma 4.2 *If $\Delta \succ 0$, then $\check{J}(\mathbf{v}_k, \bar{L}_k) = \min_{\delta,\, \mu \in \{\mu : \mu \geq 0\}} \delta + \mathbf{1}^T \mu$ subject to the following LMI in $\mathbf{v}_k$, $\bar{L}_k$, μ and δ,*

$$\begin{bmatrix} \begin{bmatrix} \delta & \frac{1}{2}\mu^T \bar{V} \\ \bar{V}^T \mu & \gamma^2 I \end{bmatrix} & \begin{bmatrix} (C_{xx}x_k + C_{xu}\mathbf{v}_k)^T \bar{Q}^{1/2} & \mathbf{v}_k^T \bar{R}^{1/2} \\ (C_{xw} + C_{xu}\bar{L}_k)^T \bar{Q}^{1/2} & \bar{L}_k^T \bar{R}^{1/2} \end{bmatrix} \\ \star & I \end{bmatrix} \succeq 0, \tag{4.37}$$

where $\bar{Q}^{1/2}$ and $\bar{R}^{1/2}$ satisfy $(\bar{Q}^{1/2})^T \bar{Q}^{1/2} = \bar{Q}$ and $(\bar{R}^{1/2})^T \bar{R}^{1/2} = \bar{R}$.

Proof If $\Delta \succ 0$, then (4.37) can be shown (by considering Schur complements) to be equivalent to the condition

$$\begin{aligned} \delta \geq\; & \|C_{xx}x_k + C_{xu}\mathbf{v}_k\|_{\bar{Q}}^2 + \|\mathbf{v}_k\|_{\bar{R}}^2 \\ & + \|\tfrac{1}{2}\mu^T \bar{V} - (C_{xx}x_k + C_{xu}\mathbf{v}_k)^T \bar{Q}(C_{xw} + C_{xu}\bar{L}_k) - \mathbf{c}_k^T \bar{R}\bar{L}_k\|_{\Delta^{-1}}^2. \end{aligned} \tag{4.38}$$

Moreover if $\Delta \succ 0$, then the equivalence of the convex QP (3.52) and its dual (3.53) implies that $\check{J}(x_k, \mathbf{v}_k, \bar{L}_k)$ is equal to the minimum of $\delta + \mathbf{1}^T \mu$ over $\mu \geq 0$ and δ subject to (4.38). □

Lemma 4.2 allows the minimization of the worst case cost (4.36) subject to (4.32a, 4.32b) with the feedback strategy of (4.29) to be formulated as a semi-definite program in $O(N^2)$ variables. This is the online optimization that forms the basis of the following disturbance-affine MPC law.

Algorithm 4.4 (*DAMPC*) At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization

$$\underset{\mathbf{v}_k, \bar{L}_k, H_k, \delta_k, \mu_k}{\text{minimize}} \quad \delta_k + \mathbf{1}^T \mu_k \text{ subject to (4.34), (4.37) and } \mu_k \geq 0. \tag{4.39}$$

(ii) Apply the control law $u_k = v^*_{0|k}$, where $\mathbf{v}^*_k = (v^*_{0|k}, \ldots, v^*_{N-1|k})$ and $\bar{L}^*_k$ are the optimal values of $\mathbf{v}_k$, $\bar{L}_k$ in (4.39). ◁

The set of feasible states for the MPC optimization (4.39) is given by[2]

$$\mathcal{F}_N \doteq \{x_k : \exists(\mathbf{v}_k, \bar{L}_k) \text{ such that } \Delta \succ 0 \text{ and (4.34) holds for some } H_k \geq 0\}.$$

For any $N \geq 1$, $\mathcal{F}_N$ can be shown to be RPI under the control law of Algorithm 4.4 by constructing $\mathbf{v}_{k+1}$ and $\bar{L}_{k+1}$ so that conditions (4.32a, 4.32b) and $\Delta \succ 0$ are satisfied at time $k + 1$. Specifically, let

$$\bar{L}_{k+1} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ L^{(2)*}_{2|k} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ L^{(2)*}_{N-1|k} & \cdots & L^{(N-1)*}_{N-1|k} & 0 & 0 \\ KA^{N-2}D & \cdots & KAD & KD & 0 \end{bmatrix}, \quad \mathbf{v}_{k+1} = \begin{bmatrix} v^*_{1|k} + L^{(1)*}_{1|k} w_k \\ v^*_{2|k} + L^{(1)*}_{2|k} w_k \\ \vdots \\ v^*_{N-1|k} + L^{(1)*}_{N-1|k} w_k \\ v_{N|k+1} + KA^{N-1} D w_k \end{bmatrix},$$

with $v_{N|k+1} = KA^N x_k + K[A^{N-1}B \ \cdots \ B]\mathbf{v}^*_k$. Then the sequence of control inputs given by $(u_{0|k+1}, \ldots, u_{N-1|k+1}) = \bar{L}_{k+1}\mathbf{w}_{k+1} + \mathbf{v}_{k+1}$ satisfies

$$u_{i|k+1} = \begin{cases} \tilde{u}_{i+1|k} & i = 0, 1, \ldots, N-2 \\ K\tilde{x}_{N|k} & i = N-1 \end{cases}$$

where $\tilde{u}_{i+1|k}$ and $\tilde{x}_{N|k}$ are elements of the optimal predicted input and state sequences at time k that would be obtained with $w_{0|k}$ equal to the actual disturbance realization, w_k, and which satisfy (4.32a, 4.32b) by construction. It can also be shown [25] that this choice of $\bar{L}_{k+1}$ satisfies

$$\gamma^2 I - (C_{xw} + C_{xu}\bar{L}_{k+1})^T \bar{Q}(C_{xw} + C_{xu}\bar{L}_{k+1}) + \bar{L}^T_{k+1}\bar{R}\bar{L}_{k+1} \succ 0,$$

whenever $\Delta \succ 0$. Therefore $x_{k+1} \in \mathcal{F}_N$ for all $w_k \in \mathcal{W}$, implying that Algorithm 4.4 is recursively feasible.

The MPC law of Algorithm 4.4 guarantees a bound, which depends on the parameter γ, on the l_2 gain from the disturbance input to the state and control input.

Corollary 4.1 *For any $x_0 \in \mathcal{F}_N$ and all $n \geq 0$, the closed-loop trajectories of (4.1) under Algorithm 4.4 satisfy the bound*

$$\sum_{k=0}^{n}\left(\|x_k\|^2_Q + \|u_k\|^2_R\right) \leq \check{J}(x_0, \mathbf{v}^*_0, \bar{L}^*_0) + \gamma^2 \sum_{k=0}^{n} \|w_k\|^2. \tag{4.40}$$

[2] There is no need to include the LMI (4.37) in the conditions defining the feasible set $\mathcal{F}_N$ because δ and $\mu \geq 0$ can always be found satisfying (4.38) (and hence also (4.37)) whenever $\Delta \succ 0$.

Proof This can be shown similarly to the proof of Theorem 3.4. □

Example 4.5 In this example we consider the set of feasible initial conditions and the optimal value of the objective of the DAMPC optimization in step (i) of Algorithm 4.4 for the system dynamics, constraints and terminal set of Example 4.2. The feasible initial condition sets $\mathcal{F}_N$ of DAMPC for $N = 4, 5, \ldots, 10$ are shown in Fig. 4.10 projected onto the plane $\{x : [0\ 1\ 0]x = 0\}$. This figure also shows the projections of $\mathcal{X}^{(N)}$, the N-step controllable sets to $\mathcal{X}_T$, onto the same plane, for the same values of N. Clearly $\mathcal{F}_N$ must be a subset of $\mathcal{X}^{(N)}$ since $\mathcal{X}^{(N)}$ is the largest feasible set for any controller parameterization with a prediction horizon of N. The Figure shows that $\mathcal{F}_N$ provides a good approximation of $\mathcal{X}^{(N)}$ for small values of N, but the approximation accuracy decreases as N increases, and $\mathcal{F}_N$ is much smaller than $\mathcal{X}^{(N)}$ for $N \geq 8$.

The degree of suboptimality of the DAMPC optimization at any given feasible point can be determined by comparing the value of the objective of the DAMPC optimization (4.39) with the optimal cost (4.8). Since the objective of (4.39) is equivalent to the min-max cost (4.35), the optimal value (4.8) is the minimum cost that can be obtained for this problem with any controller parameterization. Table 4.1 shows the percentage suboptimality of (4.39) relative to this ideal optimal cost. The average and maximum suboptimality are given for 50 values of the model state randomly selected on the boundary of $\mathcal{F}_N$ for $N = 10, 11, \ldots, 15$. For this example, the DAMPC opti-

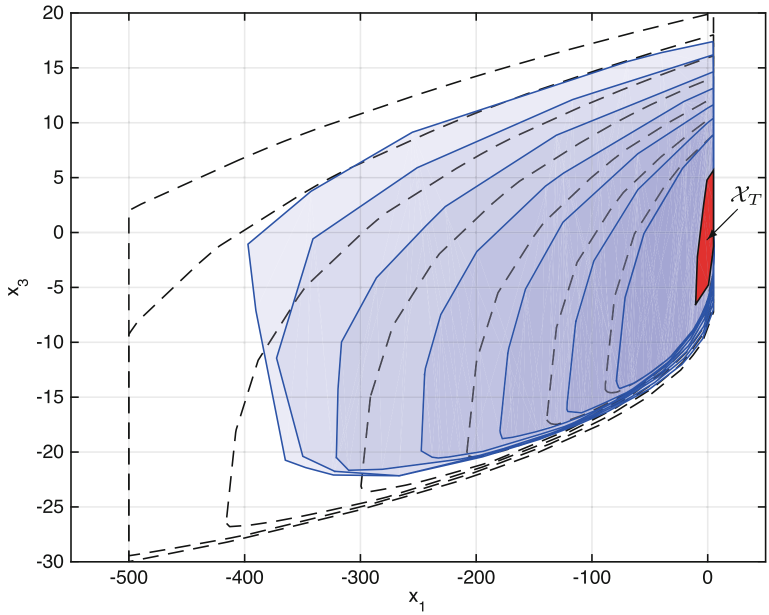

Fig. 4.10 The projections onto the (x_1, x_3)-plane of the feasible sets $\mathcal{F}_N$ for $N = 4, 5, \ldots, 10$ and the terminal set $\mathcal{X}_T$. The *dashed lines* show the projections of the N-step controllable sets to $\mathcal{X}_T$, for $N = 4, 5, \ldots, 10$

Table 4.1 The relative difference between the objective of the DAMPC optimization (4.39) and the ideal optimal cost (4.8) for varying N

Horizon N	Suboptimality	
	Average (%)	Maximum (%)
10	0.1	0.6
11	1.1	6.6
12	2.6	14.0
13	3.8	18.1
14	6.5	31.3
15	9.7	43.3

For each N the average and maximum percentage differences are given for 50 plant states randomly selected on the boundary of the feasible set $\mathcal{F}_N$

mization has negligible suboptimality for $N \leq 10$, but the degree of suboptimality rises quickly as N increases above 10.

To give an indication of the computation required by Algorithm 4.4, the average execution time[3] of (4.39) for $N = 10$ was 0.55 s and the maximum was 0.67 s, whereas for $N = 15$ the execution times were 2.1 s (average) and 2.6 s (maximum). For comparison, solving (4.8) by dynamic programming using Algorithm 4.2 with cold starting required 0.05 s (average) and 0.20 s (maximum) for $N = 10$, and 0.13 s (average) and 0.49 s (maximum) for $N = 15$. Although for this example the online computational requirement of the active set DP implemented by Algorithm 4.2 is considerably less than that of DAMPC, it has to be remembered that DAMPC has a much lower offline computational burden because it does not need the controllable sets $\mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(N)}$ to be determined offline. ♢

We close this section by noting that the disturbance-affine structure of the feedback strategy (4.29) provides compensation for future disturbances that will be known to the controller when the control law is implemented. Clearly it could be advantageous to extend this compensation beyond the future horizon consisting of the first N prediction time steps, and this indeed is proposed in [26]. This approach is cast in the context of stochastic MPC and is described in Chap. 8 (see Sect. 8.2). An application of this idea to a closed-loop prediction strategy that addresses the robust MPC problem considered in this chapter is described in Sect. 4.2.3.

4.2.2 Parameterized Tube MPC

The disturbance-affine feedback strategy considered in Sect. 4.2.1 assumes predicted control inputs with the structure shown in Table 4.2. Although computationally convenient, this parameterization is more restrictive than the general feedback strategy

[3] Execution times are reported here to provide an indication of the computational load—Algorithm 4.4 was implemented using Matlab and Mosek v7.

Table 4.2 The feedback structure of DAMPC

Mode 1					Mode 2	
$v_{0\|k}$	$v_{1\|k}$	$v_{2\|k}$	$\cdots$	$v_{N-1\|k}$	$Kx_{N\|k}$	$\cdots$
	$L^{(1)}_{1\|k}w_{0\|k}$	$L^{(1)}_{2\|k}w_{0\|k}$	$\cdots$	$L^{(1)}_{N-1\|k}w_{0\|k}$		
		$L^{(2)}_{2\|k}w_{1\|k}$	$\cdots$	$L^{(2)}_{N-1\|k}w_{1\|k}$		
			$\ddots$	$\vdots$		
				$L^{(N-1)}_{N-1\|k}w_{N-2\|k}$		
$u_{0\|k}$	$u_{1\|k}$	$u_{2\|k}$	$\cdots$	$u_{N-1\|k}$	$u_{N\|k}$	$\cdots$

The i steps-ahead predicted control input at time k, $u_{i|k}$, is the sum of the entries that lie above the horizontal line in the $(i+1)$th column of the table (zero entries are left blank)

discussed in Sect. 4.1.2 because it forces the predicted control inputs to depend linearly on disturbance inputs. As a result, the i-steps-ahead predicted control input $u_{i|k}$ is determined for all disturbance sequences $(w_{0|k}, \ldots, w_{i-1|k}) \in \mathcal{W} \times \cdots \times \mathcal{W}$ by its values at only $in_w + 1$ vertices of the set $\mathcal{W} \times \cdots \times \mathcal{W}$. On the other hand, the general feedback policy of Sect. 4.1.2 allows $u_{i|k}$ to be chosen independently at each of the m^i vertices of $\mathcal{W} \times \cdots \times \mathcal{W}$ (recall that m is the number of vertices of the disturbance set $\mathcal{W} \subset \mathbb{R}^{n_w}$). Clearly the number of optimization variables required by the general feedback policy must therefore grow exponentially with horizon length (as illustrated for example by the tree structure of Fig. 4.8), making the approach intractable for many problems. However, it is often possible to achieve the same performance as the general feedback strategy, or at least a good approximation of it, using a much more parsimonious parameterization of predicted trajectories.

This is the motivation behind the parameterized tube MPC (PTMPC) formulation of [21], which allows for more general predicted feedback laws than the disturbance-affine strategy of Sect. 4.2.1 while requiring, like DAMPC, a number of optimization variables that grows quadratically with horizon length. PTMPC defines a predicted control tube in terms of the vertices of the sets that form the tube cross sections at each prediction time step over a horizon of N steps. The predicted control trajectories of PTMPC are taken to be convex combinations of these vertices, a subset of which are designated as optimization variables. As is the case for the general feedback strategy in Sect. 4.1.2, the linearity of the system model and constraints (4.1)–(4.4) imply that the conditions for robust constraint satisfaction depend on the vertices of the state and control tubes, but not on the interpolation parameters that define specific trajectories for a given set of vertices. Unlike the general feedback policy, however, PTMPC does not assign an optimization variable to every possible sequence of disturbance vertices over the N-step prediction horizson; instead the predicted state and control tubes are constructed from the Minkowski sum of tubes that model separately the effects of disturbances at individual future time instants.

The decomposition of predicted trajectories into responses to individual disturbances is key to the complexity reduction achieved by PTMPC relative to the general feedback strategy of Sect. 4.1.2, and it also explains why PTMPC cannot in general

Table 4.3 The control tube structure of PTMPC

Mode 1					Mode 2	
$u^{(0)}_{0\|k}$	$u^{(0)}_{1\|k}$	$u^{(0)}_{2\|k}$	$\cdots$	$u^{(0)}_{N-1\|k}$	$Kx^{(0)}_{N\|k}$	$\cdots$
	$U^{(1)}_{1\|k}$	$U^{(1)}_{2\|k}$	$\cdots$	$U^{(1)}_{N-1\|k}$	$KX^{(1)}_{N\|k}$	$\cdots$
		$U^{(2)}_{2\|k}$	$\cdots$	$U^{(2)}_{N-1\|k}$	$KX^{(2)}_{N\|k}$	$\cdots$
			$\ddots$	$\vdots$	$\vdots$	
				$U^{(N-1)}_{N-1\|k}$	$KX^{(N-1)}_{N\|k}$	$\cdots$
					$KX^{(N)}_{N\|k}$	$\cdots$
						$\ddots$
$U_{0\|k}$	$U_{1\|k}$	$U_{2\|k}$	$\cdots$	$U_{N-1\|k}$	$U_{N\|k}$	$\cdots$

The i steps-ahead predicted control input at time k, $u_{i|k}$, is contained in the set $U_{i|k}$, which is formed from the Minkowski sum of all the entries above the horizontal line in the $(i+1)$th column of the table

provide the same performance as control laws that are determined using dynamic programming. This decomposition is illustrated by the triangular tube structure in Table 4.3. For each $i < N$, the set $U_{i|k}$ that defines the tube cross section containing the predicted input $u_{i|k}$ is the (Minkowski) sum of a feedforward term $u^{(0)}_{i|k}$ and sets $U^{(l)}_{i|k}$, $l = 1, 2, \ldots i$:

$$u_{i|k} \in U_{i|k} = \{u^{(0)}_{i|k}\} \oplus U^{(1)}_{i|k} \oplus \cdots \oplus U^{(i)}_{i|k}.$$

Each set $U^{(l)}_{i|k}$ is a compact convex polytope with as many vertices as the disturbance set $\mathcal{W}$:

$$U^{(l)}_{i|k} = \mathrm{Co}\{u^{(l,j)}_{i|k},\ j = 1, \ldots, m\},$$

where the vertices $u^{(l,j)}_{i|k}$, for each $i < N$, $l \le i$ and $j \le m$ are optimization variables. The tube cross section $U_{N|k}$ containing $u_{N|k}$ is similarly given by the sum of sets that are obtained by applying the fixed linear feedback law $u = Kx$ to each of a sequence of state tube cross sections.

The sets $X_{i|k}$ defining the cross sections of the tube containing the predicted state trajectories are decomposed similarly (Table 4.4) into the sum of a nominal predicted state $x^{(0)}_{i|k}$ and sets $\mathcal{W}$ and $X^{(l)}_{i|k}$ for $l = 1, \ldots, i-1$:

$$x_{i|k} \in X_{i|k} = \{x^{(0)}_{i|k}\} \oplus X^{(1)}_{i|k} \oplus \cdots \oplus X^{(i)}_{i|k}. \tag{4.41}$$

Table 4.4 The state tube structure of PTMPC

Mode 1					Mode 2		
$x^{(0)}_{0\|k}$	$x^{(0)}_{1\|k}$	$x^{(0)}_{2\|k}$	$\cdots$	$x^{(0)}_{N-1\|k}$	$x^{(0)}_{N\|k}$	$\Phi x^{(0)}_{N\|k}$	$\cdots$
	$D\mathcal{W}$	$X^{(1)}_{2\|k}$	$\cdots$	$X^{(1)}_{N-1\|k}$	$X^{(1)}_{N\|k}$	$\Phi X^{(1)}_{N\|k}$	$\cdots$
		$D\mathcal{W}$	$\cdots$	$X^{(2)}_{N-1\|k}$	$X^{(2)}_{N\|k}$	$\Phi X^{(2)}_{N\|k}$	$\cdots$
			$\ddots$	$\vdots$	$\vdots$	$\vdots$	
				$D\mathcal{W}$	$X^{(N-1)}_{N\|k}$	$\Phi X^{(N-1)}_{N\|k}$	$\cdots$
					$D\mathcal{W}$	$\Phi D\mathcal{W}$	$\cdots$
						$D\mathcal{W}$	$\cdots$
							$\ddots$
$X_{0\|k}$	$X_{1\|k}$	$X_{2\|k}$	$\cdots$	$X_{N-1\|k}$	$X_{N\|k}$	$X_{N+1\|k}$	$\cdots$

The i steps-ahead predicted state at time k, $x_{i|k}$, is contained in the set $X_{i|k}$, which is formed from the Minkowski sum of all the entries above the horizontal line in the $(i+1)$th column of the table; $\Phi = A + BK$ where K is the feedback gain appearing in Table 4.3

Each $X^{(l)}_{i|k}$ is a compact convex polytopic set defined by its vertices:

$$X^{(l)}_{i|k} = \mathrm{Co}\{x^{(l,j)}_{i|k},\ j = 1, \ldots, m\}.$$

The trajectory of $x^{(0)}_{i|k}$ is determined by the nominal model dynamics

$$x^{(0)}_{i+1|k} = Ax^{(0)}_{i|k} + Bu^{(0)}_{i|k}, \quad i = 0, 1, \ldots, N-1, \tag{4.42a}$$

and the vertices $X^{(l)}_{i|k}$ are paired with those of $U^{(l)}_{i|k}$ so that the predicted state tube evolves as

$$x^{(l,j)}_{l|k} = Dw^{(j)}, \tag{4.42b}$$

$$x^{(l,j)}_{i+1|k} = Ax^{(l,j)}_{i|k} + Bu^{(l,j)}_{i|k}, \quad i = l, l+1, \ldots, N-1, \tag{4.42c}$$

for $j = 1, \ldots, m$. Thus $X^{(i)}_{i|k} = D\mathcal{W}$ and the sets $D\mathcal{W}, X^{(l)}_{l+1|k}, X^{(l)}_{l+2|k}, \ldots$ in the $(l+1)$th row of Table 4.4 account for the effects of the disturbance input $w_{l|k}$ on predicted state trajectories. From (4.42a) to (4.42c), it follows that $x_{i+1|k} \in X_{i+1|k}$ for all $(x_{i|k}, u_{i|k}) \in X_{i|k} \times U_{i|k}$ and $w_{i|k} \in \mathcal{W}$, at each time-step $i = 0, 1, \ldots$.

The feedback policy of DAMPC is a special case of PTMPC since the disturbance affine feedback law of Table 4.2 is obtained if every vertex of the PTMPC control tube is defined as a linear function of the vertices of $\mathcal{W}$, namely if $u^{(l,j)}_{i|k} = L^{(l)}_{i|k} w^{(j)}$ for each i, j, l. In general, however, the feedback policy of PTMPC will be piecewise affine since the control input corresponding to any $x_{i|k}$ lying on the boundary of $X_{i|k}$ is given by a linear combination of vertices of $U_{i|k}$. At points on the boundary of its feasible set in state space therefore, the control law of PTMPC is, like the dynamic programming solution discussed in Sect. 4.1.1, a piecewise affine function of the plant state. However, except for some special cases (discussed in [21]), PTMPC is not equivalent to the general feedback policy of Sect. 4.1 since $u_{i|k}$ is specified by a linear combination of only $im + 1$ rather than m^i free variables.

Before considering the definition of a cost index for PTMPC, we first discuss how to impose the constraints (4.4) on the predicted state and control trajectories of PTMPC for all future disturbance realizations. An equivalent and computationally convenient formulation of the constraints $FX_{i|k} + GU_{i|k} \le \mathbf{1}$ for $i = 0, 1, \ldots, N-1$ is given by

$$Fx^{(0)}_{0|k} + Gu^{(0)}_{0|k} \le \mathbf{1} \tag{4.43a}$$

together with the following conditions for $i = 1 \ldots N-1$,

$$Fx^{(0)}_{i|k} + Gu^{(0)}_{i|k} + \sum_{l=1}^{i} f^{(l)}_{i|k} \le \mathbf{1} \tag{4.43b}$$

$$Fx^{(l,j)}_{i|k} + Gu^{(l,j)}_{i|k} \le f^{(l)}_{i|k}, \quad j = 1, \ldots, m \tag{4.43c}$$

where $f^{(l)}_{i|k}$ for $l = 1, \ldots, i$ are slack variables. The satisfaction of the condition $FX_{N|k} + GU_{N|k} \le \mathbf{1}$ is ensured by the terminal constraint, $X_{N|k} \subseteq \mathcal{X}_T$, where $\mathcal{X}_T$ is a robustly positively invariant set for (4.1), (4.2) and (4.4) under $u = Kx$. Assuming that this terminal set is given by $\mathcal{X}_T = \{x : V_T x \le \mathbf{1}\}$, the conditions for $X_{N|k} \subseteq \mathcal{X}_T$ are equivalent to

$$V_T x^{(0)}_{N|k} + \sum_{l=1}^{i} f^{(l)}_T \le \mathbf{1} \tag{4.44a}$$

$$V_T x^{(l,j)}_{N|k} \le f^{(l)}_T, \quad j = 1, \ldots, m, \tag{4.44b}$$

where $f^{(l)}_T$, $l = 1, \ldots, N$ are slack variables.

Let $\mathbf{u}_k$ denote the vector of optimization variables defining the predicted control tubes:

$$\mathbf{u}_k = \Big\{ (u^{(0)}_{0|k}, \ldots, u^{(0)}_{N-1|k}),\ (u^{(l,j)}_{i|k},\ i = 1, \ldots, N-1,\ l = 1, \ldots, i,\ j = 1, \ldots, m) \Big\}.$$

Then the set $\mathcal{F}_N$ of feasible initial conditions for (4.43a–4.43c) and (4.44a, 4.44b) can be expressed

$$\mathcal{F}_N = \Big\{ x_k : \exists\, \mathbf{u}_k \text{ such that for some } f^{(l)}_{i|k},\ i < N,\ l \le i, \text{ and } f^{(l)}_T,\ l \le N, \\ \text{(4.42a–4.42c), (4.43a–4.43c) and (4.44a, 4.44b) hold with } x^{(0)}_{0|k} = x_k \Big\}. \tag{4.45}$$

Lemma 4.3 *For any $N > 0$, $\mathcal{F}_N$ is RPI for the dynamics (4.1), disturbance set (4.2) and constraints (4.4) if $u_k = u^{(0)}_{0|k}$.*

Proof This can be shown by constructing $\mathbf{u}_{k+1}$ such that the conditions defining $\mathcal{F}_N$ are satisfied at time $k+1$, given $x_k \in \mathcal{F}_N$ and $u_k = u^{(0)}_{0|k}$. Specifically, let $w_k = \sum_{j=1}^m \lambda_j w^{(j)}$ for scalars $\lambda_j \geq 0$ satisfying $\sum_{j=1}^m \lambda_j = 1$, and define

$$u^{(0)}_{i|k+1} = \begin{cases} u^{(0)}_{i+1|k} + \sum_{j=1}^m \lambda_j u^{(1,j)}_{i+1|k}, & i = 0, \ldots, N-2 \\ K x^{(0)}_{N|k} + \sum_{j=1}^m \lambda_j K x^{(1,j)}_{N|k}, & i = N-1 \end{cases} \tag{4.46a}$$

and

$$U^{(l)}_{i|k+1} = \begin{cases} U^{(l+1)}_{i+1|k}, & i = 1, \ldots, N-2, \ \ l = 1, \ldots, i \\ K X^{(l+1)}_{N|k}, & i = N-1, \ \ l = 1, \ldots, N-1 \end{cases} \tag{4.46b}$$

Then the sets $X_{i|k+1} = \{x^{(0)}_{i|k+1}\} \oplus X^{(1)}_{i|k+1} \oplus \cdots \oplus X^{(i-1)}_{i|k+1} \oplus D\mathcal{W}$ generated by the tube dynamics (4.42a, 4.42b) with $x^{(0)}_{0|k+1} = x_{k+1}$ satisfy $X_{i|k+1} \subseteq X_{i+1|k}$ for $i = 0, 1, \ldots, N-1$. Hence conditions (4.43a–4.43c) hold at time $k+1$ by convexity. Moreover (4.44a, 4.44b) hold at $k+1$ because $X_{N|k+1} = \Phi X_{N|k} + D\mathcal{W}$ and $X_{N|k} \in \mathcal{X}_T$ where $\mathcal{X}_T$ is by assumption RPI. □

The parameterized tubes of Tables 4.3 and 4.4 can be combined with various alternative performance indices in order to define recursively feasible receding horizon control laws. For example, a quadratic nominal cost similar to that of Sect. 3.3, which involves only the nominal predicted sequences $x^{(0)}_{i|k}$ and $u^{(0)}_{i|k}$ for $i = 0, 1, \ldots, N-1$ and the nominal terminal state $x^{(0)}_{N|k}$, results in a robustly stabilizing MPC law that ensures a finite l_2 gain from the disturbance input to the closed-loop state and control sequences. This approach is described in the context of a related parameterized tube MPC formulation in [22, 27] and also is discussed in Sect. 4.2.3. Here, however, we consider a performance index that ensures exponential convergence to a predefined target set.

The definition of the vertices of predicted state and control tubes as optimization variables in PTMPC motivates the use of a piecewise linear cost, since in this case the optimal state and control sequences are given by sequences of tube vertices. For example, a piecewise linear stage cost of the form

$$l(x, u) = \|Qx\|_\infty + \|Ru\|_\infty$$

is proposed in [21], where the weighting matrices Q and R are chosen so that the sets $\{x : \|Qx\|_\infty \leq 1\}$ and $\{u : \|Ru\|_\infty \leq 1\}$ are compact and contain the origin in their (non-empty) interiors. This stage cost is used in conjunction with a state decomposition in order to penalize the distance of predicted state trajectories from desired target set in [21]. However, for ease of exposition, we consider here a conceptually simpler approach that is discussed in [22], in which the distance of the predicted state from a target set is evaluated directly through a piecewise linear stage cost.

We denote the target set into which the controller is required to steer the state as $\mathcal{S}$, and we assume $\mathcal{S}$ is a convex polytopic subset of the terminal set $\mathcal{X}_T$ that contains the origin in its (non-empty) interior and has the representation

$$\mathcal{S} = \{x : Hx \le \mathbf{1}\}.$$

We further assume that $\mathcal{S}$ is robustly positively invariant for (4.1) and (4.2) under the linear feedback law $u = Kx$, namely that

$$\Phi\mathcal{S} \oplus D\mathcal{W} \subseteq \mathcal{S}$$

and $Fx + GKx \le \mathbf{1}$ for all $x \in \mathcal{S}$. A suitable target set is provided by the maximal RPI set (determined for example using Theorem 3.1), and as explained below it is convenient to choose $\mathcal{S}$ equal to the terminal set $\mathcal{X}_T$. We define a measure of the distance of a point $x \in \mathbb{R}^{n_x}$ from the set $\mathcal{S}$ as

$$|x|_{\mathcal{S}} \doteq \begin{cases} 0, & \text{if } x \in \mathcal{S} \\ \max\{Hx\} - 1, & \text{otherwise} \end{cases}$$

where $\max\{Hx\} \doteq \max\{H_1 x, H_2 x, \dots, H_{n_H} x\}$ with H_i for $i = 1, \dots, n_H$ denoting the rows of H. A straightforward extension of this definition provides a measure of the distance of a closed set $X \subset \mathbb{R}^{n_x}$ from $\mathcal{S}$ as

$$|X|_{\mathcal{S}} \doteq \max_{x \in X} |x|_{\mathcal{S}}.$$

Note that these measures of distance from a point x to $\mathcal{S}$ and from a set X to $\mathcal{S}$ are consistent with the requirements that $|x|_{\mathcal{S}} = 0$ if and only if $x \in \mathcal{S}$ and $|X|_{\mathcal{S}} = 0$ if and only if $X \subseteq \mathcal{S}$.

The PTMPC predicted cost can be defined in terms of the distances of the cross sections of the predicted state tube from $\mathcal{S}$ as

$$J(x_{0|k}^{(0)}, \mathbf{u}_k) = \sum_{i=0}^{N-1} |X_{i|k}|_{\mathcal{S}} + |X_{N|k}|_{\mathcal{S}_T}. \tag{4.47}$$

The terminal term $|X_{N|k}|_{\mathcal{S}_T}$ is assumed to be defined so that $|x|_{\mathcal{S}_T}$ satisfies, for all $x \in \mathcal{X}_T$,

$$|x|_{\mathcal{S}_T} \ge |\Phi x + Dw|_{\mathcal{S}_T} + |x|_{\mathcal{S}}.$$

This condition requires that $|x|_{\mathcal{S}_T}$ is a piecewise linear Lyapunov function for (4.1) under the feedback law $u = Kx$ for $x \in \mathcal{X}_T$; methods of computing such Lyapunov functions are discussed in [28]. For convenience and without loss of generality, we assume here that $\mathcal{S}$ is equal to the terminal set $\mathcal{X}_T$, in which case $\mathcal{S}_T = \mathcal{S}$ is a valid choice, and the cost of (4.47) reduces to

$$J(x^{(0)}_{0|k}, \mathbf{u}_k) = \sum_{i=0}^{N-1} |X_{i|k}|_{\mathcal{S}} \tag{4.48}$$

as a result of the terminal constraint $X_{N|k} \subseteq \mathcal{X}_T$. Note that terms of the form $|U_{i|k}|_{K\mathcal{S}}$ could also be included in the stage cost in order to place an explicit penalty on the deviation of predicted control sequences from the set $K\mathcal{S} = \{Kx : x \in \mathcal{S}\}$, but for simplicity we consider the cost of (4.47) without this modification.

From the expression for $X_{i|k}$ in (4.41), each term appearing in the cost (4.48) can be expressed as

$$|X_{i|k}|_{\mathcal{S}} = \max\{Hx^{(0)}_{i|k}\} + \sum_{l=1}^{i} \max_j\{Hx^{(l,j)}_{i|k}\} - 1$$

(or $|X_{i|k}|_{\mathcal{S}} = 0$ if this is negative), where $\max_j\{Hx^{(j)}\} \doteq \max_j \max_i\{H_i x^{(j)}\}$. For implementation purposes, each stage cost can therefore be equivalently replaced by a (tight) upper bound given in terms of a slack variable:

$$J(x^{(0)}_{0|k}, \mathbf{u}_k) = \sum_{i=0}^{N-1} d_{i|k}.$$

Here $d_{i|k}$ is a slack variable satisfying the linear constraints

$$\begin{aligned} d_{i|k} &\geq h^{(0)}_{i|k} + \sum_{l=1}^{i} h^{(l)}_{i|k} - 1, \\ d_{i|k} &\geq 0, \end{aligned}$$

where $h^{(l)}_{i|k}$ for $i = 0, \ldots, N-1$ and $l = 0, \ldots, i$ are slack variables satisfying

$$Hx^{(0)}_{i|k} \leq h^{(0)}_{i|k}\mathbf{1}$$

for $i = 0, \ldots, N-1$, and

$$Hx^{(l,j)}_{i|k} \leq h^{(l)}_{i|k}\mathbf{1}, \quad j = 1, \ldots, m$$

for $i = 1, \ldots, N-1$ and $l = 1, \ldots, i$.

We can now state the PTMPC algorithm. This is based on an online optimization which is a linear program involving $O(\frac{1}{2}n_u m N^2)$ optimization variables and $O\big(\frac{1}{2}(n_c+1)N^2\big)$ slack variables, and which has $O\big(\frac{1}{2}(n_c+n_H)mN^2\big)$ linear inequality constraints.

Algorithm 4.5 (*PTMPC*) At each time $k = 0, 1, \ldots$:

(i) If $x_k \notin \mathcal{S}$:

(a) Perform the optimization

$$\begin{aligned} &\underset{\mathbf{u}_k}{\text{minimize}} \quad J(x_k, \mathbf{u}_k) \\ &\text{subject to} \quad (4.42\text{a–}4.42\text{c}), (4.43\text{a–}4.43\text{c}), (4.44\text{a}, 4.44\text{b}). \end{aligned} \tag{4.49}$$

(b) Apply the control law $u_k = u^{(0)*}_{0|k}$, where $\mathbf{u}^*_k = \big\{(u^{(0)*}_{0|k}, \ldots, u^{(0)*}_{N-1|k}), (u^{(l,j)*}_{i|k}, i < N, l \le i, j \le m)\big\}$ is the minimizing argument of (4.49).

(ii) Otherwise (i.e. if $x_k \in \mathcal{S}$), apply the control law $u_k = Kx_k$. ◁

Before discussing closed-loop stability of PTMPC, we first establish that (4.49) is recursively feasible and give a monoticity property of the optimal predicted cost, $J^*(x_k) \doteq J(x_k, \mathbf{u}^*_k)$.

Lemma 4.4 *For the system (4.1) with the control law of Algorithm 4.5 the feasible set $\mathcal{F}_N$ of (4.49) is RPI. Furthermore for any $x_0 \in \mathcal{F}_N$ the optimal objective satisfies, for all $k \ge 0$,*

$$J^*(x_{k+1}) \le J^*(x_k) - |x_k|_{\mathcal{S}}. \tag{4.50}$$

Proof Robust invariance of $\mathcal{F}_N$ under $u_k = u^{(0)*}_{0|k}$ is a direct consequence of Lemma 4.3. Hence the PTMPC optimization (4.49) is feasible at all times $k \ge 1$ if $x_0 \in \mathcal{F}_N$. For $x_k \in \mathcal{S}$, the assumption that $\mathcal{S}$ is RPI implies that (4.50) is trivially satisfied. To derive a bound on the optimal cost $J^*(x_{k+1})$ in terms of $J^*(x_k)$ when $x_k \notin \mathcal{S}$, consider the state tubes that are generated at time $k+1$ by the feasible but suboptimal control tubes given by (4.46a, 4.46b). From (4.42a to 4.42c) we obtain

$$\begin{aligned} x^{(0)}_{i|k+1} &= x^{(0)}_{i+1|k} + \sum_{j=1}^{m} \lambda_j x^{(1,j)}_{i+1|k}, && i = 0, \ldots, N-1 \\ X^{(l)}_{i|k+1} &= X^{(l+1)}_{i+1|k}, && i = 1, \ldots, N-1, \ \ l = 1, \ldots, i \end{aligned}$$

and the corresponding stage cost of (4.48) for $i = 0, \ldots, N-1$ is given by

$$|X_{i|k+1}|_{\mathcal{S}} = \max\Big\{H\big(x^{(0)}_{i+1|k} + \sum_{j=1}^{m} \lambda_j x^{(1,j)}_{i+1|k}\big)\Big\} + \sum_{l=1}^{i} \max_j\{Hx^{(l+1,j)}_{i+1|k}\} - 1$$

(or $|X_{i|k+1}|_{\mathcal{S}} = 0$ if this expression is negative). Moreover, since any pair of vectors a, b necessarily satisfies $\max\{a + b\} \le \max\{a\} + \max\{b\}$, and since $\lambda_j \ge 0$ and $\sum_{j=1}^{m} \lambda_j = 1$, we therefore obtain

$$|X_{i|k+1}|_{\mathcal{S}} \le \max\{Hx^{(0)}_{i+1|k}\} + \sum_{l=1}^{i+1} \max_j\{Hx^{(l,j)}_{i+1|k}\} - 1$$

(or $|X_{i|k+1}|_{\mathcal{S}} = 0$ if this bound is negative). Hence $|X_{i|k+1}|_{\mathcal{S}} \le |X_{i+1|k}|_{\mathcal{S}}$ for all $i = 0, \ldots, N-1$ and the optimal predicted cost at time $k+1$ satisfies $J^*(x_{k+1}) \le \sum_{i=1}^{N-1} |X_{i|k}|_{\mathcal{S}} = J^*(x_k) - |x_k|_{\mathcal{S}}$, where $X_{0|k} = \{x^{(0)}_{0|k}\} = \{x_k\}$ has been used. □

The following result gives the closed-loop stability property of PTMPC.

Theorem 4.2 *Under Algorithm 4.5 the set $\mathcal{S}$ is robustly exponentially stable for the system (4.1), (4.2) and (4.4), with a region of attraction equal to $\mathcal{F}_N$.*

Proof Since $\mathcal{S}$ is RPI for (4.1) under linear feedback $u = Kx$, the optimal value of the cost is given by $J^*(x) = 0$ for all $x \in \mathcal{S}$. Furthermore $J^*(x) \ge |x|_{\mathcal{S}}$ since the stage costs $|X_{i|k}|_{\mathcal{S}}$ for $i \ge 1$ are non-negative. In addition, the optimal cost $J^*(x)$ is the optimal value of a (right-hand side) parametric linear program, and $J^*(x)$ is therefore a continuous piecewise affine function of x on the feasible set $\mathcal{F}_N$ [29]. It follows that constants α, β with $\beta \ge \alpha \ge 1$ exist such that $J^*(x)$ is bounded for all $x \in \mathcal{F}_N$ by

$$\alpha|x|_{\mathcal{S}} \le J^*(x) \le \beta|x_k|_{\mathcal{S}}.$$

If $x_0 \in \mathcal{F}_N$, then from Lemma 4.4 we therefore obtain, for all $k \ge 1$

$$J^*(x_k) \le \left(1 - \frac{1}{\beta}\right)^k J^*(x_0)$$

where $1 - \dfrac{1}{\beta} \in (0, 1)$, and hence the bound

$$|x_k|_{\mathcal{S}} \le \frac{\beta}{\alpha}\left(1 - \frac{1}{\beta}\right)^k |x_0|_{\mathcal{S}} \tag{4.51}$$

holds for all $k \ge 1$. □

Note that the control law of Algorithm 4.5 is a dual mode control law in the strict sense, namely that it implements the linear feedback law $u_k = Kx_k$ whenever $x_k \in \mathcal{S}$. The implied switch to linear feedback is desirable because the minimizing argument of (4.49) is not uniquely defined for all $x_k \in \mathcal{S}$. Furthermore, although it has been designed with the objective of robustly stabilizing the target set $\mathcal{S}$, Algorithm 4.5 can also ensure convergence of the state to the minimal RPI set, $\mathcal{X}^{\text{mRPI}}$, of (4.1) and (4.2) under $u_k = Kx_k$. Specifically, suppose that $\mathcal{S}$ (and hence also the terminal set $\mathcal{X}_T$) is chosen to be a proper subset of the maximal RPI set $\mathcal{X}^{\text{MRPI}}$, and that

Algorithm 4.5 is modified so that the linear feedback law $u = Kx$ is implemented whenever $x_k \in \mathcal{X}^{\mathrm{MRPI}}$. Then Theorem 4.2 implies that x_k converges to $\mathcal{X}^{\mathrm{MRPI}}$ in a finite number of steps and subsequently converges exponentially to $\mathcal{X}^{\mathrm{mRPI}}$.

The online optimization problem of Algorithm 4.5 is an efficiently solvable linear program. Furthermore the number of free variables (including the slack variables) and the number of inequality constraints both depend quadratically with the prediction horizon. Thus PTMPC avoids the exponential growth in computation experienced by a MPC law with the general feedback policy described in Sect. 4.1.2, and its computational requirement is comparable to that of DAMPC. However, the prediction formulation of PTMPC includes that of DAMPC as a special case, and it is therefore to be expected that PTMPC can achieve a larger region of attraction than DAMPC, for the same length of horizon and terminal set. Finally, we note that PTMPC is in general more restricted in the definition of its predicted control trajectories than the general feedback MPC strategy of Sect. 4.1.2 and the active set DP approach of Sect. 4.1.1, and its feasible set for a horizon of N steps is therefore smaller than the N-step controllable set to $\mathcal{X}_T$. The following numerical example illustrates these properties.

Example 4.6 For the system defined by the triple integrator dynamics and the disturbance set and constraints of Example 4.2, the feasible set $\mathcal{F}_N$ of the online optimization of Algorithm 4.5 is shown projected onto the (x_1, x_3)-plane for horizon lengths $4 \le N \le 10$ in Fig. 4.11. The linear feedback gain K and the terminal set

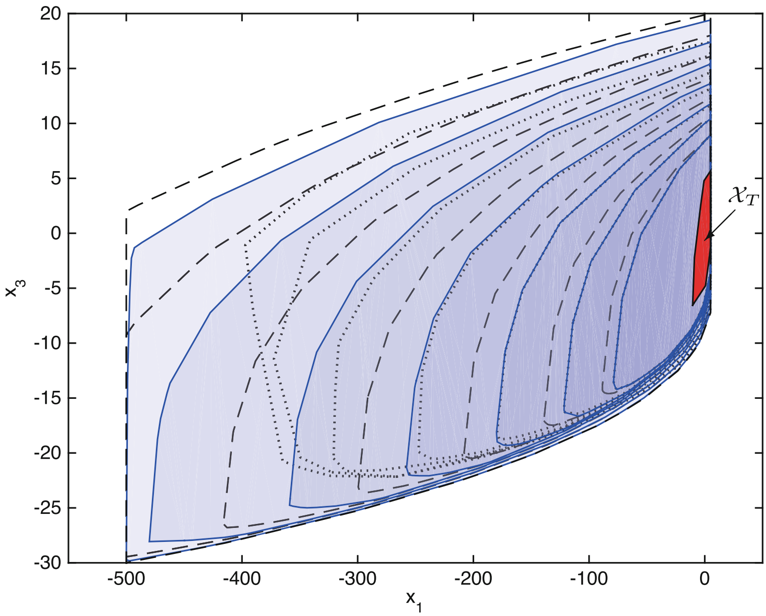

Fig. 4.11 The projections onto the (x_1, x_3)-plane of the feasible sets $\mathcal{F}_N$ for $N = 4, 5, \ldots, 10$ and the terminal set $\mathcal{X}_T$. The *dashed lines* show the projections of the N-step controllable sets to $\mathcal{X}_T$ and the *dotted lines* show the feasible sets for DAMPC

$\mathcal{X}_T$ are defined here as in Examples 4.2, 4.3 and 4.5, and $\mathcal{S}$ is taken to be equal to $\mathcal{X}_T$. It can be seen from this figure that the PTMPC feasible set is smaller than the N-step controllable set to $\mathcal{X}_T$ for each horizon length N in this range. As expected however, the projection of the PTMPC feasible set contains that of DAMPC for each N, and, although they are similar in size for $N \leq 6$, the PTMPC feasible set grows significantly faster than that of DAMPC for $N \geq 8$ in this example.

4.2.3 Parameterized Tube MPC Extension with Striped Structure

The online computational load of PTMPC can be reduced, as reported in [30], by designing some of the slack variables that appear in the constraints of (4.43) and (4.44) offline. Clearly this results in an open-loop strategy, however, and in general will therefore be conservative. An effective alternative is proposed in [22, 27], which uses the same tube parameterization as PTMPC but constrains the component $\{U^{(l)}_{l|k}, U^{(l)}_{l+1|k}, \ldots\}$ of the predicted control tube that appears in the $(l+1)$th row of Table 4.3 to be identical for each row $l = 1, 2, \ldots$. This strategy results in a striped (rather than triangular) structure for both state and control tubes, and for this reason the approach is referred to as Striped PTMPC (SPTMPC).

The key difference in this development is that the free variables defining predicted control tubes are allowed to affect directly the predicted response in both the mode 1 horizon consisting of the first N prediction time steps, and the mode 2 horizon containing the subsequent prediction instants. This can result in a relaxation of the terminal constraints and can thus allow regions of attraction that are larger than those obtained through the use of PTMPC. Furthermore, unlike PTMPC, for which the number of optimization variables and constraints grows quadratically with the prediction horizon, the corresponding growth for SPTMPC is linear, and this can result in significant reductions in computational load. As a consequence, it is possible to further increase the size of the region of attraction of SPTMPC using a longer horizon N while retaining an online computational load no greater than that of PTMPC.

The idea of allowing the effect of partial sequences to persist beyond mode 1 and into the mode 2 horizon was explored in the context the design of controlled invariant sets rather than MPC in [31]. Invariance in this setting was obtained through the use of a contraction variable, which was deployed in a prediction structure with the particular form illustrated in Table 4.5. As for the PTMPC state tubes in Table 4.4, the entries in the table indicate the components the cross sections of predicted state tubes; the corresponding input tubes have a similar structure.

Comparing Tables 4.5 and 4.4, the triangular prediction tube structure employed by PTMPC clearly introduces a greater number degrees of freedom for a given

Table 4.5 Striped prediction tube structure extending into the mode 2 horizon

Mode 1				Mode 2						
$X'_{0\|k}$	$X'_{1\|k}$	$\cdots$	$X'_{N-1\|k}$	$X'_{N\|k}$	$\alpha X'_{0\|k}$	$\cdots$	$\alpha X'_{N-1\|k}$	$\alpha X'_{N\|k}$	$\alpha^2 X'_{0\|k}$	$\cdots$
	$X'_{0\|k}$	$\cdots$	$X'_{N-2\|k}$	$X'_{N-1\|k}$	$X'_{N\|k}$	$\cdots$	$\alpha X'_{N-2\|k}$	$\alpha X'_{N-1\|k}$	$\alpha X'_{N\|k}$	$\cdots$
		$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	
			$X'_{0\|k}$	$X'_{1\|k}$	$X'_{2\|k}$	$\cdots$	$X'_{N\|k}$	$\alpha X'_{1\|k}$	$\alpha X'_{2\|k}$	$\cdots$
				$X'_{0\|k}$	$X'_{1\|k}$	$\cdots$	$X'_{N-1\|k}$	$X'_{N\|k}$	$\alpha X'_{1\|k}$	$\cdots$
					$\ddots$		$\vdots$	$\vdots$	$\vdots$	
$X_{0\|k}$	$X_{1\|k}$	$\cdots$	$X_{N-1\|k}$	$X_{N\|k}$	$X_{N+1\|k}$	$\cdots$	$X_{2N\|k}$	$X_{2N+1\|k}$	$X_{2N+2\|k}$	$\cdots$

The tube cross section $X_{i|k}$ containing $x_{i|k}$ is formed from the Minkowski sum of all the entries above the horizontal line in the $(i+1)$th column of the table. The contraction variable $\alpha \in (0, 1)$ is introduced for the purposes of controller invariance

horizon N, and these are available for expanding the region of attraction. However this is gained at the expense of additional online computation. Also it can be seen from the control input tubes in Table 4.3 that the PTMPC predicted control law in mode 2 assumes the form $u = Kx$, and hence PTMPC makes no degrees of freedom available for direct disturbance compensation over this part of the prediction horizon.

On the other hand, SPTMPC, like [31], allows the optimization variables to directly determine the cross sections of state and control tubes beyond the initial N-step horizon of mode 1. This is can be seen from the predicted state tube structure in Table 4.6 and the predicted control tube structure in Table 4.7. However, rather than repeating the sequence $X'_{1|k}, X'_{2|k}, \ldots, X'_{N|k}$, contracted by α as in [31], SPTPMC

Table 4.6 The state tube structure of SPTMPC which allows for the direct compensation to extend to mode 2

Mode 1					Mode 2		
$x^{(0)}_{0\|k}$	$x^{(0)}_{1\|k}$	$x^{(0)}_{2\|k}$	$\cdots$	$x^{(0)}_{N-1\|k}$	$x^{(0)}_{N\|k}$	$\Phi x^{(0)}_{N\|k}$	$\cdots$
	$\mathcal{DW}$	$X'_{2\|k}$	$\cdots$	$X'_{N-1\|k}$	$X'_{N\|k}$	$\Phi X'_{N\|k}$	$\cdots$
		$\mathcal{DW}$	$\cdots$	$X'_{N-2\|k}$	$X'_{N-1\|k}$	$X'_{N\|k}$	$\cdots$
			$\ddots$	$\vdots$	$\vdots$		
				$\mathcal{DW}$	$X'_{2\|k}$	$X'_{3\|k}$	$\cdots$
					$\mathcal{DW}$	$X'_{2\|k}$	$\cdots$
						$\ddots$	
$X_{0\|k}$	$X_{1\|k}$	$X_{2\|k}$	$\cdots$	$X_{N-1\|k}$	$X_{N\|k}$	$X_{N+1\|k}$	$\cdots$

The predicted state $x_{i|k}$ lies in the tube cross section $X_{i|k}$ formed from the Minkowski sum of all entries above the horizontal line in the $(i+1)$th column of the table

Table 4.7 The control tube structure of SPTMPC

Mode 1					Mode 2		
$u^{(0)}_{0\|k}$	$u^{(0)}_{1\|k}$	$u^{(0)}_{2\|k}$	$\cdots$	$u^{(0)}_{N-1\|k}$	$Kx^{(0)}_{N\|k}$	$K\Phi x^{(0)}_{N\|k}$	$\cdots$
	$U'_{1\|k}$	$U'_{2\|k}$	$\cdots$	$U'_{N-1\|k}$	$KX'_{N\|k}$	$K\Phi X'_{N\|k}$	$\cdots$
		$U'_{1\|k}$	$\cdots$	$U'_{N-2\|k}$	$U'_{N-1\|k}$	$KX'_{N\|k}$	$\cdots$
			$\ddots$	$\vdots$	$\vdots$		
				$U'_{1\|k}$	$U'_{2\|k}$	$U'_{3\|k}$	$\cdots$
					$U'_{1\|k}$	$U'_{2\|k}$	$\cdots$
						$\ddots$	
$U_{0\|k}$	$U_{1\|k}$	$U_{2\|k}$	$\cdots$	$U_{N-1\|k}$	$U_{N\|k}$	$U_{N+1\|k}$	$\cdots$

allows the predicted state and control tubes to decay in mode 2 through the closed-loop dynamics of (4.1) under $u = Kx$. Thus the state and control tube cross sections at prediction time i can be expressed

$$x_{i|k} \in X_{i|k} = x^{(0)}_{i|k} \oplus \bigoplus_{j=1}^{i} X'_{j|k}$$

$$u_{i|k} \in U_{i|k} = u^{(0)}_{i|k} \oplus \bigoplus_{j=1}^{i} U'_{j|k},$$

where $\mathcal{X}'_{1|k} = D\mathcal{W}$ and for $j = N, N+1, \ldots,$

$$X'_{j|k} = \Phi^{j-N} X'_{N|k}$$
$$U'_{j|k} = K\Phi^{j-N} X'_{N|k}.$$

State and input constraints are imposed on the predicted tubes of SPTMPC through the introduction of slack variables, similarly to the handling of constraints in PTMPC. However, since $X'_{1|k}, \ldots, X'_{N|k}$ appear at every prediction time step of the infinite mode 2 horizon, it is clear that a different terminal constraint is required in order to obtain a guarantee of recursive feasibility. In [22, 27], the set of feasible initial states is made invariant through the use of a supplementary horizon, N_2, within mode 2, over which system constraints are invoked under the mode 2 dynamics and a pair of terminal constraints.

For given N_2, the terminal conditions can be expressed

$$\Phi^{N_2} x^{(0)}_{N|k} \in \mathcal{X}_0 \tag{4.52}$$

and

$$\Phi^{N_2} X'_{N|k} \subseteq \mathcal{X}_1, \tag{4.53}$$

where $\mathcal{X}_0, \mathcal{X}_1$ are polytopic sets that contain the origin in their interior, and where $\mathcal{X}_0$ is RPI for the dynamics $x_{k+1} = \Phi x_k + \xi_k$ for all $\xi_k \in \Phi \mathcal{X}_1$, namely

$$\Phi \mathcal{X}_0 \oplus \Phi \mathcal{X}_1 \subseteq \mathcal{X}_0. \tag{4.54}$$

From (4.52) and (4.54), it follows that $\Phi^{N_2+i} x^{(0)}_{N|k} \subseteq \mathcal{X}_0$ for $i = 1, 2, \ldots$, and if (4.53) also holds, then (4.52) will be invariant for the system (4.1) and (4.2) under $u_k = u^{(0)}_{0|k}$ in the sense that, for all $w_k \in \mathcal{W}$,

$$\Phi^{N_2} x^{(0)}_{N|k+1} \in \mathcal{X}_0 \tag{4.55}$$

The reason (4.55) is implied by the conditions (4.52) and (4.53) and the property (4.54) is that (similarly to the proof of Lemma 4.3) a feasible nominal predicted state trajectory at time $k+1$ is given by the sum of the first row of Table 4.6 and a trajectory contained in the tube defined by the second row. This feasible nominal trajectory is given for $i = 0, 1, \ldots,$ by

$$x^{(0)}_{i|k+1} = \begin{cases} x^{(0)}_{i+1|k} + \sum_{j=1}^{m} \lambda_j x'^{(j)}_{i+1|k}, & i = 0, \ldots, N-1 \\ \Phi^{i+1-N}\Big(x^{(0)}_{N|k} + \sum_{j=1}^{m} \lambda_j x'^{(j)}_{N|k}\Big), & i = N, N+1, \ldots \end{cases}$$

where $\mathrm{Co}\{x'^{(1)}_{i|k}, \ldots, x'^{(m)}_{i|k}\} = X'_{i|k}$ and λ_j are non-negative scalars satisfying $w_k = \sum_{j=1}^{m} \lambda_j w^{(j)}$ and $\sum_{j=1}^{m} \lambda_j = 1$. Therefore

$$x^{(0)}_{N|k+1} \in \{\Phi x^{(0)}_{N|k}\} \oplus \Phi X'_{N|k}$$

so that (4.52) and (4.53) imply $\Phi^{N_2} x^{(0)}_{N|k+1} \in \Phi \mathcal{X}_0 \oplus \Phi \mathcal{X}_1$, and (4.55) then follows from (4.54).

With the conditions (4.52) and (4.53), it is now possible to formulate a condition that guarantees constraint satisfaction at all prediction times $i \geq N + N_2$, and which ensures recursive feasibility. To do this, we consider the components of the tube cross section X_{N+N_2+r}, for arbitrary $r \geq 0$, as specified by the terms appearing in the corresponding column of Table 4.6. Writing the entries in this column as a sequence:

$$\underbrace{\Phi^{N_2+r} x^{(0)}_{N|k}}_{\text{(A)}},\ \underbrace{\Phi^{N_2+r} X'_{N|k}, \ldots, \Phi^{N_2} X'_{N|k}}_{\text{(B)}},\ \underbrace{\Phi^{N_2-1} X'_{N|k}, \ldots, \Phi X'_{N|k}, X'_{N|k} \ldots X'_{1|k}}_{\text{(C)}}$$

we find that the first term (labelled A) lies in $\mathcal{X}_0$ by (4.52) and (4.54), while the block labelled C is common to the $(N + N_2 + r)$th column of the table for all $r \geq 0$. Thus the only challenge is presented by the block labelled B and to deal with this, we make use of the following bound

$$\bigoplus_{r=1}^{i} \Phi^{N_2+r} X'_{N|k} \subseteq \Omega_\infty(\mathcal{X}_1) \tag{4.56}$$

where $\Omega_\infty(\mathcal{X}_1)$ denotes a RPI set for the dynamics of $x_{k+1} = \Phi x_k + w_k$ with $w_k \in \mathcal{X}_1$. This is a direct consequence of (4.52), which implies that $\Phi^{N_2+r} X'_{N|k} \subset \Phi^r \mathcal{X}_1$. In this setting, it is convenient to represent the constraints $Fx + Gu \leq \mathbf{1}$, for $u = Kx$ in the format of a set inclusion:

$$x \in \mathcal{X} = \{x : (F + GK)x \leq \mathbf{1}\}. \tag{4.57}$$

Then, given the sets $\mathcal{X}_0$, $\mathcal{X}_1$ and $\Omega_\infty(\mathcal{X}_1)$, the condition

$$\mathcal{X}_0 \oplus \bigoplus_{i=1}^{N} X'_{i|k} \oplus \bigoplus_{i=1}^{N_2-1} \Phi^i X'_{N|k} \oplus \Omega_\infty(\mathcal{X}_1) \subseteq \mathcal{X} \tag{4.58}$$

ensures satisfaction of (4.57) at all prediction instants $i \geq N + N_2$ by construction. Furthermore condition (4.58) is invariant in the sense that it will be feasible at $k + 1$ if it is satisfied at time k.

In the following analysis, we make the assumption that the minimal robustly positively invariant set, $\mathcal{X}^{\mathrm{mRPI}}$, for the system $x_{k+1} = Ax_k + Bu_k + w_k$ with $u_k = Kx_k$ and $w_k \in \mathcal{W}$ lies in the interior of $\mathcal{X}$. Note that $\mathcal{X}^{\mathrm{mRPI}}$ must lie inside $\mathcal{X}$ in order that the linear feedback law $u = Kx$ is feasible in some neighbourhood of the origin; hence this is a mild assumption to make. In addition it ensures that (4.58) is necessarily feasible for sufficiently large N, N_2.

Lemma 4.5 *There exist N, N_2, $\mathcal{X}_0$, $\mathcal{X}_1$ such that (4.58) is feasible.*

Proof This result can be proved by construction. Thus let $\mathcal{X}_1 = \Phi^{N+N_2-1}\mathcal{W}$ and suppose that the control law in both mode 1 and mode 2 is chosen to be $u = Kx$. For this selection, (4.58) becomes

$$\mathcal{X}_0 \oplus \mathcal{X}^{\mathrm{mRPI}} \subseteq \mathcal{X} \tag{4.59}$$

which, by assumption, will be feasible for sufficiently small $\mathcal{X}_0$. Such a choice for $\mathcal{X}_0$ is possible provided $N + N_2$ is chosen to be large enough. □

Note that the constraint (4.58) can be relaxed through the use of smaller $\mathcal{X}_0$, but at the same time the constraint (4.52) becomes more stringent. A sensible compromise between these two effects can be reached by introducing a tuning parameter α and defining $\mathcal{X}_0$ by

$$\mathcal{X}_0 = \alpha\tilde{\mathcal{X}} \oplus \tilde{\Omega}_\infty(\Phi\mathcal{X}_1) \tag{4.60}$$

where $\tilde{\mathcal{X}}$ is an invariant set for the dynamics $x_{k+1} = \Phi x_k$ and where $\tilde{\Omega}_\infty(\Phi\mathcal{X}_1)$ is the minimal invariant set (or an invariant outer approximation) for the dynamics $x_{k+1} = \Phi x_k + \xi_k$, $\xi_k \in \Phi\mathcal{X}_1$.

We are now able to construct a robust MPC strategy with the objective of steering the state to the minimal robust invariant set $\mathcal{X}^{\mathrm{mRPI}}$. This is achieved by replacing (4.58) with the condition

$$\mathcal{X}_0 \oplus \bigoplus_{i=1}^{N} X'_{i|k} \oplus \bigoplus_{i=1}^{N_2-1} \Phi^i X'_{N|k} \oplus \Omega_\infty(\mathcal{X}_1) \subseteq \mathcal{S} \tag{4.61}$$

where $\mathcal{S}$ is a robustly positively invariant polytopic set the system (4.1) and (4.2) and constraints under $u = Kx$.

We define the online objective function to penalize the distance of each of the tube cross sections, $X_{i|k}$, $i = 0, 1, \ldots$, from $\mathcal{S}$. Since (4.61) ensures $X_{i|k} \subseteq \mathcal{S}$ for all $i \geq N + N_2$, this cost has the form

$$J(x^{(0)}_{0|k}, \mathbf{u}_k) = \sum_{i=0}^{N+N_2-1} |X_{i|k}|_{\mathcal{S}}.$$

As in the case of the cost for the PTMPC algorithm discussed in Sect. 4.2.2, this cost is non-negative and its minimum over the optimization variables $\mathbf{u}_k = \{(u^{(0)}_{0|k}, \ldots, u^{(0)}_{N-1}), (U'_{1|k}, \ldots, U'_{N+N_2-1|k})\}$ is zero if and only if $x^{(0)}_{0|k} \in \mathcal{S}$. The minimization of this cost therefore forms the basis of a receding horizon strategy for steering the state of (4.1) into $\mathcal{S}$ while robustly satisfying constraints.

For $\mathcal{S}$ described by the inequalities $\mathcal{S} = \{x : Hx \leq \mathbf{1}\}$, the minimization of the cost $J(x, \mathbf{u}_k)$ can be performed by minimizing a sum of slack variables. Using again the approach of Sect. 4.2.2 we have

$$J(x^{(0)}_{0|k}, \mathbf{u}_k) = \sum_{i=0}^{N+N_2-1} d_{i|k}$$

where for $i = 0, \ldots, N + N_2 - 1$, the parameters $d_{i|k}$ satisfy

$$d_{i|k} \geq h^{(0)}_{i|k} + \sum_{l=1}^{i} h'_{i|k} - 1,$$
$$d_{i|k} \geq 0,$$

and $h^{(0)}_{i|k}$, $h'_{i|k}$ satisfy

$$h^{(0)}_{i|k}\mathbf{1} \geq \begin{cases} Hx^{(0)}_{i|k}, & i = 0, \ldots, N-1 \\ H\Phi^{i-N}x^{(0)}_{N|k}, & i = N, \ldots, N+N_2-1 \end{cases}$$

$$h'_{i|k}\mathbf{1} \geq \begin{cases} Hx'^{(j)}_{i|k}, & i = 1, \ldots, N-1, \quad j = 1, \ldots, m \\ H\Phi^{i-N}x'^{(j)}_{N|k}, & i = N, \ldots, N+N_2-1, \quad j = 1, \ldots, m \end{cases}$$

As discussed for the case of PTMPC in Sect. 4.2.2, the constraints of (4.4) at prediction times $i = 0, \ldots, N + N_2 - 1$ can be invoked through the use of slack variables. Likewise the constraints of (4.52), (4.53) and (4.61) constitute a set of linear inequalities that can be implemented using slack variables.

Algorithm 4.6 (*SPTMPC*) At each time $k = 0, 1, \ldots$:

(i) If $x_k \notin \mathcal{S}$:

(a) Perform the optimization

$$\begin{aligned} &\underset{\mathbf{u}_k}{\text{minimize}} \quad J(x_k, \mathbf{u}_k) \\ &\text{subject to} \quad FX_{i|k} + GU_{i|k} \leq \mathbf{1}, \; i = 0, \ldots, N + N_2 - 1 \\ &\qquad\qquad\quad (4.52), (4.53) \text{ and } (4.61). \end{aligned} \tag{4.62}$$

(b) Apply the control law $u_k = u^{(0)*}_{0|k}$, where $\mathbf{u}^*_k = \{(u^{(0)*}_{0|k}, \ldots, u^{(0)*}_{N-1|k}), (U'^*_{1|k}, \ldots, U'^*_{N-1|k})\}$ is the minimizing argument of (4.62).

(ii) Otherwise (i.e. if $x_k \in \mathcal{S}$), apply the control law $u_k = Kx_k$. ◁

The closed-loop stability properties of Algorithm 4.6 can be stated in terms of the set of feasible states x_k for the online optimization (4.62), which we denote as $\mathcal{F}_{N,N_2}$.

Corollary 4.2 *For the system (4.1) and (4.2), constraints (4.4), and the control law of Algorithm 4.6, the feasible set $\mathcal{F}_{N,N_2}$ is RPI and $\mathcal{S}$ is exponentially stable with region of attraction equal to $\mathcal{F}_{N,N_2}$.*

Proof By construction the constraints of (4.62) are recursively feasible, namely feasibility at time $k = 0$ implies feasibility at all times $k = 1, 2, \ldots$. Exponential stability of $\mathcal{S}$ can be demonstrated using the same arguments as in the proofs of Lemma 4.4 and Theorem 4.2. In particular, the bound (4.50) holds along closed-loop trajectories and exponential convergence (4.51) therefore holds. □

Note that the state of (4.1) converges exponentially to the minimal RPI set $\mathcal{X}^{\text{mRPI}}$ under $u = Kx$ if $\mathcal{S}$ is chosen to be a proper subset of the RPI set on which the control law of Algorithm 4.6 switches to $u = Kx$.

The structure of SPTMPC indicated in Table 4.6 allows for only a single sequence $X'_{1|k}, \ldots, X'_{N|k}$ to be used in the definition of the predicted partial tubes. However, if it is desired to introduce more degrees of freedom into the SPTMPC algorithm, then it would be possible to implement a hybrid scheme. For example, this could be realized by introducing tubes $X^{(l)}_{1|k}, \ldots, X^{(l)}_{N|k}$, for $l = 1, \ldots, \nu$, with the full triangular structure of PTMPC into the upper rows of Table 4.6 before switching to a fixed sequence $X'_{1|k}, \ldots, X'_{N|k}$ to generate a striped tube structure in the remainder of the table. The implied algorithm can be shown to inherit the feasibility and stability properties of SPTMPC.

The advantage of SPTMPC over PTMPC is that it allows for a reduction of the online computational load and, at the same time, extends disturbance compensation into mode 2. Using the same prediction horizons for PTMPC and SPTMPC is likely to result in the constraints of PTMPC that apply to the mode 1 prediction horizon being less stringent than those of SPTMPC on account of the extra degrees of freedom introduced by PTMPC. For SPTMPC, however, the constraints of mode 2 are likely to be less stringent than for PTMPC, since SPTMPC allows for direct disturbance compensation in mode 2. In general, it is not possible to state which of the two methodologies will result in the larger region of attraction. We give next an illustrative example showing an instance of STMPC outperforming PTMPC.

Example 4.7 Consider the system defined by the model (4.1) with parameters

$$A = \begin{bmatrix} 0.787 & 1.02 \\ -0.93 & 1.03 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.331 \\ -1.01 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

and disturbance set

$$\mathcal{W} = \left\{ w : \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right.$$

The constraints (4.4) are given by

$$\begin{aligned} \{(x, u) : \ \pm[-0.044 \ \ 0.092]x &\le 1 \\ \pm[0.009 \ \ 0.093]x &\le 1 \\ \pm u &\le 1\} \end{aligned}$$

The set Ω_0 is constructed using (4.60), with

$$\Omega_0 = 0.01\tilde{X} \oplus \tilde{\Omega}_\infty(\Phi \mathcal{X}_1)$$

and to ensure satisfaction of (4.58), we take $N_2 = 5$.

The areas of the domains of attraction are given in Table 4.8 for three variant strategies: (i) the SPTMPC strategy of Algorithm 4.6; the SPTMPC strategy with a nominal cost, and with the constraint (4.61) replaced with (4.58); (iii) PTMPC. The table also gives the numbers of online optimization variables, numbers of equality constraints and numbers of inequality constraints for each algorithm.

Table 4.8 Areas of domains of attraction and numbers of variables, inequality constraints and inequality constraints for: (i) SPTMPC (Algorithm 4.6); (ii) SPTMPC (Algorithm 4.6 with nominal cost); (iii) PTMPC (Algorithm 4.5)

N	A_N			Variables			Inequalities			Equalities		
	(i)	(ii)	(iii)	(i)	(ii)	(iii)	(i)	(ii)	(iii)	(i)	(ii)	(iii)
1	2.38	2.38	2.38	93	67	24	296	186	61	2	2	10
2	3.01	2.87	3.01	120	89	71	347	216	158	12	12	28
3	3.57	3.57	3.66	147	111	144	398	246	303	22	22	54
4	4.44	4.53	4.19	174	133	243	449	276	496	32	32	88
5	5.02	5.14	4.59	201	155	368	500	306	737	42	42	130
6	5.25	5.39	4.88	228	177	519	551	336	1026	52	52	180
7	5.44	5.83	5.13	255	199	696	602	366	1363	62	62	238
8	5.57	5.87	5.34	282	221	899	653	396	1748	72	72	304
9	5.66	5.95	5.48	309	243	1128	704	426	2181	82	82	378
10	5.72	5.95	5.59	336	265	1383	755	456	2662	92	92	460

For the same value of N both variants of SPTMPC yield larger domains of attraction than PTMPC when $N \geq 4$. This is largely a result of the disturbance compensation that SPTMPC provides in mode 2. The full triangular structure of PTMPC is more general and can therefore outperform SPTMPC. However, the price of this triangular structure is that it implies a number of online optimization variables that grows quadratically with N, whereas the number of optimization variables required by SPTMPC grows only linearly with N. Thus SPTMPC can use longer horizons, thereby enlarging the size of the domain of attraction, at a computational cost which is still less than that required by PTMPC. For example, while for $N < 4$ both variants of SPTMPC have more variables and inequality constraints, for $N \geq 4$ the SPTMPC algorithms both provide a larger domain of attraction while using fewer variables and constraints. In particular, both variants of SPTMPC have fewer variables with $N = 10$ than PTMPC with $N = 5$.

The SPTMPC strategy with a nominal performance index does not constrain the predicted state to lie inside $\mathcal{S}$ at any instant and hence it achieves larger domains of attraction with fewer variables than Algorithm 4.6. However, the latter enjoys stronger stability properties which guarantee convergence to the minimal RPI set $\mathcal{X}^{\mathrm{mRPI}}$. ◊

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Chapter 5
Robust MPC for Multiplicative and Mixed Uncertainty

In this chapter, we consider constrained linear systems with imprecisely known parameters, namely systems that are subject to multiplicative uncertainty. Although of real concern in most applications, the unknown additive disturbances considered in Chaps. 3 and 4 are not the only form of uncertainty that may be present in a system model. Even if additive disturbances are not present, it is rarely the case that a linear time-invariant model is able to reproduce exactly the behaviour of a physical system. This may be a consequence of imprecise knowledge of model parameters, or it may result from inherent uncertainty in the system in the form of stochastic parameter variations. Model error can also arise through the use of reduced order models that neglect the high-order dynamics of the system. Moreover, the system dynamics may be linear but time-varying, or they may be nonlinear, or both time-varying and nonlinear.

In all such cases, it may be possible to capture the key features of system behaviour, albeit conservatively, using a linear model whose parameters are assumed to lie in a given uncertainty set. Thus for example in some instances it is possible to capture the behaviour of an uncertain nonlinear system using a linear difference inclusion (LDI). A linear difference inclusion is a discrete-time system consisting of a family of linear models, the parameters of which belong to the convex hull of a known set of vertices. An LDI model has the property that the trajectories of the actual system are contained in the convex hull of the trajectories that are generated by the linear models defined by these vertices (e.g. [1]).

Robust MPC techniques for systems subject to constraints and multiplicative model uncertainty are the focus of this chapter. As in earlier chapters, we examine receding horizon control methodologies that guarantee feasibility and stability while ensuring bounds on closed-loop performance. Also following earlier chapters, the emphasis is on computationally tractable approaches. Towards the end of the chapter, consideration is given to the situation most prevalent in practice, in which systems are subject to a mix of multiplicative uncertainty and unknown additive disturbances.

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5.1 Problem Formulation

Early work on MPC for systems with multiplicative uncertainty was of a heuristic nature and considered the case of a finite impulse response (FIR) model containing an uncertain scaling factor [2]. More general uncertain impulse response models in which the vector of impulse response coefficients is assumed to lie, at each instant, in a known polytopic set were considered in [3, 4]. These approaches posed the robust MPC problem in a min–max framework and hence minimized, over the trajectory of predicted future inputs, the maximum over all model uncertainty of a predicted cost. By defining the predicted cost in terms of the infinity-norms of tracking errors, the MPC optimization could be expressed as a linear program, with objective and constraints depending linearly on the optimization variables.

The computational requirement of the approach of [3] grows exponentially with the number of vertices used to model uncertainty. This growth can be reduced considerably by introducing slack variables, as proposed in [4], but the method remains restricted to FIR models. The convenience of FIR system descriptions is that, as mentioned in Chap. 2, they enable the implicit implementation of equality terminal conditions and for this reason FIRs have proved popular in the analysis and design of robust MPC methods for the case of multiplicative model uncertainty (e.g. [5, 6]).

To avoid excessive computational demands that make online optimization impracticable, an impulse response model must be truncated to a finite impulse response with a limited number of coefficients. Such truncation can render the model unrealistic, particularly when slow poles are present in the system dynamics. It is also clear that approaches based on FIR representations can only be applied to systems that are open-loop stable (although for the purposes of prediction alone this limitation may be overcome through the use of bicausal FIR models [7]). For these reasons, subsequent developments considered more general linear state-space models:

$$x_{k+1} = A_k x_k + B_k u_k \tag{5.1}$$

where $x_k \in \mathbb{R}^{n_x}$ and $u_k \in \mathbb{R}^{n_u}$ denote the system state and control input.

The parameters (A_k, B_k) of (5.1) are assumed to belong for all k to a convex compact polytopic set, Ω, defined by the convex hull of a known set of vertices

$$\Omega = \text{Co}\{(A^{(1)}, B^{(1)}), \ldots, (A^{(m)}, B^{(m)})\}.$$

Therefore

$$(A_k, B_k) = \sum_{j=1}^{m} q_k^{(j)} \big(A^{(j)}, B^{(i)}\big) \tag{5.2}$$

for some scalars $q_k^{(1)}, \ldots, q_k^{(m)}$ that are unknown at time k and which satisfy

$$\sum_{j=1}^{m} q_k^{(j)} = 1 \;\text{ and }\; q_k^{(j)} \geq 0, \quad j = 1, \ldots, m.$$

As in earlier chapters, the state and control inputs are assumed to be subject to linear constraints:

$$Fx_k + Gu_k \leq \mathbf{1}, \tag{5.3}$$

for given $F \in \mathbb{R}^{n_C \times n_x}$, $G \in \mathbb{R}^{n_C \times n_u}$, and we consider receding horizon strategies that aim to minimize predicted performance expressed as a sum of quadratic stage costs over a future horizon. The predicted cost may be defined in terms of either a nominal cost or a worst-case cost. A nominal cost is based on a nominal system description given by a known set of model parameters $(A^{(0)}, B^{(0)})$. Suitable nominal parameters could be defined, for example, by the expected value of the parameter set Ω or alternatively its centroid,

$$\big(A^{(0)}, B^{(0)}\big) = \frac{1}{m}\sum_{j=1}^{m}\big(A^{(j)}, B^{(j)}\big). \tag{5.4}$$

Conversely, worst-case performance is defined as the maximum value of the predicted cost over all model parameters in the uncertainty set Ω.

This chapter summarizes significant developments in robust MPC methodologies that provide guarantees of feasibility and stability over the entire uncertainty class Ω. We first discuss the approach of [8], which is based on Linear Matrix Inequalities (LMIs) and does not employ a mode 1 horizon but handles the cost and constraints implicitly through recursive quadratic bounds. Later work [9] reduced the conservativeness of this approach by constructing state tubes that take into account all possible predicted realizations of model uncertainty. This made it possible to introduce a mode 1 horizon over which the predicted cost and constraints are handled explicitly. However, [9] uses tubes that are minimal in the sense that the tube cross sections provide tight bounds on the predicted model states (i.e. every point in the tube cross section is a future predicted state for some realization of model uncertainty). This makes the computational complexity of the approach depend exponentially on the length of the mode 1 prediction horizon.

Computation can be greatly reduced, at the expense of a degree of conservativeness, if ellipsoidal bounds are used to invoke the constraints on predicted states and control inputs. This is the approach taken in [10, 11], where predicted state and input trajectories are generated by a dynamic feedback controller, and the degrees of freedom in predictions are incorporated as additional states of the prediction model. Related work [12] showed how to optimize the prediction system dynamics, similarly to the approach described in Sect. 2.9, so as to maximize ellipsoidal regions of attraction in robust MPC. An alternative form of optimized dynamics was introduced in [13] in order to improve robustness to multiplicative model uncertainty.

The conservativeness that is inherent in enforcing constraints using ellipsoidal tubes can be reduced or even avoided altogether using tubes with variable polytopic cross sections to bound predicted trajectories. Early work on this restricted tube cross sections to low-complexity polytopic sets (e.g. [14–17]). The approach was subsequently extended using an application of Farkas' Lemma to allow general polytopic sets [18–20].

After describing these developments, the chapter concludes by discussing MPC for systems that are subject to both multiplicative model uncertainty and additive disturbances [21–23]. In this case, the model of (5.1) and (5.2) becomes

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k \tag{5.5}$$

with polytopic uncertainty descriptions,

$$(A_k, B_k, D_k) \in \mathrm{Co}\{(A^{(1)}, B^{(1)}, D^{(1)}), \ldots, (A^{(m)}, B^{(m)}, D^{(m)})\}, \tag{5.6a}$$

$$w_k \in \mathrm{Co}\{w^{(1)}, \ldots, w^{(q)}\}, \tag{5.6b}$$

where the vertices $(A^{(j)}, B^{(j)}, D^{(j)})$ and $q^{(l)}$, $j = 1, \ldots, m$, $l = 1, \ldots, q$ are assumed to be known. As in earlier chapters, the nominal value of the additive disturbance is taken to be zero.

5.2 Linear Matrix Inequalities in Robust MPC

Linear matrix inequalities were encountered in earlier chapters in the context of ellipsoidal constraint approximations (Sects. 2.7 and 2.9) and worst-case quadratic performance indices (Sects. 3.4 and 4.2). We begin this section by considering analogous LMI conditions that ensure constraint satisfaction and performance bounds for systems subject to multiplicative model uncertainty. The simplest setting for this is the robust MPC strategy of [8], in which the predicted control trajectories are generated by a linear feedback law:

$$u_{i|k} = K_k x_{i|k}, \tag{5.7}$$

where $u_{i|k}$ and $x_{i|k}$ are the predictions at time k of the i steps ahead input and state variables. The corresponding predicted state trajectories satisfy

$$x_{i|k} = (A_{i|k} + B_{i|k} K_k) x_{i|k}, \quad (A_{i|k}, B_{i|k}) \in \Omega \tag{5.8}$$

for all $i \geq 0$.

The convexity property of linear matrix inequalities discussed in Sect. 2.7.3 is particularly useful in the context of robust MPC. Consider for example the function

$$M(x) \doteq M_0 + M_1 x_1 + \cdots + M_n x_n,$$

where $M_0, \dots, M_n$ are given matrices, and suppose that x assumes values in a given convex compact polytope $\mathcal{X}$. Then $M(x) \succeq 0$ for all $x \in \mathcal{X}$ if and only if $M(x) \succeq 0$ holds at the vertices of $\mathcal{X}$, namely

$$M(x) \succ 0,\ \forall x \in \mathrm{Co}\{x^{(1)}, \dots, x^{(m)}\} \quad \Longleftrightarrow \quad M(x^{(j)}) \succ 0,\ j = 1, \dots, m.$$

This property of LMIs makes it possible to express in terms of the vertices of the uncertainty set Ω the conditions under which an ellipsoidal set, defined for $P \succ 0$ by

$$\mathcal{E} \doteq \{x : \|x\|_P^2 \le 1\},$$

is robustly positively invariant for the system (5.8). Clearly, $x_{i+1|k} \in \mathcal{E}$ for all $x_{i|k} \in \mathcal{E}$ if and only if $\|(A + BK_k)x\|_P^2 \le \|x\|_P^2$ for all $(A, B) \in \Omega$ and all $x \in \mathbb{R}^{n_x}$, or equivalently

$$P - (A + BK_k)^T P(A + BK_k) \succeq 0,\ \forall (A, B) \in \Omega. \tag{5.9}$$

Although the dependence of (5.9) on the uncertain parameters (A, B) is quadratic, this condition can be rewritten in terms of linear conditions through the use of Schur complements, which are defined for general partitioned matrices in Sect. 2.7.3. Since $P \succ 0$, and hence also $P^{-1} \succ 0$, a necessary and sufficient condition for the quadratic inequality (5.9) is therefore

$$\begin{bmatrix} P & (A + BK_k)^T \\ A + BK_k & P^{-1} \end{bmatrix} \succeq 0, \quad \forall (A, B) \in \Omega,$$

and the linear dependence of this condition on $A + BK_k$ implies that it is equivalent to the conditions

$$\begin{bmatrix} P & (A^{(j)} + B^{(j)}K_k)^T \\ A^{(j)} + B^{(j)}K_k & P^{-1} \end{bmatrix} \succeq 0, \quad j = 1, \dots, m \tag{5.10}$$

involving only the vertices of Ω.

We next consider conditions for robust satisfaction of the constraints (5.3). Following the approach of Sect. 2.7.1, a necessary and sufficient condition for the constraints (5.3) to hold under (5.7) for all $x_{i|k} \in \mathcal{E} = \{x : \|x\|_P^2 \le 1\}$ is that there exists a symmetric matrix H satisfying

$$\begin{bmatrix} H & F + GK_k \\ (F + GK_k)^T & P \end{bmatrix} \succeq 0, \quad e_i^T He_i \le 1, \quad i = 1, 2, \dots, n_C \tag{5.11}$$

where e_i is the ith column of the identity matrix. This can be shown using the argument that was used in the proof of Theorem 2.9, namely that, for $x \in \mathcal{E}$, the ith element of $(F + GK_k)x$ has upper bound

$$e_i^T (F + GK_k)x \le \|(F + GK_k)^T e_i\|_{P^{-1}}, \tag{5.12}$$

and hence the Schur complements of (5.11) ensure that $(F + GK_k)x \le \mathbf{1}$. Conversely, the bound (5.12) is satisfied with equality for some $x \in \mathcal{E}$, so if there exists no H satisfying (5.11), then $(F + GK_k)x \nleq \mathbf{1}$ for some $x \in \mathcal{E}$. If $\mathcal{E}$ is robustly invariant, then clearly (5.11) ensures that the constraints of (5.3) hold along all predicted trajectories of (5.7) and (5.8) starting from any initial state $x_{0|k}$ lying in $\mathcal{E}$.

A cost index forming the objective of a robust MPC law with guaranteed stability can be constructed from a suitable upper bound on the worst-case predicted cost, which is defined for given weights $Q \succ 0$ and $R \succ 0$ by

$$\check{J}(x_{0|k}, K_k) \doteq \max_{(A_{i|k}, B_{i|k}) \in \Omega,\, i=0,1,\ldots} \sum_{i=0}^{\infty} \bigl(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\bigr). \tag{5.13}$$

For this purpose [8] derived bounds on (5.13) using a quadratic function that, for computational convenience, was determined by strengthening the conditions defining the invariant set $\mathcal{E}$. Following a similar approach, if (5.9) is replaced by the condition

$$P - (A + BK_k)^T P (A + BK_k) \succeq \gamma^{-1}(Q + K_k^T R K_k), \quad \forall (A, B) \in \Omega \tag{5.14}$$

for some scalar $\gamma > 0$, then the cost of (5.13) necessarily satisfies the bound

$$\check{J}(x_{0|k}, K_k) \le \gamma, \quad \forall x_{0|k} \in \mathcal{E} = \{x : \|x\|_P^2 \le 1\}. \tag{5.15}$$

This is shown by the following lemma, which uses the variable transformations

$$P = S^{-1}, \quad K_k = Y S^{-1} \tag{5.16}$$

to express (5.14) equivalently in terms of LMIs in variables Y, S and γ.

Lemma 5.1 *If $Y \in \mathbb{R}^{n_u \times n_x}$, symmetric $S \in \mathbb{R}^{n_x \times n_x}$ and scalar γ satisfy*

$$\begin{bmatrix} \begin{bmatrix} S & (A^{(j)}S + B^{(j)}Y)^T \\ A^{(j)}S + B^{(j)}Y & S \end{bmatrix} & \begin{bmatrix} SQ^{1/2} & Y^T R^{1/2} \\ 0 & 0 \end{bmatrix} \\ \star & \gamma I \end{bmatrix} \succeq 0 \tag{5.17}$$

for $j = 1, \ldots, m$, then the bound (5.15) holds with $P = S^{-1}$ and $K_k = YS^{-1}$.

Proof Condition (5.17) is linear in the parameters $A^{(j)}, B^{(j)}$. By considering all convex combinations of the matrix appearing on the LHS of (5.17), it can therefore be shown that (5.17) holds with $(A^{(j)}, B^{(j)})$ replaced by any $(A, B) \in \Omega$. Hence, by Schur complements, (5.17) implies $S \succ 0$, $\gamma > 0$ and

$$S - (AS + BY)^T S^{-1}(AS + BY) \succeq \gamma^{-1}(SQS + Y^T RY), \quad \forall (A, B) \in \Omega.$$

Pre- and post-multiplying both sides of this inequality by P and using (5.16), we obtain (5.14), so from (5.7) it follows that

$$\|x_{i|k}\|_P^2 - \|Ax_{i|k} + Bu_{i|k}\|_P^2 \geq \gamma^{-1}\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big), \ \forall (A, B) \in \Omega,$$

Summing over all $i \geq 0$ and making use of (5.8), we therefore obtain

$$\check{J}(x_{0|k}, K_k) \leq \gamma \|x_{0|k}\|_P^2,$$

which implies $\check{J}(x_{0|k}, K_k) \leq \gamma$ for all $x_{0|k} \in \mathcal{E}$. □

The cost bound provided by (5.15) could be conservative because it is constant on the boundary of the invariant set $\mathcal{E} = \{x : \|x\|_P^2 \leq 1\}$. Clearly, this may conflict with the requirement that it provides a tight bound on the sublevel sets of the cost (5.13) since the invariant set was constructed in order to approximate the set of feasible initial conditions. On the boundary of the feasible set in particular, (5.15) is likely to be a very conservative cost bound. To reduce the level of suboptimality therefore, the parameters (K_k, P, γ) are assigned as variables in the MPC optimization and recomputed online at each time step $k = 0, 1, \ldots$. The lemma below establishes the optimality conditions for the feedback gain K_k.

Lemma 5.2 *If the following optimization is feasible,*

$$(Y_k^*, S_k^*, \gamma_k^*) = \arg \min_{\substack{S=S^T \succ 0, \\ H=H^T \succ 0, \\ Y, \gamma}} \gamma \tag{5.18a}$$

subject to (5.17) and, for $x = x_k$ and some symmetric H, the conditions

$$\begin{bmatrix} 1 & x^T \\ x & S \end{bmatrix} \succeq 0, \tag{5.18b}$$

$$\begin{bmatrix} H & FS + GY \\ (FS + GY)^T & S \end{bmatrix} \succeq 0, \quad e_i^T H e_i \leq 1, \quad i = 1, \ldots, n_C \tag{5.18c}$$

then, with $K_k = Y_k^*(S_k^*)^{-1}$, the predicted trajectories of (5.7) and (5.8) satisfy $\check{J}(x_{0|k}, K_k) \leq \gamma_k^*$ and meet the constraints $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$ for all $i \geq 0$.

Proof Using Schur complements, (5.18b) implies $x_k \in \mathcal{E}$, so by Lemma 5.1 the optimal objective in (5.18a) is an upper bound on the predicted cost $\check{J}(x_k, K_k)$. On the other hand, pre- and post-multiplying (5.18c) by the block diagonal matrix $\operatorname{diag}\{I, P\}$ (which is unitary since (5.18b) implies $P \succ 0$) yields (5.11), and hence (5.18c) ensures $(F + GK_k)x_{i|k} \leq \mathbf{1}$ for all $i \geq 0$. □

The optimization of Lemma 5.2 is the basis of the following robust MPC strategy, which requires the online solution of an SDP in $O(n_x^2)$ variables.

Algorithm 5.1 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization of Lemma 5.2.
(ii) Apply the control law $u_k = K_k x_k$. ◁

Theorem 5.1 *For the system (5.1)–(5.3) and control law of Algorithm 5.1:*

(a) The optimization in step (i) is feasible for all times $k > 0$ *if it is feasible at time* $k = 0$.
(b) The origin of the state space of (5.1) is robustly asymptotically stable with region of attraction

$$\mathcal{F} \doteq \{x \in \mathbb{R}^{n_x} : \textit{(5.17), (5.18b), (5.18c) are feasible}\}, \tag{5.19}$$

and for $x_0 \in \mathcal{F}$ *the trajectories of the closed-loop system satisfy the bound*

$$\sum_{k=0}^{\infty} \big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) \le \gamma_0^*. \tag{5.20}$$

Proof We first demonstrate that, if the optimization of Lemma 5.2 is feasible at time k, then a feasible solution at time $k+1$ is given by

$$(Y, S, \gamma) = (\alpha Y_k^*, \alpha S_k^*, \alpha \gamma_k^*) \tag{5.21}$$

where $\alpha = \|x_{k+1}\|_{P_k}^2$ for $P_k = (S_k^*)^{-1}$. Considering each of the constraints (5.17), (5.18b) and (5.18c) at time $k+1$:

- With $S^{-1} = \alpha^{-1} P_k$, we obtain $\|x_{k+1}\|_{S^{-1}}^2 = \alpha^{-1}\|x_{k+1}\|_{P_k}^2 = 1$, and hence (5.18b) is necessarily satisfied with $x = x_{k+1}$.
- If $(Y_k^*, S_k^*, \gamma_k^*)$ is feasible for (5.17) then, since the inequality in (5.17) is unchanged if the matrix on the LHS is multiplied by $\alpha > 0$, the solution (Y, S, γ) in (5.21) must be feasible for (5.17), for any $\alpha > 0$.
- The feasibility of (5.17) and (5.18b) at time k implies $\|x_{k+1}\|_{P_k}^2 \le \|x_k\|_{P_k}^2$ and hence $\alpha \le 1$. Therefore, if (Y_k^*, S_k^*) satisfies (5.18c) with $H = H_k$, then (Y, S) in (5.21) must also satisfy (5.18c) with $H = \alpha H_k$.

It follows that the optimization in step (i) of Algorithm 5.1 is recursively feasible since if x_0 lies in the feasible set $\mathcal{F}$, then $x_k \in \mathcal{F}$ for all $k \ge 1$.

Turning next to closed-loop stability, from the optimality of the solution at time $k+1$ and (5.17) we have

$$\begin{aligned}\gamma_{k+1}^* \le \alpha\gamma_k^* &= \gamma_k^* \|x_{k+1}\|_{P_k}^2 \\ &\le \gamma_k^* \|x_k\|_{P_k}^2 - \big(\|x_k\|_Q^2 + \|u_k\|_R^2\big),\end{aligned}$$

but the constraint (5.18b) is necessarily active at the optimal solution, so $\|x_k\|_{P_k} = 1$ and

$$\gamma_{k+1}^* - \gamma_k^* \leq -\big(\|x_k\|_Q^2 + \|u_k\|_R^2\big).$$

This inequality implies that $x = 0$ is stable because $\gamma_k^* \geq \check{J}(x_k, K_k)$, while (5.13) implies $\check{J}(x, K)$ is positive definite in x since $Q \succ 0$. Summing both sides of this inequality over $k = 0, 1, \dots$ yields the bound (5.20), which implies that $(\|x_k\|_Q^2 + \|u_k\|_R^2) \to 0$ as $k \to \infty$ for any $x_0 \in \mathcal{F}$, and hence $\lim_{k\to\infty}(x_k, u_k) = (0, 0)$, since Q and R are positive definite matrices. □

The online optimization posed in Algorithm 5.1 constitutes a convex program (see e.g. [24]) which can be solved efficiently (in polynomial time) using semidefinite programming solvers. However it should be noted that its computational burden increases considerably with the system dimension. For fast sampling applications (such as applications involving electromechanical systems), this algorithm is therefore viable only for small-scale models.

Example 5.1 An uncertain system is described by the model (5.1)–(5.2) with parameters

$$A^{(1)} = \begin{bmatrix} -0.7 & 0.15 \\ -0.35 & -0.6 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} -0.75 & -0.1 \\ 0.15 & -0.65 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} -0.65 & -0.35 \\ -0.1 & -0.55 \end{bmatrix}$$
$$B^{(1)} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} 0.2 \\ 1.4 \end{bmatrix}, \quad B^{(3)} = \begin{bmatrix} 0.3 \\ 0.6 \end{bmatrix}.$$

The system is subject to state and input constraints

$$\begin{aligned} -10 &\leq \begin{bmatrix} 0 & 1 \end{bmatrix} x_k \leq 10 \\ -5 &\leq u_k \leq 5 \end{aligned}$$

which are equivalent to (5.3) with

$$F = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ -0.2 \end{bmatrix}.$$

Figure 5.1 shows the feasible set, $\mathcal{F}$, for Algorithm 5.1. This set is formed from a union of ellipsoidal sets, and for this example is clearly not itself ellipsoidal. Figure 5.1 also shows that $\mathcal{F}$ contains points that lie outside the largest area ellipsoidal set that is robustly invariant under any given linear feedback law. This is to be expected since $\mathcal{F}$ contains every ellipsoidal set that is RPI for (5.1)–(5.3) under a linear feedback law.

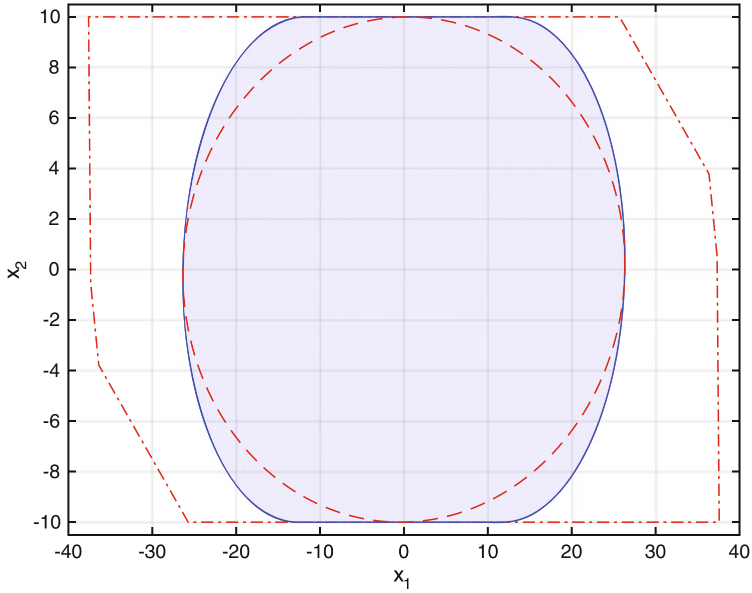

Fig. 5.1 The set of feasible initial states for Algorithm 5.1 applied to the uncertain system of Example 5.1 (*solid line*). For comparison, this figure also shows the maximal ellipsoidal RPI set under any linear feedback law (*dashed line*) and the maximal robustly controlled invariant set (*dash-dotted line*)

For 100 values of x evenly distributed on the dashed ellipsoid in Fig. 5.1, the average optimal value of the online MPC optimization (5.18) is 11,779 and the maximum is 69,505. However, the value of the minimum quadratic bound on the worst-case cost computed for the solution of (5.18) and averaged over this set of initial conditions is 939 and the corresponding maximum value is 1660. The discrepancy between these worst-case cost bounds is a result of the conservativeness of the cost bound that forms the objective of (5.18), which as previously discussed is constructed by scaling the ellipsoidal set $\mathcal{E}$ in order to ensure that the objective (5.18a) is an upper bound on the worst-case cost. $\lozenge$

5.2.1 Dual Mode Predictions

By the criteria of Chaps. 3 and 4, Algorithm 5.1 is a feedback MPC strategy since the predicted control trajectories depend on the realization of future uncertainty through a feedback gain computed online. Despite this, the parameterization of predicted trajectories in terms of a single linear feedback gain over the prediction horizon can be restrictive. Furthermore, for computational convenience the cost and constraints are approximated in Algorithm 5.1 using potentially conservative quadratic bounds.

A conceptually straightforward way to avoid these shortcomings is to extend Algorithm 5.1 by adjoining a mode 1 horizon containing additional degrees of freedom over which the cost and constraints are evaluated explicitly. This approach is proposed in [9], in which the predicted control trajectory is specified as

$$u_{i|k} = K_{i|k}x_{i|k} + c_{i|k} \quad \begin{cases} K_{i|k} = K_{k+i}, & i = 0, \ldots, N-1 \\ K_{i|k} = K_{N|k},\ c_{i|k} = 0, & i = N, N+1, \ldots \end{cases} \tag{5.22}$$

where $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ and $K_{N|k}$ are optimization variables at time k. In order to be able to make use of the conditions of Lemmas 5.1 and 5.2 for the computation of $K_{N|k}$ while retaining a convex online optimization, the feedback gains $K_k, \ldots, K_{N-1+k}$ are fixed at time k, their values being carried over from the optimization at a previous time instant.

A polytopic tube $\mathcal{X}_{0|k}, \ldots, \mathcal{X}_{N|k}$ containing the future state trajectories for all realizations of model uncertainty can be defined in terms of the vertices, $v_{i|k}^{(l)}$, of $\mathcal{X}_{i|k}$:

$$\mathcal{X}_{i|k} = \mathrm{Co}\{v_{i|k}^{(l)},\ l = 1, \ldots, m^i\},$$

where $\mathcal{X}_{0|k} = \{x_k\}$ and $x_{i|k} \in \mathcal{X}_{i|k}$. The tube cross sections $\mathcal{X}_{i|k}$ for $i = 1, \ldots, N$, are computed in [9] using the recursion

$$v_{i+1|k}^{(q)} = (A^{(j)} + B^{(j)}K_{k+i})v_{i|k}^{(l)} + B^{(j)}c_{i|k} \tag{5.23}$$

for $j = 1, \ldots, m$, $l = 1, \ldots, m^i$ and $q = 1, \ldots, m^{i+1}$. Clearly the number of vertices defining the tube cross sections increases exponentially with the length of the prediction horizon in this approach, limiting the approach to short horizons. The reason for this very rapid growth in complexity is that state tubes defined in this way are minimal in the sense that $\mathcal{X}_{i|k}$ is the smallest set containing the i-steps ahead predicted state $x_{i|k}$ for all possible realizations of the model uncertainty.

Let $\boldsymbol{\gamma}_k \doteq (\gamma_{0|k}, \ldots, \gamma_{N|k})$ denote a sequence of upper bounds on the predicted stage costs of the worst-case predicted performance index (5.13):

$$\gamma_{i|k} \geq \|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2.$$

By expressing these bounds as LMIs in the predicted states, they can be imposed for all $x_{i|k} \in \mathcal{X}_{i|k}$ through conditions on the vertices $v_{i|k}^{(l)}$, namely for $i = 1, \ldots, N-1$:

$$\begin{bmatrix} \gamma_{i|k} & \begin{bmatrix} v_{i|k}^{(l)\,T} Q^{1/2} & (K_{k+i}v_{i|k}^{(l)} + c_{i|k})^T R^{1/2} \end{bmatrix} \\ \star & I \end{bmatrix} \succeq 0, \quad l = 1, \ldots, m^i. \tag{5.24}$$

For all $i \geq N$, the predicted control inputs are defined in (5.22) by the feedback law $u_{i|k} = K_{N|k}x_{i|k}$, and an upper bound on the predicted cost over the mode 2 prediction horizon can be computed using the approach of Sect. 5.2. In the current context, however, the N-step ahead predicted state, $x_{N|k}$, is not known exactly, but instead is known to lie in the polytopic set $\mathcal{X}_{N|k}$. An upper bound on the cost to go over the mode 2 prediction horizon is therefore obtained by invoking the LMI of (5.18b) at each of the vertices defining $\mathcal{X}_{N|k}$:

$$\begin{bmatrix} 1 & v_{N|k}^{(l)\,T} \\ v_{N|k}^{(l)} & S \end{bmatrix} \succeq 0, \quad l = 1, \ldots, m^N. \tag{5.25}$$

Then, by an obvious extension of Lemma 5.1, the infinite horizon cost of (5.13) satisfies the bound

$$\check{J}(x_k, \mathbf{c}_k, K_{N|k}) \le \sum_{i=0}^{N} \gamma_{i|k} \tag{5.26}$$

if (5.17) is invoked with $\gamma = \gamma_{N|k}$ and $K_{N|k} = YS^{-1}$.

Using the arguments of Lemma 5.2, it can be shown that (5.18c) ensures constraint satisfaction in mode 2, whereas in mode 1, given the linear nature of the constraints (5.3), a necessary and sufficient condition is that

$$(F + GK_{k+i})v_{i|k}^{(l)} + Gc_{i|k} \le \mathbf{1}, \quad l = 1, \ldots, m^i, \;\; i = 0, \ldots, N-1. \tag{5.27}$$

On the basis of this development, it is possible to state the following robust MPC algorithm, which requires the online solution of a SDP problem involving $O(n_x m^N)$ variables.

Algorithm 5.2 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{\mathbf{c}_k, Y, S, \gamma_k}{\text{minimize}} \sum_{i=0}^{N} \gamma_{i|k} \;\text{ subject to (5.24), (5.25), (5.27),}$$
$$\text{(5.17) with } \gamma = \gamma_{N|k} \text{ and (5.18c)} \tag{5.28}$$

(ii) Apply the control law $u_k = K_k x_k + c^*_{0|k}$ and set $K_{k+N} = Y^*_k (S^*_k)^{-1}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$, S^*_k and Y^*_k are, respectively, the optimal values of $\mathbf{c}_k$, S and Y in (5.28). ◁

This algorithm must be initialized at $k = 0$ by computing the feedback gains $K_0, \ldots, K_{N-1}$ offline. Assuming knowledge of the initial state x_0 (or alternatively knowledge of a polytopic set $\mathrm{Co}\{v_{0|0}^{(1)}, \ldots, v_{0|0}^{(r)}\}$ containing x_0), this can be done by performing the optimization of Lemma 5.2 with $x = x_0$ in (5.18b) (or alternatively performing this optimization with

$$\begin{bmatrix} 1 & v_{0|0}^{(l)\,T} \\ v_{0|k}^{(l)} & S \end{bmatrix} \succeq 0, \quad l = 1, \ldots, m^N.$$

in place of (5.18b)) and by setting $K_i = K_0$ for all $i = 0, \ldots, N-1$.

Theorem 5.2 *Algorithm 5.2 is recursively feasible and robustly asymptotically stabilizes the origin of the state space of the system (5.1)–(5.3).*

Proof Assuming feasibility at time k, a feasible but suboptimal solution to (5.28) at time $k+1$ is given by

$$\begin{aligned} \mathbf{c}_{k+1} &= (c_{1|k}^*, \ldots, c_{N-1|k}^*, 0), \\ \gamma_{k+1} &= \Bigl(\gamma_{1|k}^*, \ldots, \gamma_{N-1|k}^*, \max_{x \in \mathcal{X}_{N|k}} (\|x\|_Q^2 + \|K_{k+N}x\|_R^2), \alpha\gamma_{N|k}^*\Bigr), \\ Y &= \alpha Y_k^*, \quad S = \alpha S_k^*, \quad \alpha = \max_{x \in \mathcal{X}_{N|k}, j \in \{1,\ldots,m\}} \|(A^{(j)} + B^{(j)}K_{k+N})x\|_{P_k}^2 \end{aligned}$$

where $P_k = (S_k^*)^{-1}$. Feasibility of the constraints (5.24) and (5.27) follows directly from the inclusion property $\mathcal{X}_{i|k+1} \subseteq \mathcal{X}_{i+1|k}$, $i = 0, \ldots, N-1$, while feasibility of (5.25), (5.17) with $\gamma = \alpha\gamma_{N|k}^*$ and (5.18b) follows from the argument used in the proof of Theorem 5.1 and the property that $\mathcal{X}_{N|k+1} \subseteq \mathrm{Co}\{(A^{(j)} + B^{(j)}K_{k+N})\mathcal{X}_{N|k}, j = 1, \ldots, m\}$, as a consequence of $u_{N|k+1} = K_{k+N}x_{N|k+1}$ and $\mathcal{X}_{N-1|k+1} \subseteq \mathcal{X}_{N|k}$. The proof of Theorem 5.1 also shows that $\alpha\gamma_{N|k}^* \le \gamma_{N|k}^* - (\|x\|_Q^2 + \|K_{k+N}x\|_R^2)$ for all $x \in \mathcal{X}_{N|k}$. Hence the sum of the elements of γ_{k+1} is no greater than $\sum_{i=1}^N \gamma_{i|k}^*$, and optimality at time $k+1$ therefore implies

$$\sum_{i=0}^N \gamma_{i|k+1}^* \le \sum_{i=1}^N \gamma_{i|k}^* = \sum_{i=0}^N \gamma_{i|k}^* - (\|x_k\|_Q^2 + \|u_k\|_R^2).$$

Since the bound (5.26) and $Q \succ 0$ and $R \succ 0$ imply that $\sum_{i=0}^N \gamma_{i|k}^*$ is positive definite in x_k, it follows that $x = 0$ is asymptotically stable. □

5.3 Prediction Dynamics in Robust MPC

In the presence of multiplicative uncertainty, robust MPC algorithms employing minimal tubes to bound predicted trajectories suffer from the same disadvantage as Algorithm 5.2, namely that the number of constraints, and hence also the computational demand, grows rapidly (in general, exponentially) with the prediction horizon N. Therefore the use of minimal tubes in this context is generally impractical for anything other than short horizons and descriptions of model uncertainty with small numbers of vertices. To avoid this problem, it is necessary to bound predicted state and control trajectories using non-minimal tubes with lower complexity cross sections.

An approach that is extremely computationally efficient, though somewhat conservative, is based on ellipsoidal sets used in conjunction with autonomous prediction dynamics [10, 11]. As in the nominal case considered in Sect. 2.7, an ellipsoidal invariant set can be computed offline for a prediction system that incorporates the degrees of freedom in predicted trajectories into its state vector. This section extends the methods of Sects. 2.7 and 2.9 to the case of robust MPC, and considers the design of the prediction dynamics in order to enlarge feasible sets and to reduce the sensitivity of predicted performance to multiplicative uncertainty.

Following the approach of Sect. 2.7, the predicted control trajectory at time k is defined for all $i \geq 0$ by

$$u_{i|k} = Kx_{i|k} + c_{i|k},$$

where $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ is a vector of variables in the MPC optimization at time k and $c_{i|k} = 0$ for $i \geq N$. The feedback gain K is fixed and is assumed to be the unconstrained LQ optimal feedback gain associated with the nominal cost

$$J(s_{0|k}, \mathbf{c}_k) = \sum_{i=0}^{\infty} \bigl(\|s_{i|k}\|_Q^2 + \|v_{i|k}\|_R^2\bigr), \tag{5.29}$$

which is evaluated along state and control trajectories of the nominal model:

$$\begin{aligned} s_{i+1|k} &= \Phi^{(0)} s_{i|k} + B^{(0)} c_{i|k} \\ v_{i|k} &= K s_{i|k} + c_{i|k} \end{aligned}$$

with $\Phi^{(0)} = A^{(0)} + B^{(0)} K$. We further assume that $u = Kx$ robustly quadratically stabilizes the uncertain system (5.1) and (5.2) in the absence of constraints. This assumption (which will be removed in Sect. 5.3.2) requires that a quadratic Lyapunov function exists for the unconstrained system (5.1) and (5.2) under $u = Kx$, namely that there exists $P \succ 0$ satisfying

$$P - \Phi^{(j)T} P \Phi^{(j)} \succ 0, \quad j = 1, \ldots, m \tag{5.30}$$

where $\Phi^{(j)} = A^{(j)} + B^{(j)} K$.

A prediction system incorporating the uncertain model (5.1) and (5.2) is described for all $i \geq 0$ by

$$z_{i+1|k} = \Psi_{i|k} z_{i|k}, \quad \Psi_{i|k} \in \mathrm{Co}\{\Psi^{(1)}, \ldots, \Psi^{(m)}\}, \tag{5.31a}$$

where the vertices of the parameter uncertainty set are given by

$$\Psi^{(j)} = \begin{bmatrix} \Phi^{(j)} & B^{(j)} E \\ 0 & M \end{bmatrix}, \quad \Phi^{(j)} = A^{(j)} + B^{(j)} K \tag{5.31b}$$

for $j = 1, \ldots, m$. The initial prediction system state and the predicted state and input trajectories are defined, as in Sect. 2.7, by

$$z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}, \quad \begin{aligned} u_{i|k} &= \begin{bmatrix} K & E \end{bmatrix} z_{i|k} \\ x_{i|k} &= \begin{bmatrix} I & 0 \end{bmatrix} z_{i|k} \end{aligned} \tag{5.32}$$

and matrices E and M are as given in (2.26b), so that $E\mathbf{c}_k = c_{0|k}$ and $M\mathbf{c}_k = (c_{1|k}, \ldots, c_{N-1|k}, 0)$.

The predicted state and input trajectories generated by (5.31a, 5.31b) and (5.32) necessarily satisfy the constraints (5.3) at all prediction times $i \geq 0$ if the initial prediction system state, $z_{0|k}$, is constrained to lie in a set that is robustly invariant for (5.31a, 5.31b) and feasible with respect to (5.3). This is demonstrated by the following extension of Theorem 2.9.

Corollary 5.1 *The ellipsoidal set $\mathcal{E}_z \doteq \{z : z^T P_z z \leq 1\}$, with $P_z \succ 0$, is robustly positively invariant for the dynamics (5.31a, 5.31b) and constraints $\begin{bmatrix} F + GK & GE \end{bmatrix} z \leq \mathbf{1}$ if and only if P_z satisfies*

$$P_z - \Psi^{(j)T} P_z \Psi^{(j)} \succeq 0, \quad j = 1, \ldots, m \tag{5.33}$$

and

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & GE \end{bmatrix} \\ \begin{bmatrix} (F+GK)^T \\ (GE)^T \end{bmatrix} & P_z \end{bmatrix} \succeq 0, \quad e_i^T H e_i \leq 1, \quad i = 1, \ldots, n_C \tag{5.34}$$

for some symmetric matrix H, where e_i is the ith column of the identity matrix.

Proof Using Schur complements, (5.33) is equivalent to

$$\begin{bmatrix} P_z & \Psi^{(j)T} \\ \Psi^{(j)} & P_z^{-1} \end{bmatrix} \succeq 0, \quad j = 1, \ldots, m.$$

Since this is an LMI in $\Psi^{(j)}$, it is equivalent, again using Schur complements, to $P_z - \Psi^T P_z \Psi \succeq 0$ for all $\Psi \in \mathrm{Co}\{\Psi^{(1)}, \ldots, \Psi^{(m)}\}$. Therefore, the sufficiency and necessity of (5.33) and (5.34) can be shown using the same argument as the proof of Theorem 2.9. □

Adopting the nominal cost of (5.29) as the objective of the MPC online optimization, we have, from Lemma 2.1 and Theorem 2.10,

$$J(x_k, \mathbf{c}_k) = \|z_{0|k}\|_W^2 = \|x_k\|_{W_x}^2 + \|\mathbf{c}_k\|_{W_c}^2, \tag{5.35a}$$

where W is the solution of the Lyapunov equation

$$W - {\Psi^{(0)}}^T W \Psi^{(0)} = \hat{Q}, \quad \hat{Q} = \begin{bmatrix} Q + K^T R K & K^T R E \\ E^T R K & E^T R E \end{bmatrix}. \tag{5.35b}$$

The block diagonal structure of $W = \mathrm{diag}\{W_x, W_c\}$ in (5.35a, 5.35b) follows from the definition of K as the unconstrained LQ optimal feedback gain. Furthermore, W_x is the solution of the Riccati equation (2.9) with $(A, B) = (A^{(0)}, B^{(0)})$, and W_c satisfies

$$W_c - M^T W_c M = E^T \big({B^{(0)}}^T W_x B^{(0)} + R \big) E. \tag{5.36}$$

Hence, by Theorem 2.10, W_c is block diagonal with diagonal blocks equal to $(B^{(0)})^T W_x B^{(0)} + R$. Clearly, the problem of minimizing $J(x_k, \mathbf{c}_k)$ for given x_k is equivalent to that of minimizing $\|\mathbf{c}_k\|^2_{W_c}$. Therefore the MPC algorithm can be stated in terms of W_c and an ellipsoidal set $\mathcal{E}_z$ satisfying the conditions of Corollary 5.1 as follows.

Algorithm 5.3 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \|\mathbf{c}_k\|^2_{W_c} \ \text{ subject to } \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{E}_z. \tag{5.37}$$

(ii) Apply the control law $u_k = K x_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ in (5.37). ◁

Along the trajectories of the closed-loop system,

$$x_{k+1} = \Phi_k x_k + B_k c^*_{0|k}, \quad (\Phi_k, B_k) \in \mathrm{Co}\big\{(\Phi^{(1)}, B^{(1)}), \ldots, (\Phi^{(m)}, B^{(m)})\big\}, \tag{5.38}$$

the optimal value of the objective in (5.37) is not necessarily non-increasing at successive time instants since the predicted cost in (5.29) is computed assuming that model parameters are equal to their nominal values. However, Algorithm 5.3 can be shown to be robustly stabilizing using a method similar to the analysis in Sect. 3.3 of robust MPC based on a nominal cost in the presence of additive disturbances. We first give an l_2 stability property of the closed-loop system (5.38), based on the assumption that $u = Kx$ quadratically stabilizes the model (5.1) and (5.2).

Lemma 5.3 *If the quadratic stability condition (5.30) holds, then the state of the closed-loop system (5.38) satisfies the quadratic bound*

$$\sum_{k=0}^{\infty} \|x_k\|^2 \leq \|x_0\|^2_P + \gamma^2 \sum_{k=0}^{\infty} \|c^*_{0|k}\|^2 \tag{5.39}$$

for some matrix $P \succ 0$ and a scalar γ.

Proof Suppose that $P_0 - \Phi^{(j)\,T} P_0 \Phi^{(j)} \succ 0$ for some $P_0 \succ 0$ and $j = 1, \dots, m$. Then $P_0 - \Phi^{(j)\,T} P_0 \Phi^{(j)} \succ \epsilon I_{n_x}$ for some $\epsilon > 0$ and hence $P - \Phi^{(j)\,T} P \Phi^{(j)} \succ I_{n_x}$ for $j = 1, \dots, m$, where $P = \epsilon^{-1} P_0 \succ 0$. Using Schur complements, this implies that there must exist $\gamma > 0$ satisfying

$$\begin{bmatrix} P & \Phi^{(j)^T} P \\ P\Phi^{(j)} & P \end{bmatrix} \succ \begin{bmatrix} I_{n_x} & 0 \\ 0 & \gamma^{-2} P B^{(j)} B^{(j)^T} P \end{bmatrix},$$

and using Schur complements again, this is equivalent to the condition

$$\begin{bmatrix} P & \Phi^{(j)^T} P & 0 \\ \star & P & P B^{(j)} \\ \star & \star & \gamma^2 I_{n_u} \end{bmatrix} \succ \begin{bmatrix} I_{n_x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this condition is an LMI in the parameters $\Phi^{(j)}$ and $B^{(j)}$, it must hold with $(\Phi^{(j)}, B^{(j)})$ replaced by any $(\Phi, B) \in \mathrm{Co}\{(\Phi^{(1)}, B^{(1)}), \dots, (\Phi^{(m)}, B^{(m)})\}$. Using Schur complements once more, we therefore have

$$\begin{bmatrix} P & 0 \\ 0 & \gamma^2 I_{n_u} \end{bmatrix} - \begin{bmatrix} \Phi^T \\ B^T \end{bmatrix} P \begin{bmatrix} \Phi & B \end{bmatrix} \succeq \begin{bmatrix} I_{n_x} & 0 \\ 0 & 0 \end{bmatrix},$$

for all $(\Phi, B) \in \mathrm{Co}\{(\Phi^{(j)}, B^{(j)}), j = 1, \dots, m\}$. Pre- and post-multiplying both sides of this inequality by $z_k = (x_k, c^*_{0|k})$ and using (5.38) gives

$$\|x_k\|_P^2 + \gamma^2 \|c^*_{0|k}\|^2 - \|x_{k+1}\|_P^2 \geq \|x_k\|^2$$

and summing both sides of this inequality over $k \geq 0$ yields the bound (5.39). □

Theorem 5.3 *For the system (5.1)–(5.3) with the control law of Algorithm 5.3, the optimization (5.37) is recursively feasible and the origin of state space is robustly asymptotically stable with region of attraction equal to the feasible set* $\mathcal{F} = \{x : \exists \mathbf{c}$ *such that* $(x, \mathbf{c}) \in \mathcal{E}_z\}$.

Proof If x_k lies in the feasible set $\mathcal{F}$, then a feasible but suboptimal solution of (5.37) at time $k+1$ is given by $\mathbf{c}_{k+1} = M\mathbf{c}_k^*$ since the robust positive invariance of $\mathcal{E}_z$ implies that $\Psi[x_k^T \ (\mathbf{c}_k^*)^T]^T \in \mathcal{E}_z$ for all $\Psi \in \mathrm{Co}\{\Psi^{(1)}, \dots, \Psi^{(m)}\}$. Hence (5.36) implies that the optimal solution of (5.37) at time $k+1$ satisfies

$$\|\mathbf{c}_{k+1}^*\|_{W_c}^2 \leq \|M\mathbf{c}_k^*\|_{W_c}^2 \leq \|\mathbf{c}_k^*\|_{W_c}^2 - \|c^*_{0|k}\|^2_{R + B^{(0)T} W_x B^{(0)}}.$$

Summing this inequality over all $k \geq 0$ yields the bound

$$\sum_{k=0}^{\infty} \|c^*_{0|k}\|^2 \leq \frac{1}{\underline{\lambda}(R)} \|\mathbf{c}_0^*\|_{W_c}^2$$

where $\underline{\lambda}(R)$ is the smallest eigenvalue of R. Therefore Lemma 5.3 implies

$$\sum_{k=0}^{\infty} \|x_k\|^2 \leq \|x_0\|_P^2 + \frac{\gamma^2}{\underline{\lambda}(R)} \|\mathbf{c}_0^*\|_{W_c}^2 \tag{5.40}$$

and hence $x_k \to 0$ as $k \to \infty$. Finally, we note that $x = 0$ is Lyapunov stable since Algorithm 5.3 coincides with the feedback law $u_k = Kx_k$ (which is robustly stabilizing by assumption) for all x_k in the set $\{x : (x, 0) \in \mathcal{E}_z\}$, and this set necessarily contains $x = 0$ in its interior since $P_z \succ 0$. □

To maximize the volume of the feasible set, $\mathcal{F} = \{x : \exists \mathbf{c} \text{ such that } (x, \mathbf{c}) \in \mathcal{E}_z\}$, for Algorithm 5.3, the matrix P_z defining $\mathcal{E}_z$ can be designed offline analogously to the case of nominal MPC considered in Sect. 2.7.3. Thus, rewriting (5.33) and (5.34) in terms of $S = P_z^{-1}$ gives the equivalent conditions

$$\begin{bmatrix} S & \Psi^{(j)} S \\ S\Psi^{(j)T} & S \end{bmatrix} \succeq 0, \quad j = 1, \ldots, m \tag{5.41a}$$

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & GE \end{bmatrix} S \\ S \begin{bmatrix} (F + GK)^T \\ (GE)^T \end{bmatrix} & S \end{bmatrix} \succeq 0, \quad e_i^T H e_i \leq 1, \quad i = 1, \ldots, n_C \tag{5.41b}$$

which are LMIs in the variables S and H. Hence the volume of $\mathcal{F}$ is maximized if $P_z = S^{-1}$ where S is the solution of the SDP problem

$$\underset{S,H}{\text{maximize}} \ \log\det\left(\begin{bmatrix} I_{n_x} & 0 \end{bmatrix} S \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix} \right) \text{ subject to (5.41a, 5.41b).} \tag{5.42}$$

In concluding this section, we note that the online computation required by Algorithm 5.3 is identical to that for the nominal MPC law of Algorithm 2.2. Therefore the computational advantages of Algorithm 2.2 also apply to Algorithm 5.3. In particular, the online minimization of Algorithm 5.3 can be performed extremely efficiently by solving for the unique negative real root of a well-behaved polynomial using the Newton–Raphson iteration described in Sect. 2.8.

5.3.1 Prediction Dynamics Optimized to Maximize the Feasible Set

The system (5.31a, 5.31b) and (5.32) that generates the predicted state and control trajectories underpinning Algorithm 5.3 can be interpreted in terms of a dynamic feedback law applied to the uncertain model (5.1) and (5.2). The initial state of this dynamic controller is defined by the vector, $\mathbf{c}_k$, of degrees of freedom in the state and input predictions at time k. In order to maximize the ellipsoidal region of attraction of Algorithm 5.3, it is possible to optimize the predicted controller dynamics simultaneously with the invariant set $\mathcal{E}_z$ by solving a convex optimization problem [12]. This is achieved through an extension of the method of optimizing prediction dynamics described in Sect. 2.9 to the case of systems with uncertain dynamics.

Let the vertices $\Psi^{(j)}$ of the uncertainty set in (5.31a) be redefined as

$$\Psi^{(j)} = \begin{bmatrix} \Phi^{(j)} & B^{(j)}C_c \\ 0 & A_c \end{bmatrix}, \quad j = 1, \ldots, m \tag{5.43}$$

where $A_c \in \mathbb{R}^{\nu_c \times \nu_c}$ and $C_c \in \mathbb{R}^{n_u \times \nu_c}$ are to be designed offline together with the matrix P_z defining the ellipsoid $\mathcal{E}_z = \{z \in \mathbb{R}^{n_x+\nu_c} : z^T P_z z \le 1\}$. The conditions for robust invariance of $\mathcal{E}_z$ can then be obtained by restating Corollary 5.1 in terms of these uncertainty set vertices. However the resulting conditions are nonconvex when A_c, C_c and P_z are treated as variables. We therefore use the transformation (2.60) to reformulate them as equivalent convex conditions in terms of variables X, Y, U, V parameterizing P_z and variables Ξ, Γ parameterizing A_c, C_c. Using the approach of Sect. 2.9, we then obtain, analogously to (2.62a, 2.62b), the LMI conditions:

$$\begin{bmatrix} \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi^{(j)}Y + B^{(j)}\Gamma & \Phi^{(j)}X \\ \Xi + \Phi^{(j)}Y + B^{(j)}\Gamma & \Phi^{(j)}X \end{bmatrix} \\ \star & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} \succeq 0 \tag{5.44a}$$

$$\begin{bmatrix} H & \begin{bmatrix} (F+GK)Y + G\Gamma & (F+GK)X \end{bmatrix} \\ \star & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} \succeq 0, \quad e_i^T H e_i \le 1, \quad i = 1, \ldots, n_C \tag{5.44b}$$

for $j = 1, \ldots, m$. Therefore, matrices A_c, C_c, P_z satisfying (5.33), (5.34) and (5.43) exist only if the LMIs (5.44a, 5.44b) hold for some X, Y, Ξ, Γ. Furthermore, the conditions (5.44a, 5.44b) are sufficient as well as necessary if the dimension of $\mathbf{c}$ is equal to that of x, namely if $\nu_c = n_x$, since in this case, by the argument of Sect. 2.9, the inverse transformation

$$P_z = \begin{bmatrix} X^{-1} & X^{-1}U \\ U^T X^{-1} & -U^T X^{-1} Y V^{-T} \end{bmatrix}, \quad A_c = U^{-1}\Xi V^{-T}, \quad C_c = \Gamma V^{-T} \tag{5.45}$$

where $UV^T = X - Y$, necessarily exists and defines A_c, C_c and P_z uniquely.

The set $\mathcal{F}$ of feasible initial conditions for Algorithm 5.3 is the projection of $\mathcal{E}_z$ onto the x-subspace (i.e. $\mathcal{F} = \{x : \exists \mathbf{c} \text{ such that } (x, \mathbf{c}) \in \mathcal{E}_z\}$). Therefore $\mathcal{F}$ can be maximized offline by solving the SDP problem:

$$\underset{\Xi,\Gamma,X,Y}{\text{maximize}} \quad \log\det(Y) \quad \text{subject to (5.44a, 5.44b)},$$

and then determining A_c, C_c, P_z using (5.45). With these parameters the nominal cost $J(x_k, \mathbf{c}_k)$ is given by (5.35a), where W_x is the solution of the Riccati equation (2.9) with $(A, B) = (A^{(0)}, B^{(0)})$ and W_c is the solution of the Lyapunov equation

$$W_c - A_c^T W_c A_c = C_c^T \big(B^{(0)^T} W_x B^{(0)} + R\big) C_c.$$

A robust MPC law can then be defined analogously to Algorithm 5.3 with the difference that the control law is given by

$$u_k = K x_k + C_c \mathbf{c}_k^*$$

where $\mathbf{c}_k^*$ is the optimal solution of the online MPC optimization. Through a straightforward extension of Theorem 5.3, it can be shown that the closed-loop system is robustly asymptotically stable with region of attraction $\mathcal{F}$.

If there is no model uncertainty, then, as discussed in Sect. 2.9, the offline optimization of the prediction dynamics results in the remarkable property that the feasible set $\mathcal{F}$ (namely the projection of $\mathcal{E}_z$ onto the x-subspace) is equal to the maximal ellipsoidal invariant set under any linear feedback law subject to the constraints (5.3). Since this property holds regardless of the choice of feedback gain K, the MPC law is able to deploy a highly tuned linear feedback law close to the origin without reducing the set of initial conditions that are feasible for the online optimization. This result does not carry over to the case of polytopic model uncertainty if a single matrix A_c is employed in (5.43). It does, however, hold for the robust case if A_c is allowed to assume values in the convex hull of a set of vertices, each vertex being associated with one of the vertices of the model parameter uncertainty set and optimized offline [12]. Thus we let

$$A_c \in \mathrm{Co}\{A_c^{(1)}, \ldots, A_c^{(m)}\}, \quad \Psi^{(j)} = \begin{bmatrix} \Phi^{(j)} & B^{(j)} C_c \\ 0 & A_c^{(j)} \end{bmatrix}, \quad j = 1, \ldots, m \tag{5.46a}$$

and

$$\Xi^{(j)} = U A_c^{(j)} V^T, \quad j = 1, \ldots, m. \tag{5.46b}$$

Then, replacing (5.44a) with the condition

$$\begin{bmatrix} \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi^{(j)} Y + B^{(j)} \Gamma & \Phi^{(j)} X \\ \Xi^{(j)} + \Phi^{(j)} Y + B^{(j)} \Gamma & \Phi^{(j)} X \end{bmatrix} \\ \star & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} \succeq 0, \tag{5.47}$$

the projection of $\mathcal{E}_z$ onto the x-subspace is maximized subject to (5.33) and (5.34) by solving the SDP problem:

$$\underset{\Xi^{(1)},\ldots,\Xi^{(m)},\Gamma,X,Y}{\text{maximize}} \quad \log\det(Y) \ \text{ subject to (5.47) and (5.44b)} \tag{5.48}$$

and applying the inverse transformation (5.45) with $A_c^{(j)} = U^{-1}\Xi^{(j)}V^{-T}$ for $j = 1, \ldots, m$. By considering the conditions for feasibility of (5.47) and (5.44b), it can be shown that the solution of (5.48) defines an ellipsoidal set $\mathcal{E}_z$ whose projection onto the x-subspace is equal to the maximal ellipsoidal invariant set for the uncertain model (5.1)–(5.3) under any linear feedback law. For a proof of this result, we refer the interested reader to [12].

From $z_{i+1|k} \in \mathrm{Co}\{\Psi^{(1)}z_{i|k}, \ldots, \Psi^{(m)}z_{i|k}\}$ and (5.46a), we obtain

$$\mathbf{c}_{i+1|k} \in \mathrm{Co}\{A_c^{(1)}\mathbf{c}_{i|k}, \ldots, A_c^{(m)}\mathbf{c}_{i|k}\}$$

along predicted trajectories. Hence, the controller dynamics are subject to polytopic uncertainty and the value of A_c is unknown at each prediction time step since it depends, implicitly through (5.46a), on the realization of the uncertain model parameters. A robust MPC law based on these predictions falls into the category of feedback MPC strategies since the evolution of the predicted control trajectory depends on the realization of future model uncertainty. Despite the future evolution of the controller state being uncertain, the implied MPC law is implementable since only the value of $\mathbf{c}_k$ need be known in order to evaluate $u_k = Kx_k + C_c\mathbf{c}_k$.

The objective of the MPC online optimization can be defined as in (5.35a), but to ensure closed-loop stability we require that the weighting matrix W_c satisfies

$$W_c - A_c^{(j)^T} W_c A_c^{(j)} \succeq C_c^T \big(B^{(0)^T} W_x B^{(0)} + R\big) C_c, \quad j = 1, \ldots, m \tag{5.49}$$

where W_x is the solution of (2.9) with $(A, B) = (A^{(0)}, B^{(0)})$. Thus W_c can be computed by minimizing trace(W_c) subject to (5.49). The MPC algorithm, which requires the same online computation as Algorithm 5.3, is stated next.

Algorithm 5.4 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \|\mathbf{c}_k\|_{W_c}^2 \ \text{ subject to } \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{E}_z. \tag{5.50}$$

(ii) Apply the control law $u_k = Kx_k + C_c\mathbf{c}_k^*$, where $\mathbf{c}_k^*$ is the optimal value of $\mathbf{c}_k$ in (5.50). ◁

Theorem 5.4 *For the system (5.1)–(5.3) under the control law of Algorithm 5.4, the optimization (5.50) is recursively feasible and $x = 0$ is asymptotically stable with region of attraction $\mathcal{F} = \{x : \exists \mathbf{c}$ such that $(x, \mathbf{c}) \in \mathcal{E}_z\}$.*

Proof Recursive feasibility is a consequence of the robust invariance of $\mathcal{E}_z$, which implies that a feasible solution to (5.50) at time $k+1$ is given by $\mathbf{c}_{k+1} = A_{c,k}\mathbf{c}_k$, for some $A_{c,k} \in \mathrm{Co}\{A_c^{(1)}, \ldots, A_c^{(m)}\}$. Asymptotic stability can be shown using a similar argument to the proof of Theorem 5.3. In particular, (5.30) implies that the bound

$$\sum_{k=0}^{\infty} \|x_k\|^2 \leq \|x_0\|_P^2 + \gamma^2 \sum_{k=0}^{\infty} \|C_c \mathbf{c}_k^*\|^2 \tag{5.51}$$

holds for the closed-loop system $x_{k+1} = \Phi_k x_k + B_k C_c \mathbf{c}_k^*$, for some $P \succ 0$ and scalar γ. Also, from (5.49) and feasibility of $\mathbf{c}_{k+1} = A_{c,k}\mathbf{c}_k^*$ the optimal solution at time $k+1$ necessarily satisfies

$$\|\mathbf{c}_{k+1}^*\|_{W_c}^2 \leq \|\mathbf{c}_k^*\|_{W_c}^2 - \|C_c \mathbf{c}_k^*\|_{R+B^{(0)T} W_x B^{(0)}}^2.$$

Summing this inequality over all $k \geq 0$ and using (5.51) gives the asymptotic bound (5.40), which implies $\lim_{k\to\infty} x_k = 0$ for all $x_0 \in \mathcal{F}$. Stability of $x = 0$ follows from the fact that the control law of Algorithm 5.4 is equal to $u_k = K x_k$ for all $x_k \in \{x : (x, 0) \in \mathcal{E}_z\}$, and this feedback law is robustly stabilizing by (5.30). □

In order to minimize a worst-case predicted cost instead of the cost of Algorithm 5.4, the optimization (5.50) can be replaced with

$$\underset{\mathbf{c}_k}{\text{minimize}} \ \|z_k\|_{\check{W}}^2 \text{ subject to } z_k = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{E}_z$$

where $\check{W}$ satisfies the following LMIs for $j = 1, \ldots, m$

$$\check{W} - \Psi^{(j)T} \check{W} \Psi^{(j)} \succeq \begin{bmatrix} I & K^T \\ 0 & C_c^T \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ K & C_c \end{bmatrix}. \tag{5.52}$$

With this modification, Algorithm 5.4 minimizes the upper bound on predicted performance:

$$\|z_k\|_{\check{W}}^2 \geq \max_{\substack{\Psi_{i|k} \in \mathrm{Co}\{\Psi^{(1)},\ldots,\Psi^{(m)}\} \\ i=0,1,\ldots}} \sum_{i=0}^{\infty} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big),$$

and the closed-loop system has the properties given in Theorem 5.4.

Example 5.2 For the uncertain system and constraints defined in Example 5.1, the unconstrained optimal feedback law for the nominal model (5.4) and cost weights $Q = I$ and $R = 1$ is $u = Kx$, $K = [0.19\ 0.34]$, and the offline optimization (5.48) yields

$$A_c^{(1)} = \begin{bmatrix} -0.69 & 0.20 \\ -0.27 & -0.14 \end{bmatrix}, \quad A_c^{(2)} = \begin{bmatrix} -0.74 & -0.01 \\ 0.26 & -0.01 \end{bmatrix}, \quad A_c^{(3)} = \begin{bmatrix} -0.63 & -0.21 \\ -0.05 & -0.27 \end{bmatrix},$$

and $C_c = [0.12\ -0.12]$.

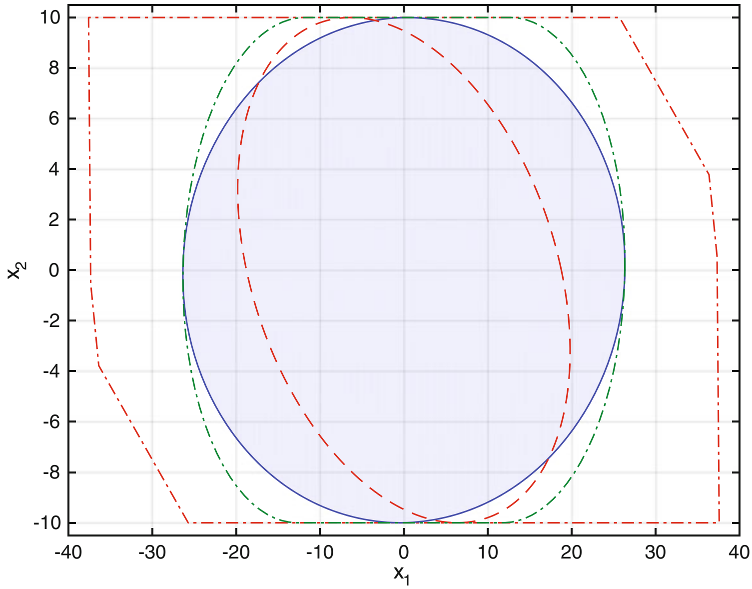

Fig. 5.2 The set of feasible initial states for Algorithm 5.4 applied to the system of Example 5.1. Also shown are the maximal ellipsoidal RPI set under the LQ-optimal feedback law $u = Kx$ (*dashed line*), the feasible set for Algorithm 5.1 (*inner dash-dotted line*) and the maximal robustly controlled invariant set (*outer dash-dotted line*)

Figure 5.2 shows the set, $\mathcal{F}$, of feasible initial conditions for Algorithm 5.4. Comparing Figs. 5.2 and 5.1, it can be seen that, as expected, this feasible set is identical to the maximum area robustly invariant ellipsoidal set under any linear feedback law. The average quadratic bound on the predicted worst-case cost computed for the solution of (5.50) at 100 initial conditions evenly distributed on the boundary of $\mathcal{F}$ is 1002, while the maximum worst-case cost bound for these states is 1823. These bounds are on average 4 % higher than the worst-case cost bounds obtained using Algorithm 5.1 with the same set of initial conditions (discussed in Example 5.1); however, neither approach results in cost bounds that are consistently lower for all initial conditions. The conservativeness in this case results from the offline computation of the parameters K, C_c and $A_c^{(1)}, \ldots, A_c^{(m)}$ defining the predicted control law. On the other hand, the online computation of Algorithm 5.4 is typically orders of magnitude lower than that of Algorithm 5.1[1] ◊

[1]For this example, the time required to solve (5.50) using the Newton–Raphson method is between one and two orders of magnitude less than the computation time for (5.18) using the Mosek.

5.3.2 Prediction Dynamics Optimized to Improve Worst-Case Performance

Polytopic controller dynamics were introduced in Sect. 5.3.1 with the aim of maximizing the volume of the ellipsoidal region of attraction. This was achieved under the assumption that the feedback law $u_k = Kx_k$ robustly stabilizes the system (5.1) and (5.2) in the absence of constraints. Moreover K was assumed to be the LQ optimal feedback gain for the unconstrained nominal system. However this feedback law may not be robustly stabilizing, and in this case it is necessary either to resort to a different feedback gain, or to retain the nominally optimal unconstrained feedback law in the problem formulation and ensure through other means that the predicted control law is robustly stabilizing [13]. The latter can be achieved through the use of predicted polytopic controller dynamics introduced as

$$u_{i|k} = Kx_{i|k} + v_{i|k} + c_{i|k} \tag{5.53a}$$

where $c_{i|k} = 0$ for all $i \geq N$ and

$$v_{i+1|k} \in \mathrm{Co}\{L^{(j)}x_{i|k} + N^{(j)}v_{i|k},\ j = 1, \ldots, m\} \tag{5.53b}$$

for all $i \geq 0$.

The perturbations $c_{i|k}$ serve the same purpose here as in earlier sections of this chapter, namely to ensure satisfaction of the constraints (5.3), while the polytopic dynamics of (5.53b) are designed with the aim of improving robustness and reducing the sensitivity of closed-loop performance to model uncertainty. This is done by invoking the following conditions for $W \succ 0$:

$$W - \begin{bmatrix} \Phi^{(j)} & B^{(j)} \\ L^{(j)} & N^{(j)} \end{bmatrix}^T W \begin{bmatrix} \Phi^{(j)} & B^{(j)} \\ L^{(j)} & N^{(j)} \end{bmatrix} \succeq \begin{bmatrix} Q + K^T RK & K^T R \\ RK & R \end{bmatrix}, \quad j = 1, \ldots, m \tag{5.54}$$

where $\Phi^{(j)} = A^{(j)} + B^{(j)}K$, and choosing $L^{(j)}$, $N^{(j)}$, $j = 1, \ldots, m$ and W by solving

$$(L^{(1)}, N^{(1)}, \ldots, L^{(m)}, N^{(m)}, W) = \arg \min_{L^{(1)}, N^{(1)}, \ldots, L^{(m)}, N^{(m)}, W} \bar{\lambda}(W_x - W_{xv}W_v^{-1}W_{vx}) \tag{5.55}$$

subject to (5.54), where $\bar{\lambda}(P)$ denotes the maximum eigenvalue of the matrix P, and where W_x, W_{xv} and W_v are the blocks of the partition

$$W = \begin{bmatrix} W_x & W_{xv} \\ W_{xv}^T & W_v \end{bmatrix}.$$

Although the optimization is nonconvex as stated in (5.55), its solution can be determined by solving the following equivalent (convex) SDP problem:

$$(Y^{(1)}, \ldots, Y^{(m)}, S) = \arg \max_{Y^{(1)},\ldots,Y^{(m)},S,\lambda} \lambda \quad \text{subject to}$$

$$\begin{bmatrix} I_{n_x} & 0 \end{bmatrix} S \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix} \succeq \lambda I_{n_x}$$

$$\begin{bmatrix} S & \begin{bmatrix} S \begin{bmatrix} \Phi^{(j)T} \\ B^{(j)T} \end{bmatrix} & Y^{(j)T} \end{bmatrix} & S \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} & S \begin{bmatrix} K^T R^{1/2} \\ R^{1/2} \end{bmatrix} \\ \star & S & 0 & 0 \\ \star & \star & I & 0 \\ \star & \star & \star & I \end{bmatrix} \succeq 0, \quad j = 1, \ldots, m \tag{5.56}$$

and then computing $W = S^{-1}$ and $\begin{bmatrix} L^{(j)} & N^{(j)} \end{bmatrix} = Y^{(j)} W$, for $j = 1, \ldots, m$.

The rationale behind this strategy is that, if $L^{(j)}, N^{(j)}$ for $j = 1, \ldots, m$ satisfy (5.54) for some $W \succ 0$, then the control law $u_{i|k} = K x_{i|k} + v_{i|k}$ robustly stabilizes the system

$$\begin{aligned} x_{i+1|k} &= A_{i|k} x_{i|k} + B_{i|k} u_{i|k}, \\ v_{i+1|k} &= L_{i|k} x_{i|k} + N_{i|k} v_{i|k}, \end{aligned} \quad \begin{bmatrix} A_{i|k} & B_{i|k} \\ L_{i|k} & N_{i|k} \end{bmatrix} \in \mathrm{Co}\left\{ \begin{bmatrix} \Phi^{(j)} & B^{(j)} \\ L^{(j)} & N^{(j)} \end{bmatrix}, \ j = 1, \ldots, m \right\}$$

Furthermore, (5.54) implies that the predicted cost for this system under the control law $u_{i|k} = K x_{i|k} + v_{i|k}$ satisfies the bound

$$\sum_{i=0}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) \leq \begin{bmatrix} x_{0|k} \\ v_{0|k} \end{bmatrix}^T W \begin{bmatrix} x_{0|k} \\ v_{0|k} \end{bmatrix} \tag{5.57}$$

for all admissible realizations of model uncertainty. If the constraints (5.3) are inactive, then the minimum, over all $v_{0|k} \in \mathbb{R}^{n_u}$, of this bound can be shown to be $x_{0|k}^T (W_x - W_{xv} W_v^{-1} W_{xv}^T) x_{0|k}$. Therefore the optimization (5.55) chooses $(L^{(1)}, N^{(1)}, \ldots, L^{(m)}, N^{(m)})$ so as to minimize the maximum, over all $x_{0|k}$ in the ball $\{x : \|x\| \leq r\}$, of the minimum value of the bound (5.57) in the absence of constraints.

Clearly, $v_{0|k}$ provides degrees of freedom for minimizing the predicted cost in the online MPC optimization subject to constraints. However the system constraints (5.3) must be satisfied and for this reason we introduce the vector of perturbations $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ as additional degrees of freedom. Under the control law of (5.53a, 5.53b), the predicted state and control trajectories are generated by the prediction system

$$z_{i+1|k} = \Psi_{i|k} z_{i|k}, \quad \Psi_{i|k} = \mathrm{Co}\{\Psi^{(1)}, \ldots, \Psi^{(m)}\} \tag{5.58}$$

where

$$z_{0|k} = \begin{bmatrix} x_{0|k} \\ v_{0|k} \\ \mathbf{c}_k \end{bmatrix}, \quad \Psi^{(j)} = \begin{bmatrix} \Phi^{(j)} & B^{(j)} & B^{(j)}E \\ L^{(j)} & N^{(j)} & 0 \\ 0 & 0 & M \end{bmatrix} \tag{5.59}$$

with E and M as defined in (5.31a, 5.31b). As before, the constraints of (5.3) can be imposed on predictions in a computationally efficient (though somewhat conservative) way by constraining the prediction system state $z_{0|k}$ to lie in a robustly invariant ellipsoidal set $\mathcal{E}_z \doteq \{z : z^T P_z z \le 1\}$. Analogously to Corollary 5.1, the conditions for invariance and constraint satisfaction are given by

$$P_z - \Psi^{(j)T} P_z \Psi^{(j)} \succeq 0, \quad j = 1, \ldots, m \tag{5.60}$$

and

$$\begin{bmatrix} H & \begin{bmatrix} F + GK & G & GE \end{bmatrix} \\ \begin{bmatrix} F + GK & G & GE \end{bmatrix}^T & P_z \end{bmatrix} \succeq 0, \quad e_i^T H e_i \le 1, \quad i = 1, \ldots, n_C \tag{5.61}$$

for some symmetric matrix H, where e_i is the ith column of the identity matrix.

Additionally, if $\check{W}$ satisfies the condition

$$\check{W} - \Psi^{(j)T} \check{W} \Psi^{(j)} \succeq \begin{bmatrix} Q + K^T R K & K^T R & K^T R E \\ RK & R & RE \\ E^T R K & E^T R & E^T R E \end{bmatrix}, \quad j = 1, \ldots, m, \tag{5.62}$$

then the worst-case cost along predicted trajectories of (5.1) and (5.2) under the control law (5.53a, 5.53b)

$$\check{J}(x_{0|k}, v_{0|k}, \mathbf{c}_k) \doteq \max_{(A_{i|k}, B_{i|k}) \in \Omega,\, i=0,1,\ldots} \sum_{i=0}^{\infty} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \big),$$

satisfies the bound $\check{J}(x, v, \mathbf{c}) \le \|(x, v, \mathbf{c})\|_{\check{W}}^2$. On the basis of this predicted performance bound, we can state the following min–max robust MPC algorithm.

Algorithm 5.5 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{v_k, \mathbf{c}_k}{\text{minimize}} \ \|(x_k, v_k, \mathbf{c}_k)\|_{\check{W}}^2 \ \text{ subject to } \begin{bmatrix} x_k \\ v_k \\ \mathbf{c}_k \end{bmatrix} \in \mathcal{E}_z \tag{5.63}$$

(ii) Apply the control law $u_k = K x_k + v_k^* + c_{0|k}^*$, where $\mathbf{c}_k^* = (c_{0|k}^*, \ldots, c_{N-1|k}^*)$ and $(v_k^*, \mathbf{c}_k^*)$ is the optimal solution of (5.63). ◁

Theorem 5.5 *For the system (5.1)–(5.3) and control law of Algorithm 5.4, the optimization (5.63) is recursively feasible and* $x = 0$ *is asymptotically stable with region of attraction* $\mathcal{F} = \{x : \exists(v, \mathbf{c})$ *such that* $(x, v, \mathbf{c}) \in \mathcal{E}_z\}$.

Proof Recursive feasibility of (5.63) follows from (5.60) and (5.61), which imply that $\Psi z_k \in \mathcal{E}_z$ for some $\Psi \in \mathrm{Co}\{\Psi^{(1)}, \dots, \Psi^{(m)}\}$ if $z_k \in \mathcal{E}_z$, and hence $(v_{k+1}, \mathbf{c}_{k+1}) = (Lx_k + Nv_k^*, Mc_k^*)$ is a feasible solution to (5.63) at time $k+1$ for some $(L, N) \in \mathrm{Co}\{L^{(1)}, N^{(1)}, \dots, L^{(m)}, N^{(m)}\}$ if $x_k \in \mathcal{F}$. From (5.62), we therefore have

$$\|z_{k+1}\|^2_{\check{W}} \le \|z_k\|^2_{\check{W}} - \big(\|x_k\|^2_Q + \|u_k\|^2_R\big),$$

where $z_k = (x_k, v_k^*, \mathbf{c}_k^*)$, and this implies that $x = 0$ is asymptotically stable since $\|z_k\|^2_{\check{W}} \ge \check{J}(x_k, v_k^*, \mathbf{c}_k^*)$ and $Q, R \succ 0$. □

Theorem 5.5 implies that the closed-loop system necessarily converges to a region of state space on which the trajectories of the nominal system model satisfy the constraints (5.3) at all future times under the unconstrained nominal LQ optimal feedback law. To allow the predicted control trajectories (5.53a, 5.53b) to realize this feedback law, the additional constraint that $(L^{(0)}, N^{(0)}) = (0, 0)$ must be included in the optimization (5.55) defining $(L^{(j)}, N^{(j)})$, where

$$(L^{(0)}, N^{(0)}) = \sum_{j=1}^{m} \mu^{(j)} (L^{(j)}, N^{(j)}) \tag{5.64}$$

and where $\mu^{(1)}, \dots, \mu^{(m)}$ are scalar constants that define the nominal model parameters $(A^{(0)}, B^{(0)})$ via

$$(A^{(0)}, B^{(0)}) = \sum_{j=1}^{m} \mu^{(j)} (A^{(j)}, B^{(j)}).$$

The condition (5.64) can be imposed in the optimization (5.56) through a constraint which is linear in $Y^{(1)}, \dots, Y^{(m)}$, namely that $\sum_{j=1}^{m} \mu^{(j)} Y^{(j)} = 0$.

To ensure that the MPC law recovers the nominal LQ optimal feedback law whenever it is feasible, a nominal cost computed analogously to (5.35b) can be used in place of the worst-case cost of Algorithm 5.5. In this case, however, the nominal cost does not have the block diagonal structure of (5.35a), and hence the predicted cost may not be monotonically non-increasing. Closed-loop stability is therefore ensured in [13] by tightening condition (5.60) by replacing the RHS of the LMI with ρP_z for some $\rho \in (0, 1)$, and by replacing the constraint in the online optimization (5.63) with the condition $(x_k, v_k, \mathbf{c}_k) \in \rho^k \mathcal{E}_z$, thus guaranteeing exponential stability of $x = 0$.

Robustness can alternatively be addressed by introducing controller dynamics through the Youla parameter [25]. Moreover, the approach can be recast in terms of state-space models, allowing constraints to be imposed in a computationally efficient way through the use of robustly invariant ellipsoidal sets. A nominal cost can be adopted for the definition of an MPC law and the free Youla parameters can be optimized so as to minimize a sensitivity transfer function, thus reducing the sensitivity of predicted trajectories to the model uncertainty and making a nominal cost more representative of system performance [13]. This is an indirect way of minimizing cost sensitivity to uncertainty, a problem that remains open, both for MPC and for constrained optimal control in general.

5.4 Low-Complexity Polytopes in Robust MPC

Although computationally convenient for robust MPC involving multiplicative model uncertainty, the ellipsoidal invariant sets discussed in Sect. 5.3 necessarily result in conservative feasible sets as a result of their implicit handling of state and control constraints. On the other hand, the exact tubes considered in Sect. 5.2.1, which enable constraints to be imposed on predicted states and inputs non-conservatively, are in general impractical because their computational requirements grow exponentially with the length of prediction horizon. To avoid these difficulties, this section considers the application of tubes with low-complexity polytopic cross sections (which were discussed in Sect. 3.6.2 in the context of robust MPC with additive model uncertainty) to the case of multiplicative model uncertainty.

5.4.1 Robust Invariant Low-Complexity Polytopic Sets

Tubes defined in terms of low-complexity polytopic sets are the basis of a family of computationally efficient methods of imposing constraints on predicted trajectories in the presence of multiplicative uncertainty [16–18, 26]. We discuss terminal constraints in this section and consider the uncertain system (5.1)–(5.3) under a given terminal feedback law $u = Kx$:

$$x_{k+1} = \Phi_k x_k, \quad \Phi_k \in \Omega_K \doteq \mathrm{Co}\{\Phi^{(1)}, \ldots, \Phi^{(m)}\} \tag{5.65}$$

where $\Phi^{(j)} = A^{(j)} + B^{(j)}K$, $j = 1, \ldots, m$. Recall that a low-complexity polytope, denoted here by $\Pi(V, \alpha)$, is defined for a non-singular matrix $V \in \mathbb{R}^{n_x \times n_x}$ and a positive vector $\alpha \in \mathbb{R}^{n_x}$ by

$$\Pi(V, \alpha) \doteq \{x : |Vx| \le \alpha\} \tag{5.66}$$

where the extraction of absolute values and the inequality sign apply on an element-by-element basis. The conditions under which $\Pi(V, \alpha)$ is a suitable terminal set, namely invariance under the dynamics of (5.65) for a given terminal control law

$u = Kx$ and feasibility with respect to the system constraints (5.3), can be stated as follows.

Theorem 5.6 *The low-complexity polytope $\Pi(V, \alpha) = \{x : |Vx| \le \alpha\}$ is robustly invariant for the dynamics (5.65) if and only if*

$$|V\Phi^{(j)}W|\alpha \le \alpha, \quad j = 1, \dots, m \tag{5.67}$$

where $W = V^{-1}$. Furthermore, under $u = Kx$ the constraints (5.3) are satisfied for all $x \in \Pi(V, \alpha)$ if and only if

$$|(F + GK)W|\alpha \le \mathbf{1}. \tag{5.68}$$

Proof The set $\Pi(V, \alpha)$ is robustly invariant if and only if $\Phi x \in \Pi(V, \alpha)$ for all $x \in \Pi(V, \alpha)$ and all $\Phi \in \text{Co}\{\Phi^{(j)},\ j = 1, \dots, m\}$. Equivalently, for all x such that $|Vx| \le \alpha$, we require $|V\Phi x| \le \alpha$. Therefore, given that

$$|V\Phi x| = |V\Phi W V x| \le |V\Phi W||Vx| \le |V\Phi W|\alpha$$

for all $x \in \Pi(V, \alpha)$, a sufficient condition for invariance is

$$|V\Phi W|\alpha \le \alpha.$$

This condition can be expressed in terms of inequalities that depend linearly on Φ, and therefore it needs to be enforced only at the vertices of the model uncertainty set, as is done in (5.67). The conditions of (5.67) are necessary as well as sufficient because, for each $i = 1, \dots, n_x$, the ith element of $|V\Phi^{(j)}x|$ satisfies $|V_i\Phi^{(j)}x| = |V_i\Phi^{(j)}WVx| = |V_i\Phi^{(j)}W|\alpha$ for x equal to one of the vertices of $\Pi(V, \alpha)$, where V_i is the ith row of V. The necessity and sufficiency of (5.68) follows similarly from the inequalities

$$(F + GK)x \le |(F + GK)x| = |(F + GK)WVx| \le |(F + GK)W|\alpha$$

for all $x \in \Pi(V, \alpha)$, where, for each i, $(F + GK)_i x = |(F + GK)_i W|\alpha$ for some x such that $|Vx| = \alpha$, with $(F + GK)_i$ denoting the ith row of $F + GK$. □

A non-symmetric low-complexity polytope is defined for $V \in \mathbb{R}^{n_x \times n_x}$ and $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}^{n_x}$, with $\underline{\alpha} < 0$ and $\overline{\alpha} > 0$, as the set

$$\tilde{\Pi}(V, \underline{\alpha}, \overline{\alpha}) \doteq \{x : \underline{\alpha} \le Vx \le \overline{\alpha}\}. \tag{5.69}$$

These sets share many of the computational advantages of symmetric low-complexity polytopic sets, but can provide larger invariant sets if the system constraints (5.3) are non-symmetric. The conditions of Theorem 5.6 for robust invariance and feasibility with respect to constraints extend to non-symmetric low-complexity polytopes in an obvious way, as we show next.

Corollary 5.2 *Let* $A^+ \doteq \max\{A, 0\}$ *and* $A^- \doteq \max\{-A, 0\}$ *for any real matrix* A*. Under the feedback law* $u = Kx$*, the non-symmetric low-complexity polytope* $\tilde{\Pi}(V, \underline{\alpha}, \overline{\alpha})$ *is robustly invariant for the dynamics (5.65) and the constraint (5.3) is satisfied for all* $x \in \tilde{\Pi}(V, \underline{\alpha}, \overline{\alpha})$ *if and only if*

$$\begin{bmatrix} \left(V\Phi^{(j)}W\right)^+ & \left(V\Phi^{(j)}W\right)^- \\ \left(V\Phi^{(j)}W\right)^- & \left(V\Phi^{(j)}W\right)^+ \end{bmatrix} \begin{bmatrix} \overline{\alpha} \\ -\underline{\alpha} \end{bmatrix} \leq \begin{bmatrix} \overline{\alpha} \\ -\underline{\alpha} \end{bmatrix}, \quad j = 1, \ldots, m \tag{5.70}$$

$$\begin{bmatrix} \left((F + GK)W\right)^+ & \left((F + GK)W\right)^- \end{bmatrix} \begin{bmatrix} \overline{\alpha} \\ -\underline{\alpha} \end{bmatrix} \leq \mathbf{1}. \tag{5.71}$$

where $W = V^{-1}$.

Proof This follows from Lemma 3.6, which implies that the following bounds hold for all x such that $\underline{\alpha} \leq Vx \leq \overline{\alpha}$:

$$\begin{aligned} V\Phi x &\leq (V\Phi W)^+\overline{\alpha} + (V\Phi W)^-(-\underline{\alpha}) \\ V\Phi x &\geq (V\Phi W)^+\underline{\alpha} + (V\Phi W)^-(-\overline{\alpha}) \\ (F + GK)x &\leq \left((F + GK)W\right)^+\overline{\alpha} + \left((F + GK)W\right)^-(-\underline{\alpha}). \end{aligned}$$

Lemma 3.6 also shows that each inequality in these conditions must hold with equality for some x such that $\underline{\alpha} \leq Vx \leq \overline{\alpha}$. A similar argument to the proof of Theorem 5.6 therefore implies that (5.70) and (5.71) are necessary and sufficient for robust invariance of $\tilde{\Pi}(V, \underline{\alpha}, \overline{\alpha})$ for the system (5.65) and for satisfaction of the constraints (5.3) under $u = Kx$ at all points in this set. □

To make it possible to compute offline a low-complexity polytopic terminal set for use in an online MPC optimization, we require that the conditions of Theorem 5.6 (or Corollary 5.2) hold for some V and $\alpha > 0$ (or V, $\underline{\alpha} < 0$ and $\overline{\alpha} > 0$). The crucial condition to be satisfied is the invariance condition (5.67), since, by linearity of the dynamics (5.65), the existence of V and $\alpha > 0$ satisfying (5.67) is necessary as well as sufficient for (5.70) to hold for some V, $\underline{\alpha} < 0$ and $\overline{\alpha} > 0$. Likewise the feasibility conditions (5.68) and (5.70) can be satisfied simply by scaling α whenever (5.67) holds.

Necessary and sufficient conditions for existence of an ellipsoidal invariant set for the uncertain system (5.65) follow directly from the discussion of Sect. 5.2, namely the system admits an ellipsoidal invariant set if and only if it is quadratically stable. Moreover, quadratic stability is relatively easy to check numerically by determining whether a set of LMIs is feasible. Likewise a polytopic invariant set defined by a finite (but arbitrary) number of vertices exists whenever the system (5.65) is exponentially stable [27]. This condition is equivalent to the requirement that the joint spectral radius defined by

$$\rho \doteq \lim_{k\to\infty} \max_{\Phi_i \in \Omega_K,\, i=1,2,\ldots} \|\Phi_1 \cdots \Phi_k\|^{1/k} \tag{5.72}$$

satisfies $\rho < 1$.[2] For the special case of low-complexity polytopic sets, nonconservative conditions for the existence of low-complexity invariant sets are not available for general systems of the form (5.65); however, the following result provides a useful sufficient condition.

Lemma 5.4 *For given V, define $\bar{\Phi}$ as the matrix with (i,k)th element*

$$[\bar{\Phi}]_{ik} = \max_{j\in\{1,\ldots,m\}} |V_i \Phi^{(j)} W_k|, \quad i = 1,\ldots,n_x, \quad k = 1,\ldots,n_x,$$

where V_i and W_k are, respectively, the ith row of V and kth column of $W = V^{-1}$. Then α exists such that $\Pi(V,\alpha)$ is robustly invariant for the dynamics (5.65) if $\lambda_{PF}(\bar{\Phi}) \le 1$ where $\lambda_{PF}(\bar{\Phi})$ is the Perron–Frobenius eigenvalue of $\bar{\Phi}$.

Proof For each $i = 1,\ldots,n_x$ and $j = 1,\ldots,m$ we have $|V_i\Phi^{(j)}W|\alpha \le \bar{\Phi}_i\alpha$ where $\bar{\Phi}_i$ is the ith row of $\bar{\Phi}$. Thus, if α is chosen as the Perron–Frobenius eigenvector of $\bar{\Phi}$, then $|V\Phi W|\alpha \le \bar{\Phi}\alpha = \lambda_{PF}(\bar{\Phi})\alpha \le \alpha$ which implies that (5.67) admits at least one feasible α. □

The volume in $\mathbb{R}^{n_x}$ of a low-complexity polytope $\Pi(V,\alpha)$, with $\alpha = (\alpha_1,\ldots,\alpha_{n_x})$, is given by

$$C_{n_x}\,|\det(V^{-1})| \prod_{i=1}^{n_x} \alpha_i$$

where C_{n_x} is independent of V and α. If $\Pi(V,\alpha)$ exists satisfying the conditions of Theorem 5.6, then the maximum volume invariant set is therefore given by $\Pi(V,\mathbf{1})$ where V is the maximizing argument of

$$\underset{V,W}{\text{maximize}}\ |\det(W)| \ \text{ subject to (5.67), (5.68) with } \alpha = \mathbf{1}, \text{ and } V = W^{-1}.$$

Although this problem is nonconvex, its constraints can be reformulated in terms of equivalent bilinear constraints:

$$\begin{aligned}
\underset{W,H^{(1)},\ldots,H^{(m)}}{\text{maximize}}\quad & |\det(W)| \\
\text{subject to}\quad & WH^{(j)} = \Phi^{(j)}W, \qquad |H^{(j)}|\mathbf{1} \le \mathbf{1}, \qquad j = 1,\ldots,m \\
& |(F+GK)W|\mathbf{1} \le \mathbf{1}
\end{aligned} \tag{5.73}$$

Expressed in this form, the problem can be solved approximately by solving a sequence of convex programs [26, 29].

To avoid the inherent nonconvexity (offline) of (5.73), the volume of $\Pi(V,\alpha)$ can be maximized instead over $\alpha > 0$ for a given fixed V by solving

[2] In general, it is not possible to compute ρ exactly. However, upper bounds on ρ can be computed to any desired accuracy, for example, using sum of squares programming [28].

$$\underset{\alpha=(\alpha_1,\ldots,\alpha_{n_x})}{\text{maximize}} \prod_{i=1}^{n_x} \alpha_i \quad \text{subject to (5.67) and (5.68).} \tag{5.74}$$

The constraints of this problem are linear in α and the objective can be expressed as the determinant of a symmetric positive-definite matrix (namely $\text{diag}(\alpha_1, \ldots, \alpha_{n_x})$). Therefore the optimization can be performed by solving an equivalent (convex) semidefinite program (see e.g. [30]). An obvious modification of this problem allows the volume of the non-symmetric low-complexity polytope $\{x : \underline{\alpha} \le Vx \le \overline{\alpha}\}$ to be maximized over $\underline{\alpha}, \overline{\alpha}$ for given V by solving a similar semidefinite program.

If α is chosen so as to maximize the volume of $\Pi(V, \alpha)$ for fixed V using (5.74), then clearly V must be designed so as to ensure that (5.67) is feasible for some α. For example, robust invariance of $\Pi(V, \alpha)$ under (5.65) can be ensured for some α by choosing V on the basis of a robustly invariant ellipsoidal set $\mathcal{E} = \{x : x^T Px \le 1\}$, computed using semidefinite programming. If P is a symmetric positive-definite matrix satisfying

$$\Phi^{(j)T} P \Phi^{(j)} \preceq P/n_x, \quad j = 1, \ldots, m,$$

then the choice $V = P^{1/2}$ ensures that (5.67) is feasible since the bounds $\|x\|_\infty \le \|x\|_2 \le \sqrt{n_x}\|x\|_\infty$ (which hold for all $x \in \mathbb{R}^{n_x}$) then imply

$$\|V\Phi^{(j)}x\|_\infty \le \|V\Phi^{(j)}x\|_2 \le \frac{1}{\sqrt{n_x}}\|Vx\|_2 \le \|Vx\|_\infty, \quad \forall x \in \mathbb{R}^{n_x}, \quad j = 1, \ldots, m. \tag{5.75}$$

It follows that $\Pi(V, \mathbf{1})$ is robustly invariant and the constraints of (5.74) are therefore necessarily feasible. Note that this approach can only be used if the dynamics (5.65) satisfy the strengthened quadratic stability condition (5.75), and of course this may not be the case if K is designed as the unconstrained LQ optimal feedback gain for the nominal dynamics.

Alternatively, V could be chosen on the basis of the nominal model parameters. In particular, in the absence of uncertainty (i.e. for $\Phi = \Phi^{(0)}$), and for the case that the eigenvalues of $\Phi^{(0)}$ are real, an obvious choice for V is the inverse of the (right) eigenvector matrix of $\Phi^{(0)}$ since this gives $V\Phi^{(0)}W = \Lambda$, where Λ is the eigenvalue matrix of $\Phi^{(0)}$ and W is the corresponding eigenvector matrix. Given that the feedback gain K is stabilizing by assumption, the elements of Λ must be no greater than 1 in absolute value, and hence (5.67) holds for any chosen $\alpha > 0$ in this case. Similarly, if $\Phi^{(0)}$ has complex eigenvalues, then V and $W = V^{-1}$ can be chosen to be real matrices such that $V\Phi^{(0)}W$ is in (real) Jordan normal form. In this case, $V\Phi^{(0)}W$ is block diagonal—for example if the eigenvalues of $\Phi^{(0)}$ are distinct and equal to

$$\lambda_1, \ldots, \lambda_p, \sigma_1 \pm j\omega_1, \ldots, \sigma_q \pm j\omega_q,$$

where $p + 2q = n_x$, then

$$V\Phi^{(0)}W = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & \begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} \sigma_q & \omega_q \\ -\omega_q & \sigma_q \end{bmatrix} \end{bmatrix}.$$

For this choice of V, the structure of $V\Phi^{(0)}W$ implies that a necessary and sufficient condition for existence of $\alpha > 0$ satisfying $|V\Phi^{(0)}W|\alpha \leq \alpha$ is that the eigenvalues of $\Phi^{(0)}$ should lie in the box in the complex plane with vertices at $\pm 1 \pm j$ shown in Fig. 3.9. For the case considered here, in which the model (5.1) and (5.2) has uncertain parameters A, B, the same approach can be used to define V on the basis of the nominal model pair $A^{(0)}$, $B^{(0)}$. If the uncertainty is sufficiently small and the eigenvalues of $\Phi^{(0)} = A^{(0)} + B^{(0)}K$ are sufficiently contractive, then the invariance condition (5.67) will hold for all $j = 1, \ldots, m$ for some $\alpha > 0$, and (5.74) could be used to maximize $\Pi(V, \alpha)$ for this choice of V.

Example 5.3 For the uncertain system (5.1)–(5.3) with model parameters defined in Example 5.1, define V as the inverse of the (right) eigenvector matrix of $\Phi^{(0)} = A^{(0)} + B^{(0)}K$ for $K = [0.19\ 0.34]$. For this choice of V, the Perron–Frobenius eigenvalue of $\tilde{\Phi}$ defined in Lemma 5.4 is 0.89, so Lemma 5.4 indicates that the low-complexity polytopic set $\Pi(V, \alpha)$ is robustly positively invariant for some $\alpha > 0$ for (5.1)–(5.3) under $u = Kx$. Choosing α so as to maximize the area of $\Pi(V, \alpha)$ by solving the convex optimization (5.74) then gives

$$\Pi(V, \alpha) = \left\{ x : \left| \begin{bmatrix} -0.997 & -0.084 \\ -0.229 & -0.973 \end{bmatrix} x \right| \leq \begin{bmatrix} 15.80 \\ 5.90 \end{bmatrix} \right\} \tag{5.76}$$

This set is shown in Fig. 5.3. ◊

5.4.2 *Recursive State Bounding and Low-Complexity Polytopic Tubes*

In order to account for the effects of model uncertainty on predicted state and control trajectories, this section considers the construction, through a recursive bounding procedure, of tubes $\{\mathcal{X}_{0|k}, \mathcal{X}_{1|k}, \ldots\}$ such that $x_{i|k} \in \mathcal{X}_{i|k}$ for all $i = 0, 1, \ldots$. For convenience, and assuming the terminal constraint to be defined by a low-complexity

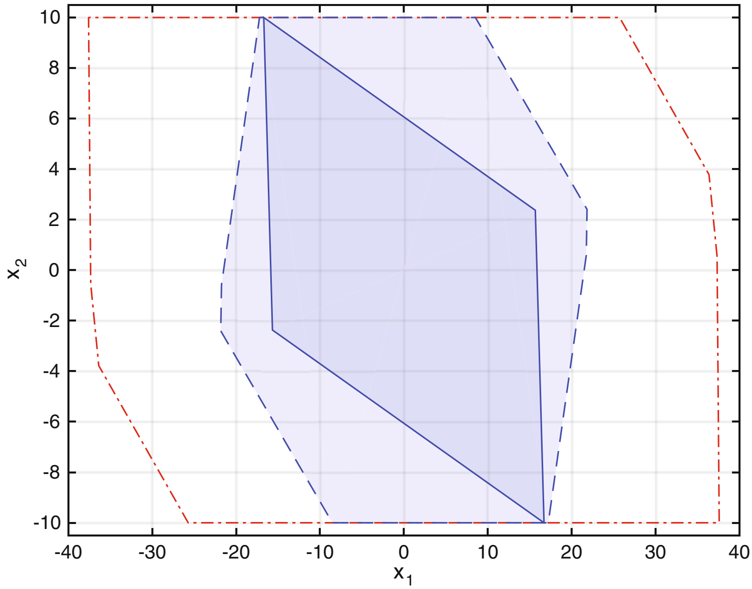

Fig. 5.3 The low-complexity polytopic set $\Pi(V,\alpha)$ defined by (5.76) (*solid line*) and the maximal RPI set (*dashed line*) under the LQ-optimal feedback law $u = Kx$ for the system of Example 5.1. Also shown is the maximal robustly controlled invariant set for this system (*dash-dotted line*)

polytopic invariant set $\Pi(V,\alpha)$, we define each tube cross section as a low-complexity polytope of the form

$$\mathcal{X}_{i|k} = \{x : \underline{\alpha}_{i|k} \le Vx \le \overline{\alpha}_{i|k}\}.$$

Here the matrix $V \in \mathbb{R}^{n_x \times n_x}$ is assumed to be determined offline so that $\Pi(V,\alpha)$ satisfies the robust invariance conditions of Theorem 5.6 for some α and for a given linear feedback gain K, and the parameters $\underline{\alpha}_{i|k}, \overline{\alpha}_{i|k}$ are variables in the online MPC optimization. We also assume that the predicted control input is given by the open-loop strategy

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \quad i = 0, 1 \ldots \tag{5.77}$$

with $c_{i|k} = 0$ for all $i \ge N$.

The tube cross sections $\mathcal{X}_{i|k}$ are computed using a sequence of one-step ahead bounds on future model states. For the transformed state variable

$$\xi_{i|k} \doteq Vx_{i|k}$$

and the predicted control law (5.77), we obtain the dynamics

$$\xi_{i+1|k} = \tilde{\Phi}\xi_{i|k} + \tilde{B}c_{i|k}, \quad (\tilde{\Phi}, \tilde{B}) \in \text{Co}\big\{(\tilde{\Phi}^{(1)}, \tilde{B}^{(1)}), \ldots, (\tilde{\Phi}^{(m)}, \tilde{B}^{(m)})\big\}$$

where $\tilde{\Phi}^{(j)} = V\Phi^{(j)}W$, $W = V^{-1}$, and $\tilde{B}^{(j)} = VB^{(j)}$ for $j = 1, \ldots, m$. Using these transformed dynamics, it is easy to show that $\underline{\alpha}_{i+1|k} \le \xi_{i+1|k} \le \overline{\alpha}_{i+1|k}$ for all $\xi_{i|k}$ satisfying $\underline{\alpha}_{i|k} \le \xi_{i|k} \le \overline{\alpha}_{i|k}$ if and only if

$$\begin{aligned}\underline{\alpha}_{i+1|k} &\le \tilde{\Phi}^+\underline{\alpha}_{i|k} + \tilde{\Phi}^-(-\overline{\alpha}_{i|k}) + \tilde{B}c_{i|k}\\ \overline{\alpha}_{i+1|k} &\ge \tilde{\Phi}^+\overline{\alpha}_{i|k} + \tilde{\Phi}^-(-\underline{\alpha}_{i|k}) + \tilde{B}c_{i|k}\end{aligned}$$

where, as in Sect. 5.4.1, $A^+ = \max\{A, 0\}$ and $A^- = \max\{-A, 0\}$ denote, respectively, the absolute values of the positive and negative elements of a real matrix A. Invoking these conditions for all $(\tilde{\Phi}, \tilde{B})$ in the uncertainty set associated with the model parameters yields, by linearity and convexity, a finite set of conditions:

$$\begin{aligned}\underline{\alpha}_{i+1|k} &\le (\tilde{\Phi}^{(j)})^+\underline{\alpha}_{i|k} - (\tilde{\Phi}^{(j)})^-\overline{\alpha}_{i|k} + \tilde{B}^{(j)}c_{i|k}\\ \overline{\alpha}_{i+1|k} &\ge (\tilde{\Phi}^{(j)})^+\overline{\alpha}_{i|k} - (\tilde{\Phi}^{(j)})^-\underline{\alpha}_{i|k} + \tilde{B}^{(j)}c_{i|k}\end{aligned} \tag{5.78}$$

for $j = 1, \ldots, m$ and $i = 0, \ldots, N-1$. Thus the linear constraints (5.78) ensure that the predicted model trajectories satisfy $x_{i|k} \in \mathcal{X}_{i|k}$ for $i = 1, \ldots, N$, with $\mathcal{X}_{i|k} = \tilde{\Pi}(V, \underline{\alpha}_{i|k}, \overline{\alpha}_{i|k})$.

Given a tube $\{\mathcal{X}_{0|k}, \ldots, \mathcal{X}_{N-1|k}\}$ bounding predicted state trajectories and the predicted control law (5.77), the constraints $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$ at prediction times $i = 0, \ldots, N-1$ can be imposed through the conditions

$$(\tilde{F} + G\tilde{K})^+\overline{\alpha}_{i|k} - (\tilde{F} + G\tilde{K})^-\underline{\alpha}_{i|k} + Gc_{i|k} \le 1, \tag{5.79}$$

for $i = 0, \ldots, N-1$, where $\tilde{F} = FW$ and $\tilde{K} = KW$. Likewise, the initial and terminal conditions

$$\underline{\alpha}_{0|k} \le Vx_k, \qquad \overline{\alpha}_{0|k} \ge Vx_k \tag{5.80}$$

$$\underline{\alpha}_{N|k} = -\alpha, \qquad \overline{\alpha}_{N|k} = \alpha \tag{5.81}$$

enforce the constraints that $x_k \in \mathcal{X}_{0|k}$ and $\mathcal{X}_{N|k} \subseteq \Pi(V, \alpha)$ (Fig. 5.4).

Consider next the definition of the MPC performance index. Assuming a nominal (rather than a worst case) approach, the predicted cost is defined by (5.29) with $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$. Thus if K is the unconstrained LQ optimal feedback gain for the nominal model, then by Theorem 2.10 we have

$$J(s_{0|k}, \mathbf{c}_k) = \|s_{0|k}\|^2_{W_x} + \|\mathbf{c}_k\|^2_{W_c} \tag{5.82}$$

where $W_c = \mathrm{diag}\{B^{(0)T}W_xB^{(0)} + R, \ldots, B^{(0)T}W_xB^{(0)} + R\}$. For the case that K is not LQ optimal for (5.29), the cost is a quadratic function of $(s_{0|k}, \mathbf{c}_k)$,

$$J(s_{0|k}, \mathbf{c}_k) = \begin{bmatrix} s_{0|k} \\ \mathbf{c}_k \end{bmatrix}^T W \begin{bmatrix} s_{0|k} \\ \mathbf{c}_k \end{bmatrix}$$

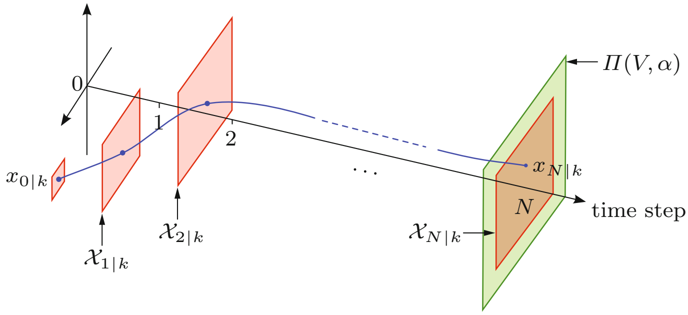

Fig. 5.4 A low-complexity polytopic tube and the predicted evolution of the state for a single realization of the model parameters

where W is the solution of the Lyapunov equation (2.34). Rather than set the initial state $s_{0|k}$ of the nominal cost index equal to x_k as was done in Sect. 5.3, we give here a more general formulation of the algorithm in which $s_{0|k}$ is an optimization variable.

The robust MPC algorithm, which requires the online solution of a QP in $O\big(N(2n_x + n_u)\big)$ variables and $O\big(N(2n_x + n_C)\big)$ inequality constraints, can be stated as follows.

Algorithm 5.6 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{\substack{s_{0|k},\mathbf{c}_k \\ \underline{\alpha}_{0|k},\ldots,\underline{\alpha}_{N|k} \\ \overline{\alpha}_{0|k},\ldots,\overline{\alpha}_{N|k}}}{\text{minimize}} \; J(s_{0|k}, \mathbf{c}_k) \text{ subject to (5.78)–(5.81) and } \underline{\alpha}_{0|k} \le V s_{0|k} \le \overline{\alpha}_{0|k} \tag{5.83}$$

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ in (5.83). ◁

The construction of the constraints (5.78)–(5.81) ensures recursive feasibility of the online MPC optimization (as demonstrated by Theorem 5.7). Therefore, if the dynamics (5.65) that govern predicted states for $i \ge N$ satisfy the quadratic stability condition (5.30), then the closed-loop system under Algorithm 5.6 can be analysed using Lemma 5.3 and the argument of Theorem 5.3. This approach (which assumes that (5.83) incorporates the additional constraint $s_{0|k} = x_k$) shows that the control law of Algorithm 5.6 asymptotically stabilizes the origin of the state space of (5.1)–(5.3).

However there is no intrinsic requirement in Algorithm 5.6 that the mode 2 prediction dynamics (5.65) should be quadratically stable. Instead it is required that $\Pi(V,\alpha)$ is robustly invariant for these dynamics. This is equivalent to the requirement that $\|V_\alpha x\|_\infty$, where $V_\alpha \doteq (\text{diag}\{\xi_1, \ldots, \xi_{n_x}\})^{-1}V$, is a (piecewise-linear) Lyapunov function for (5.65). Consequently, the additional assumption of quadratic stability may be over-restrictive here, and we therefore use a different approach to analyse stability that does not require an assumption of quadratic stability but is based instead on the robust invariance property of $\Pi(V,\alpha)$. Thus we assume that $\Pi(V,\alpha)$ is a λ-contractive set [27] for (5.65), namely that $\lambda \in [0,1)$ exists such that

$$\Phi \Pi(V,\alpha) \subseteq \lambda \Pi(V,\alpha), \quad \forall \Phi \in \Omega_K, \tag{5.84}$$

or equivalently

$$\|V_\alpha \Phi^{(j)} x\|_\infty \le \lambda \|V_\alpha x\|_\infty, \quad j = 1,\dots,m \tag{5.85}$$

for all $x \in \Pi(V,\alpha) = \{x : \|V_\alpha x\|_\infty \le 1\}$. Under this assumption, the trajectories of the closed-loop system

$$x_{k+1} = \Phi_k x_k + B_k c^*_{0|k}, \quad (\Phi_k, B_k) \in \mathrm{Co}\big\{(\Phi^{(1)}, B^{(1)}),\dots,(\Phi^{(m)}, B^{(m)})\big\}, \tag{5.86}$$

satisfy the following bound.

Lemma 5.5 *If (5.85) holds for $\lambda \in [0,1)$, then the bound*

$$\limsup_{k\to\infty} \|V_\alpha x_k\|_\infty \le \frac{1}{(1-\lambda)} \sup_{k\ge 0} \max_j \|V_\alpha B^{(j)} c^*_{0|k}\|_\infty \tag{5.87}$$

holds along trajectories (5.86).

Proof Using the triangle inequality and (5.85), we have, for all $k \ge 0$ along trajectories of (5.86),

$$\|V_\alpha x_{k+1}\|_\infty \le \lambda \|V_\alpha x_k\|_\infty + \max_j \|V_\alpha B^{(j)} c^*_{0|k}\|_\infty.$$

It follows that

$$\|V_\alpha x_k\|_\infty \le \lambda^k \|V_\alpha x_0\|_\infty + \sum_{i=0}^{k-1} \lambda^{k-i-1} \max_j \|V_\alpha B^{(j)} c^*_{0|i}\|_\infty,$$

and the bound (5.87) is obtained from this inequality in the limit as $k \to \infty$ since $\max_j \|V_\alpha B^{(j)} c^*_{0|k}\|_\infty \le \sup_{k\ge 0} \max_j \|V_\alpha B^{(j)} c^*_{0|k}\|_\infty$. □

Theorem 5.7 *The optimization (5.83) is recursively feasible and if $\Pi(V,\alpha)$ satisfies (5.85) for $\lambda \in [0,1)$, then $x = 0$ is an asymptotically stable equilibrium of the closed-loop system (5.1)–(5.3) under the control law of Algorithm 5.6, with region of attraction equal to the feasible set:*

$$\mathcal{F}_N = \big\{x_k : \exists(\mathbf{c}_k, \underline{\alpha}_{0|k},\dots,\underline{\alpha}_{N|k}, \overline{\alpha}_{0|k},\dots,\overline{\alpha}_{N|k}) \textit{ satisfying (5.78)–(5.81)}\big\}.$$

Proof Given feasibility at time k, a feasible solution at $k+1$ is obtained with $\mathcal{X}_{i|k+1} = \mathcal{X}_{i+1|k}$, $i = 0,\dots,N-2$, $\mathcal{X}_{N-1|k+1} = \mathcal{X}_{N|k+1} = \Pi(V,\alpha)$, and

$$
\begin{aligned}
s_{0|k+1} &= \Phi^{(0)} s^*_{0|k} + B^{(0)} c^*_{0|k} \\
\mathbf{c}_{k+1} &= (c^*_{1|k}, \dots, c^*_{N-1|k}, 0) \\
\underline{\alpha}_{i|k+1} &= \underline{\alpha}^*_{i+1|k}, \ i = 0, \dots, N-1, \quad \underline{\alpha}_{N|k+1} = -\alpha, \\
\overline{\alpha}_{i|k+1} &= \overline{\alpha}^*_{i+1|k}, \ i = 0, \dots, N-1, \quad \overline{\alpha}_{N|k+1} = \alpha.
\end{aligned}
$$

The feasibility of this solution follows from the constraints at time k and robust invariance of $\Pi(V, \alpha)$ (which imply that constraints (5.78), (5.79) and (5.81) are necessarily satisfied), and from $x_{1|k} \in \mathcal{X}_{1|k} = \mathcal{X}_{0|k+1}$ (which ensures that (5.80) and $s_{0|k+1} \in \mathcal{X}_{0|k+1}$ are satisfied).

The optimality of the solution of (5.83) at time $k+1$ therefore ensures that the optimal cost satisfies

$$
J(s^*_{0|k+1}, \mathbf{c}^*_{k+1}) \le J(s^*_{0|k}, \mathbf{c}^*_k) - (\|s^*_{0|k}\|^2_Q + \|K s^*_{0|k} + c^*_{0|k}\|^2_R),
$$

for all $k \ge 0$, and hence

$$
\sum_{k=0}^{\infty} (\|s^*_{0|k}\|^2_Q + \|K s^*_{0|k} + c^*_{0|k}\|^2_R) \le J(s^*_{0|k}, \mathbf{c}^*_0).
$$

Since $Q, R \succ 0$, it follows that $s^*_{0|k} \to 0$ and $c^*_{0|k} \to 0$ as $k \to \infty$, and, for any $\epsilon > 0$ there necessarily exists a finite n such that

$$
\|c^*_{0|k}\|_\infty \le \epsilon \ \ \forall k \ge n.
$$

Therefore Lemma 5.5 implies

$$
\limsup_{m \to \infty} \|V_\alpha x_{n+m}\|_\infty \le \frac{\epsilon}{(1-\lambda)} \max_j \|V_\alpha B^{(j)}\|_\infty
$$

and since this bound can be made arbitrarily small by choosing sufficiently small ϵ, it follows that $x_k \to 0$ as $k \to \infty$ for all $x_0 \in \mathcal{F}_N$. To complete the proof, we note that Algorithm 5.6 gives $u = Kx$ for all $x \in \Pi(V, \alpha)$ (since (5.82) and the robust invariance of $\Pi(V, \alpha)$ imply that $\mathbf{c} = 0$ is necessarily optimal for (5.83) in this case), and from (5.85) this feedback law is locally exponentially stabilizing for $x = 0$. □

It may be desirable to detune the state feedback gain K with a view to enlarging the terminal set and hence also the size of the region of attraction, and in such cases the justification for the nominal cost of (5.82) will no longer be valid. It is, however, possible to construct a worst-case cost [17] using the l_1-norm of the bounds $\underline{\alpha}_{i|k}, \overline{\alpha}_{i|k}$ $i = 0, \dots, N$ together with a terminal penalty term that is designed to preserve the monotonic non-increasing property of the optimized cost. The resulting robust MPC law has the same closed-loop properties as Algorithm 5.6 and its online optimization is a linear program as a result of the linear dependence of the cost on the degrees of freedom.

5.5 Tubes with General Complexity Polytopic Cross Sections

The low-complexity polytopic tubes discussed in Sect. 5.4 are convenient from the perspective of online computation but could be unduly conservative. In order to obtain tighter bounds on predicted trajectories, it is possible to remove the restriction to low-complexity polytopes and instead require that the predicted states lie in polytopic tubes with cross sections described by arbitrary but fixed numbers of linear inequalities [18–20, 23]. This modification can be beneficial in terms of the size of the feasible set of initial conditions and closed-loop performance, and it can also simplify offline computation by removing the need for a low-complexity robustly invariant terminal set.

In this section, we consider tubes $\{\mathcal{X}_{0|k}, \mathcal{X}_{1|k}, \ldots\}$ with polytopic cross sections defined in terms of linear inequalities:

$$\mathcal{X}_{i|k} = \{x : Vx \leq \alpha_{i|k}\} \tag{5.88}$$

where $V \in \mathbb{R}^{n_V \times n_x}$ is a full-rank matrix with a number of rows, n_V, that is typically greater than the number, $2n_x$, required to define a low-complexity polytope. The matrix V is to be chosen offline, whereas the parameter $\alpha_{i|k} \in \mathbb{R}^{n_V}$ is retained as a variable online MPC optimization. Since it is expressed as an intersection of half-spaces, $\mathcal{X}_{i|k}$ is necessarily convex; however, unlike a low-complexity polytope, there is no requirement here for $\mathcal{X}_{i|k}$ to be bounded.

As in Sect. 5.4, we assume an open-loop prediction strategy, and hence predicted states and inputs of the models (5.1) and (5.2) evolve according to

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \tag{5.89a}$$

$$x_{i+1|k} = \Phi_{i|k}x_{i|k} + B_{i|k}c_{i|k}, \ (\Phi_{i|k}, B_{i|k}) \in \mathrm{Co}\{(\Phi^{(j)}, B^{(j)}), \ j = 1, \ldots, m\} \tag{5.89b}$$

for $i = 0, 1, \ldots$ with $c_{i|k} = 0$ for all $i \geq N$. Although the associated predicted state and control trajectories are the same as in Sect. 5.4, the recursive bounding approach of Sect. 5.4 is no longer applicable. Essentially, this is because closed-form expressions for the extreme points of $\mathcal{X}_{i|k}$ are not generally available when the set is parameterized, as in (5.88), by an intersection of half-spaces. Instead we use the following result based on Farkas' Lemma [31, 32], to express polytopic set inclusion conditions in terms of algebraic conditions.

Lemma 5.6 *Let $\mathcal{S}_i \doteq \{x : F_i x \leq b_i\}$, $i = 1, 2$, be non-empty subsets of $\mathbb{R}^{n_x}$. Then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if there exists a nonnegative matrix H satisfying*

$$HF_1 = F_2 \tag{5.90a}$$

$$Hb_1 \leq b_2 \tag{5.90b}$$

Proof To show that the conditions (5.90a, 5.90b) are sufficient for $\mathcal{S}_1 \subseteq \mathcal{S}_2$, suppose that x satisfies $F_1 x \le b_1$. Then $HF_1 x \le Hb_1$ since $H \ge 0$, so (5.90a, 5.90b) imply $F_2 x \le b_2$ and it follows that $x \in \mathcal{S}_2$ for all $x \in \mathcal{S}_1$.

To show the necessity of (5.90a, 5.90b), assume $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then we must have $\mu \le b_2$ where, for each i, the ith element of μ is defined by

$$\mu_i \doteq \max_{x \in \mathbb{R}^{n_x}} (F_2)_i x \quad \text{subject to} \quad F_1 x \le b_1$$

with $(F_2)_i$ denoting the ith row of F_2. The optimal value of this linear program (which is feasible by assumption) is equal to the optimal value of the dual problem [33]:

$$\mu_i = \min_{h \in \mathbb{R}^{n_1}} b_1^T h \quad \text{subject to} \quad h^T F_1 = (F_2)_i \text{ and } h \ge 0 \tag{5.91}$$

where n_1 is the number of rows of F_1. Let h_i^* be the optimal solution of this linear program and let H be the matrix with ith row equal to h_i^{*T}. Then (5.91) implies that $H \ge 0$ exists satisfying (5.90a, 5.90b) whenever $\mathcal{S}_1 \subseteq \mathcal{S}_2$. □

From (5.89b) and Lemma 5.6, we have $x_{i+1|k} \in \mathcal{X}_{i+1|k}$ for all $x_{i|k} \in \mathcal{X}_{i|k}$ and $i = 0, \dots, N-1$ if there exist matrices $H^{(j)} \ge 0$, $j = 1, \dots, m$ satisfying

$$\alpha_{i+1|k} \ge H^{(j)} \alpha_{i|k} + VB^{(j)} c_{i|k}, \quad i = 0, \dots, N-1 \tag{5.92a}$$

$$H^{(j)} V = V \Phi^{(j)} \tag{5.92b}$$

for $j = 1, \dots, m$. Similarly, (5.89a) and Lemma 5.6 imply that the constraint $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$ is satisfied for all $x_{i|k} \in \mathcal{X}_{i|k}$, $i = 0, \dots, N-1$, if there exists a matrix $H_c \ge 0$ satisfying

$$H_c \alpha_{i|k} + G c_{i|k} \le \mathbf{1}, \quad i = 0, \dots, N-1 \tag{5.93a}$$

$$H_c V = F + GK. \tag{5.93b}$$

The constraints (5.92a) and (5.93a) are nonlinear if $H^{(j)}$ and H_c are treated as variables concurrently with $\alpha_{i|k}$. However these conditions become linear constraints in the online MPC optimization if $H^{(j)}$ and H_c are determined offline. This has the effect of making (5.92a) and (5.93a) only sufficient (not necessary) to ensure $x_{i+1|k} \in \mathcal{X}_{i+1|k}$ and $Fx_{i|k} + Gc_{i|k} \le \mathbf{1}$ for all $x_{i|k} \in \mathcal{X}_{i|k}$. Therefore, to relax the constraints on $\alpha_{i|k}$, a convenient criterion for the offline design of $H^{(j)}$ and H_c is to minimize the sum of the elements in each row of these matrices subject to (5.92b) and (5.93b). This suggests computing the rows, $(H_c)_i$, of H_c for $i = 1, \dots, n_C$ offline by solving the linear programs

$$h_i^* = \arg \min_{h \in \mathbb{R}^{n_V}} \mathbf{1}^T h \quad \text{subject to} \quad h^T V = (F + GK)_i \text{ and } h \ge 0 \tag{5.94}$$

and setting $(H_c)_i = h_i^{*T}$, where $(H_c)_i$ and $(F + GK)_i$ denote the ith rows of H_c and $F + GK$, respectively. Similarly, each $H^{(j)}$, for $j = 1, \ldots, m$, can be determined offline by solving the linear programs

$$h_i^{(j)*} = \arg \min_{h \in \mathbb{R}^{n_V}} \mathbf{1}^T h \ \text{ subject to } \ h^T V = V_i \Phi^{(j)} \text{ and } h \geq 0 \tag{5.95}$$

for $i = 1, \ldots, n_V$ and setting $H_i^{(j)} = h_i^{(j)*^T}$, where $H_i^{(j)}$ and V_i denote, respectively, the ith rows of $H^{(j)}$ and V.[3]

The constraints $Fx_{i|k} + Gc_{i|k} \leq \mathbf{1}$ can be imposed for $i \geq N$ through terminal constraints on $\alpha_{N|k}$. From (5.92a) and (5.93a) with $c_{i|k} = 0$, the required conditions are given by

$$\alpha_{i+1|k} \geq H^{(j)} \alpha_{i|k}, \tag{5.96a}$$

$$H_c \alpha_{i|k} \leq \mathbf{1} \tag{5.96b}$$

for $j = 1, \ldots, m$ and $i = N, N+1, \ldots$. Although these conditions are expressed in terms of inequalities, the fact that H_c and $H^{(j)}$ are nonnegative matrices implies that (5.96a, 5.96b) are feasible for all $i = N, N+1, \ldots$ if and only if they are feasible for the unique trajectory $\{\alpha_{N|k}, \alpha_{N+1|k}, \ldots\}$ defined by the piecewise-linear dynamics

$$(\alpha_{i+1|k})_l = \max_{j \in \{1,\ldots,m\}} H_l^{(j)} \alpha_{i|k}, \quad l = 1, \ldots, n_V, \tag{5.97}$$

where $(\alpha_{i|k})_l$ denotes the lth element of $\alpha_{i|k}$. The following result shows that these dynamics are stable if V is chosen so that the set $\{x : Vx \leq \mathbf{1}\}$ is contractive for the mode 2 prediction dynamics defined in (5.65).

Lemma 5.7 *If $\{x : Vx \leq \mathbf{1}\}$ is λ-contractive for some $\lambda \in [0, 1)$ under the dynamics (5.65), then (5.97) satisfies $\|\alpha_{i+1|k}\|_\infty \leq \lambda \|\alpha_{i|k}\|_\infty$, for all i.*

Proof If $\mathcal{S} = \{x : Vx \leq \mathbf{1}\}$ is λ-contractive, namely if $\Phi^{(j)}\mathcal{S} \subseteq \lambda\mathcal{S}$ for all $j = 1, \ldots, m$, then Lemma 5.6 implies $\|H^{(j)}\|_\infty \leq \lambda$. □

---

[3]With H_c and $H^{(j)}$, $j = 1, \ldots, m$ chosen so as to minimize sum of elements in each of their rows, the conditions (5.92) and (5.93) include the corresponding conditions that were derived in Sect. 5.4.2 for low-complexity polytopes as a special case. Thus, expressing the low-complexity polytopic set $\{x : \underline{\alpha} \leq V_0 x \leq \overline{\alpha}\}$ equivalently as $\{x : Vx \leq \alpha\}$ with $V = [V_0^T \ -V_0^T]^T$ and $\alpha = [\overline{\alpha}^T \ -\underline{\alpha}^T]^T$, the solutions of (5.94) and (5.95) can be obtained in closed form as

$$H_c = [(\tilde{F} + G\tilde{K})^+ \ (\tilde{F} + G\tilde{K})^-] \text{ and } H^{(j)} = \begin{bmatrix} (\tilde{\Phi}^{(j)})^+ & (\tilde{\Phi}^{(j)})^- \\ (\tilde{\Phi}^{(j)})^- & (\tilde{\Phi}^{(j)})^+ \end{bmatrix}, \ j = 1, \ldots, m.$$

Therefore conditions (5.78) and (5.79) are identical to (5.92) and (5.93) for this case.

To meet the requirement in Lemma 5.7 that the set $\{x : Vx \leq \mathbf{1}\}$ is λ-contractive for $\lambda \in [0, 1)$, V can be chosen, for example, so that this set is the maximal robust invariant set for the system $x_{k+1} = (1/\lambda)\Phi x_k$, $\Phi \in \Omega_K$ subject to $(F + GK)x_k \leq \mathbf{1}$. For any λ in the interval $(\rho, 1)$, where ρ is the joint spectral radius defined in (5.72), it can be shown that this maximal robust invariant set is defined by a set of linear inequalities of the form

$$\begin{aligned}\mathcal{X}^{(n)} = \{x : (F + GK)x \leq \mathbf{1}, (F + GK)\Phi^{(j_1)}x \\ \leq \lambda\mathbf{1}, \ \ldots, (F + GK)\Phi^{(j_1)} \cdots \Phi^{(j_n)}x \leq \lambda^n\mathbf{1}, \ j_i = 1, \ldots, m, \ i = 1, \ldots, n\}\end{aligned} \tag{5.98}$$

where n is necessarily finite if the pair $\big(\Phi, (F + GK)\big)$ is observable for some $\Phi \in \Omega_K$. Rather than prove this result (details of which can be found in [34]), we consider next the related problem of determining the maximal positively invariant set for (5.97) contained in the set on which $H_c\alpha \leq \mathbf{1}$.

Under the assumption that the set $\{x : Vx \leq \mathbf{1}\}$ is λ-contractive, which by Lemma 5.7 implies that the system describing the evolution of $\alpha_{i|k}$ in (5.97) is asymptotically stable, the constraint that $H_c\alpha_{i|k} \leq \mathbf{1}$ for all $i \geq N$ can be ensured by imposing a finite set of linear constraints on $\alpha_{N|k}$. To see this, let $\mathcal{A}^{(n)}$ denote the set

$$\begin{aligned}\mathcal{A}^{(n)} \doteq \big\{\alpha : H_c\alpha \leq \mathbf{1}, \ H_cH^{(j_1)}\alpha \leq \mathbf{1}, \ \ldots, \ H_cH^{(j_1)} \cdots H^{(j_n)}\alpha \leq \mathbf{1}, \\ j_i = 1, \ldots, m, \ i = 1, \ldots, n\big\},\end{aligned}$$

then the maximal positively invariant set defined by

$$\mathcal{A}^{\text{MPI}} \doteq \{\alpha_{N|k} : \text{ (5.96b), and (5.97) hold for } i = N, N + 1, \ldots\},$$

is given by $\mathcal{A}^{\text{MPI}} = \lim_{n\to\infty} \mathcal{A}^{(n)}$. The following result, which is an extension of Theorems 2.3 and 3.1, gives a characterisation of $\mathcal{A}^{\text{MPI}}$ in terms of a finite number of linear conditions.

Corollary 5.3 *The maximal positively invariant set for the system (5.97) and constraints (5.96b) is given by*

$$\mathcal{A}^{\text{MPI}} = \mathcal{A}^{(\nu)}$$

where ν is the smallest integer such that $\mathcal{A}^{(\nu)} \subseteq \mathcal{A}^{(\nu+1)}$. If $\mathcal{A}^{\text{MPI}}$ is bounded and $\{x : Vx \leq \mathbf{1}\}$ is λ-contractive for $\lambda \in [0, 1)$ under (5.65), then n is necessarily finite.

Proof If $\mathcal{A}^{(\nu)} \subseteq \mathcal{A}^{(\nu+1)}$, then $\mathcal{A}^{(\nu)}$ is necessarily invariant under (5.97) and is therefore a subset of $\mathcal{A}^{\text{MPI}}$ in this case. But $\mathcal{A}^{\text{MPI}}$ is by definition a subset of $\mathcal{A}^{(n)}$ for all n and it follows that $\mathcal{A}^{(\nu)} = \mathcal{A}^{\text{MPI}}$.

Furthermore, if $\mathcal{A}^{\text{MPI}}$ is bounded, then $\mathcal{A}^{(n)}$ must also be bounded for some n. From the definition of $\mathcal{A}^{(n)}$ and Lemma 5.7, we have

$$\begin{aligned}\mathcal{A}^{(n+1)} = \mathcal{A}^{(n)} \cap \bigl\{\alpha : H_c H^{(j_1)} \cdots H^{(j_{n+1})}\alpha \le \mathbf{1},\ j_i = 1, \ldots, m, \\ i = 1, \ldots, n+1\bigr\} \\ \supseteq \mathcal{A}^{(n)} \cap \{\alpha : \|H_c\|_\infty \lambda^{n+1} \|\alpha\|_\infty \le 1\},\end{aligned}$$

and $\lambda \in [0, 1)$ therefore implies $\mathcal{A}^{(n+1)} \supseteq \mathcal{A}^{(n)}$ for some finite n. □

Note that the assertion that $\mathcal{A}^{\text{MPI}}$ is bounded, which is used in Corollary 5.3 to ensure that the maximal positively invariant set for (5.97) and (5.96b) is finitely determined, can always be assumed to hold in practice. This is because unboundedness of $\mathcal{A}^{\text{MPI}}$ would indicate that some of the rows of V are redundant. Therefore, these rows could be removed without affecting the set $\{x : Vx \le \alpha\}$ for $\alpha \in \mathcal{A}^{\text{MPI}}$.

Lemma 5.7 implies that a small value of λ will make the trajectories of (5.97) converge more rapidly, but choosing λ close to the joint spectral radius ρ could require a large number of rows in V such that $\{x : Vx \le \mathbf{1}\}$ is λ-contractive. Conversely, larger values of λ typically result in smaller values of n_V but at the same time allow the tube cross sections to grow more rapidly along predicted trajectories, and this can cause the set of feasible initial conditions of an associated MPC law to increase more slowly with the prediction horizon N. The design of λ is discussed further in Example 5.4.

From the preceding discussion, it follows immediately that the predicted trajectories of (5.89a, 5.89b) necessarily satisfy the constraints $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$ for all $i \ge 0$ and for all realizations of model uncertainty if $\{\alpha_{0|k}, \ldots, \alpha_{N|k}\}$ satisfy the initial and terminal conditions

$$Vx_k \le \alpha_{0|k} \tag{5.99}$$

$$\alpha_{N|k} \in \mathcal{A}^{\text{MPI}} \tag{5.100}$$

in addition to the conditions for inclusion (5.92a) and feasibility (5.93a) for $i = 0, \ldots, N-1$. On the basis of these constraints and using the nominal cost (5.82) discussed in Sect. 5.4.2, a robust MPC algorithm can be stated as follows. The associated online optimization is a QP with $O(N(n_x + n_u))$ variables and $O(N(n_x + n_C) + m^\nu)$ constraints.

Algorithm 5.7 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{\substack{s_{0|k}, \mathbf{c}_k \\ \alpha_{0|k}, \ldots, \alpha_{N|k}}}{\text{minimize}}\ J(s_{0|k}, \mathbf{c}_k)\ \text{subject to}\quad (5.92a), (5.93a), (5.99), (5.100), \text{ and } Vs_{0|k} \le \alpha_{0|k} \tag{5.101}$$

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal value of $\mathbf{c}_k$ in (5.101). ◁

Theorem 5.8 *If $\{x : Vx \le \mathbf{1}\}$ is λ-contractive for some $\lambda \in [0, 1)$, then the optimization (5.101) is recursively feasible and the control law of Algorithm 5.7 asymptotically stabilizes the origin of the system (5.1)–(5.3), with a region of attraction equal to the feasible set:*

$$\mathcal{F}_N = \big\{x_k : \exists(\mathbf{c}_k, \alpha_{0|k}, \dots, \alpha_{N|k}) \text{ satisfying (5.92a), (5.93a), (5.99), (5.100)}\big\}.$$

Proof The proof of this result closely follows that of Theorem 5.7, and we therefore provide only a sketch of the argument here. The recursive feasibility of (5.101) follows from the feasibility of the solution at time $k + 1$ given by

$$\begin{aligned} s_{0|k+1} &= \Phi^{(0)} s^*_{0|k} + B^{(0)} c^*_{0|k} \\ \mathbf{c}_{k+1} &= (c^*_{1|k}, \dots, c^*_{N-1|k}, 0) \\ \alpha_{i|k+1} &= \alpha^*_{i+1|k}, \quad i = 0, \dots, N-1, \\ (\alpha_{N|k+1})_l &= \max_{j\in\{1,\dots,m\}} H_l^{(j)} \alpha^*_{N|k}, \quad l = 1, \dots, n_V \end{aligned}$$

since with these parameters we obtain $\mathcal{X}_{i|k+1} = \mathcal{X}_{i+1|k}$ for $i = 0, \dots, N-1$ and $\alpha_{N|k+1} \in \mathcal{A}^{\text{MPI}}$ (since $\alpha^*_{N|k} \in \mathcal{A}^{\text{MPI}}$). Thus at time $k+1$, (5.92a) and (5.93a) hold for $i = 0, \dots, N-1$ and likewise (5.100) holds, whereas (5.99) and $V s_{0|k+1} \le \mathbf{1}$ are satisfied at time $k+1$ because $x_{k+1} \in \mathcal{X}_{1|k}$ for all realizations of model uncertainty at time k.

Asymptotic convergence can be shown using the bound

$$\limsup_{k\to\infty} \max\{V x_k\} \le \frac{1}{(1-\lambda)} \sup_{k\ge 0} \max_j \max\{V B^{(j)} c^*_{0|k}\} \tag{5.102}$$

(where $\max\{\cdot\}$ indicates the maximum element of a vector). This follows, analogously to Lemma 5.5, from the assumption that $\{x : Vx \le \mathbf{1}\}$ is λ-contractive for $\lambda \in [0, 1)$, and hence

$$\begin{aligned} \max\{V x_{k+1}\} &\le \max\{V \Phi_k x_k\} + \max\{V B_k c^*_{0|k}\} \\ &\le \lambda \max\{V x_k\} + \max\{V B_k c^*_{0|k}\} \end{aligned}$$

along the trajectories of the closed-loop system. Asymptotic convergence, $x_k \to 0$, then follows from the argument of the proof of Theorem 5.7 applied to (5.102), whereas stability of $x = 0$ follows from local stability under $u = Kx$ and the property that $\mathbf{c}^*_k = 0$ if x_k is sufficiently close to zero. □

Computing offline the matrices $H^{(j)}$ and H_c that appear in the constraints of the online MPC optimization (5.101) as opposed to retaining these matrices as variables in the optimization of predicted performance at each time step could make the handling of constraints conservative. The following corollary of Lemma 5.6

gives conditions under which computing $H^{(j)}$ and H_c offline incurs no conservativeness.

Corollary 5.4 *For given $\mathcal{F}_i \in \mathbb{R}^{n_i \times n_x}$ and $b_i \in \mathbb{R}^{n_i}$, let $\mathcal{S}_i = \{x : F_i x \le b_i\}$, $i = 1, 2$. If there exists $H \ge 0$ satisfying $HF_1 = F_2$ such that each row of H has only one non-zero element, then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $Hb_1 \le b_2$.*

Proof A sufficient condition for $\mathcal{S}_1 \subseteq \mathcal{S}_2$ is $Hb_1 \le b_2$ since then $H \ge 0$ and $HF_1 = F_2$ imply that $F_2 x \le b_2$ whenever $F_1 x \le b_1$. To show that $Hb_1 \le b_2$ is also necessary if each row of H contains only one non-zero element, we first note that $\{x : F_1 x \le b_1\}$ can be assumed without loss of generality to be an irreducible representation of $\mathcal{S}_1$, i.e. for each $i \in \{1, \ldots, n_1\}$ there exists $x \in \mathcal{S}_1$ such that $(F_1)_i x = (b_1)_i$. Suppose that the jth element of the ith row of H is non-zero and choose $x \in \mathcal{S}_1$ so that $(F_1)_j x = (b_1)_j$. Then $HF_1 = F_2$ implies $(F_2)_i x = H_i b_1$, where H_i is the ith row of H. Hence $H_i b_1 \le (b_2)_i$ is needed in order that $x \in \mathcal{S}_2$; repeating this argument for each $i = 1, \ldots, n_2$ shows that $Hb_1 \le b_2$ is necessary for $\mathcal{S}_1 \subseteq \mathcal{S}_2$. □

If V is chosen so that $\{x : Vx \le \mathbf{1}\}$ is the maximal RPI set for the dynamics $x_{k+1} = (1/\lambda)\Phi x_k$, $\Phi \in \Omega_K$, with $(F + GK)x_k \le \mathbf{1}$, then from (5.98), V has the form $V = [(F + GK)^T \;\; V'^T]^T$ for some V'. Therefore it is always possible to choose $H_c \ge 0$ satisfying (5.93b) so that $H_c = [I_{n_C} \;\; 0]$. Hence, by Corollary 5.4, the conditions (5.93a) in the online optimization (5.101) are necessary as well as sufficient for satisfaction of the constraints $Fx_{i|k} + Gc_{i|k} \le \mathbf{1}$ for all $x_{i|k} \in \mathcal{X}_{i|k}, i = 0, \ldots, N-1$ in this case.

The linear dependence of conditions (5.92a), (5.93a), (5.99) and (5.100) on the variables $\{\alpha_{0|k}, \ldots, \alpha_{N|k}\}$ allows the MPC optimization (5.101) to be formulated as a QP problem. However defining the terminal condition in terms of the maximal MPI set $\mathcal{A}^{\text{MPI}}$ may introduce a large number of constraints into the online optimization. At worst this terminal constraint contributes $O(m^\nu)$ linear constraints to (5.101), and although removing redundant constraints reduces this number substantially (see for example the related approach of [35]), the possibility of rapid growth of the number of constraints with m may limit the usefulness of $\mathcal{A}^{\text{MPI}}$ as a terminal set if either the number, m, of vertices of the uncertain model or the value of ν in Corollary 5.3 is large.

To reduce the number of constraints in the online MPC optimization (5.101), it is possible to define the terminal constraint using an invariant set for (5.97) and (5.93b) that is not necessarily maximal. Provided this terminal set is invariant for the dynamics (5.97), the guarantee of recursive feasibility of the MPC optimization in Theorem 5.8 will not be affected. A simple alternative is to replace (5.100) with the terminal constraints

$$\alpha_{N|k} \ge H^{(j)}\alpha_{N|k}, \quad j = 1, \ldots, m \tag{5.103a}$$

$$H_c \alpha_{N|k} \le \mathbf{1}. \tag{5.103b}$$

These constraints are necessarily feasible for some $\alpha_{N|k}$ because of the conditions on H_c and $H^{(j)}$ in (5.92b) and (5.93b) and the asymptotic stability property of the system (5.97). To lessen the impact of this modification, it is possible to invoke the terminal constraint on $\alpha_{T|k}$ by introducing a supplementary horizon of $T - N$ and invoking the constraints (5.96a, 5.96b) for $i = N, \ldots, T - N - 1$ [19, 20]. For further details, see Question 9 on p. 237.

Finally we note that Algorithm 5.7 is based on a nominal predicted cost, but it could be reformulated as a min–max robust MPC algorithm using the worst-case cost discussed in Sect. 5.3.1. For example, if $\check{W}$ satisfies the LMIs, for $j = 1, \ldots, m$,

$$\check{W} - \Psi^{(j)T}\check{W}\Psi^{(j)} \succeq \begin{bmatrix} I & K^T \\ 0 & E^T \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} I & 0 \\ K & E \end{bmatrix}, \quad \Psi^{(j)} = \begin{bmatrix} \Phi^{(j)} & B^{(j)}E \\ 0 & M \end{bmatrix},$$

then $\|(x_k, \mathbf{c}_k)\|^2_{\check{W}}$ is an upper bound on the worst-case cost:

$$\left\| \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \right\|^2_{\check{W}} \geq \check{J}(x_k, \mathbf{c}_k) \doteq \max_{(A_{i|k}, B_{i|k}) \in \Omega,\, i=0,1,\ldots} \sum_{i=0}^{\infty} \left(\|x_{i|k}\|^2_Q + \|u_{i|k}\|^2_R \right), \tag{5.104}$$

and replacing the objective of (5.101) with $\|(x_k, \mathbf{c}_k)\|^2_{\check{W}}$ converts Algorithm 5.7 to a min–max robust MPC requiring a QP optimization online. The implied control law can be shown to be asymptotically stabilizing by demonstrating, similarly to the proof of Theorem 5.5, that $\|(x_k, \mathbf{c}_k)\|^2_{\check{W}}$ is a Lyapunov function for the closed-loop system.

Example 5.4 For the system (5.1)–(5.3) with the model parameters given in Example 5.1 and $K = [0.19\ 0.34]$, the joint spectral radius of the system $x_{k+1} = \Phi x_k$, $\Phi \in \Omega_K$, is $\rho = 0.7415$. For values of λ in the interval $(\rho, 1)$, Table 5.1 gives numbers of rows, n_V, of the matrix V such that $\{x : Vx \leq \mathbf{1}\}$ is the maximal RPI set for the dynamics $x_{k+1} = (1/\lambda)\Phi x_k$, $\Phi \in \Omega_K$ and constraints $(F + GK)x_k \leq \mathbf{1}$. The table also shows the number of terminal constraints associated with either $\mathcal{A}^{\text{MPI}}$ or the alternative conditions (5.103a, 5.103b) when the tube cross sections are given by $\mathcal{X}_{i|k} = \{x : Vx \leq \alpha_{i|k}\}$. For all values of λ in this range and with either of these definitions of the terminal constraints, the terminal set (namely the set $x_{N|k}$ such that there exists $\alpha_{N|k}$ satisfying the terminal constraints and $Vx_{N|k} \leq \alpha_{N|k}$) is equal to the maximal RPI set under $u = Kx$ (Fig. 5.5).

From Table 5.1, it can be seen that, as expected, n_V increases as λ is reduced. This causes the required number of variables and constraints in the MPC online

Table 5.1 Number of facets of tube cross sections and number of terminal constraints for varying λ

λ	Tube cross section n_V	Terminal constraints $\mathcal{A}^{\text{MPI}}$	Terminal constraints (5.103a, 5.103b)
0.999	6	12	22
0.9	8	20	28
0.8	12	60	40
0.742	28	808	88

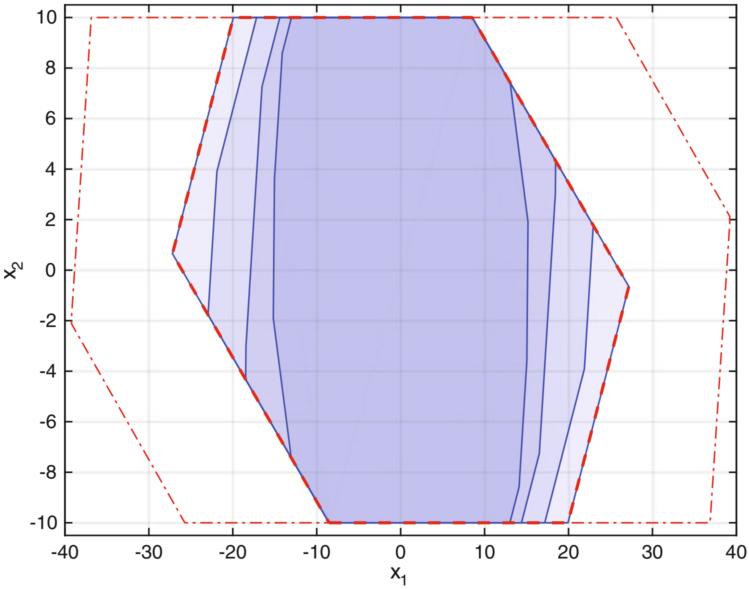

Fig. 5.5 The λ-contractive sets $\{x : Vx \leq \mathbf{1}\}$ for $\lambda = 0.742, 0.8, 0.9, 0.999$ (*solid lines*) and the corresponding terminal sets defined either by $\mathcal{A}^{\text{MPI}}$ or (5.103a, 5.103b) (*dashed line*). The maximal robustly controlled invariant set is also shown (*dash-dotted line*)

optimization (5.101) to increase, and in order to reduce online computation it is therefore desirable to choose λ so as to avoid large values of n_V. But for values of λ close to unity the set of feasible initial conditions for Algorithm 5.7 grows slowly with N, and for this example a good compromise is obtained with $\lambda = 0.9$.

The sets of feasible initial conditions, $\mathcal{F}_N$, are shown in Fig. 5.6 for Algorithm 5.7 with $\lambda = 0.9$ and with terminal constraints defined by $\mathcal{A}^{\text{MPI}}$. The feasible set $\mathcal{F}_4$ is equal to the maximal robustly controlled positively invariant (CPI) set for this example, which has an area of 1415. Hence, there can be no further increase in $\mathcal{F}_N$ for $N > 4$. For comparison, Fig. 5.7 shows the feasible sets for Algorithm 5.6 with tube cross sections and a terminal set defined by the low-complexity polytope given in Example 5.3. Using low-complexity tubes results in smaller numbers of variables and constraints in the online MPC optimization (Table 5.2), but for this example there is no increase in the feasible set for Algorithm 5.6 for $N > 5$, which implies that Algorithm 5.6 remains infeasible irrespective of the value of N for some initial conditions in the robustly CPI set.

The use of low-complexity polytopic tubes has a small but appreciable effect on performance. For a set of initial conditions consisting of 34 points lying on the boundary of $\mathcal{F}_5$ for Algorithm 5.6, the worst-case predicted cost of Algorithm 5.6 with $N = 5$ is on average 0.8 % (and at most 1.6 %) greater than the worst-case predicted cost of Algorithm 5.7 with $N = 4$. $\diamondsuit$

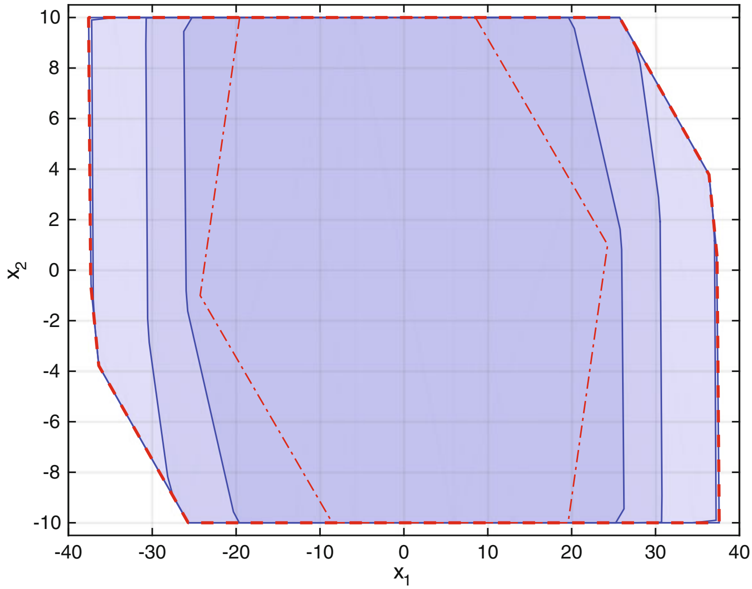

Fig. 5.6 The feasible sets, $\mathcal{F}_N$ for Algorithm 5.7 with $N = 1, 2, 3, 4$ (*solid lines*), and the maximal robustly controlled invariant set (*dashed line*). The terminal set is shown by the *dash-dotted lines*

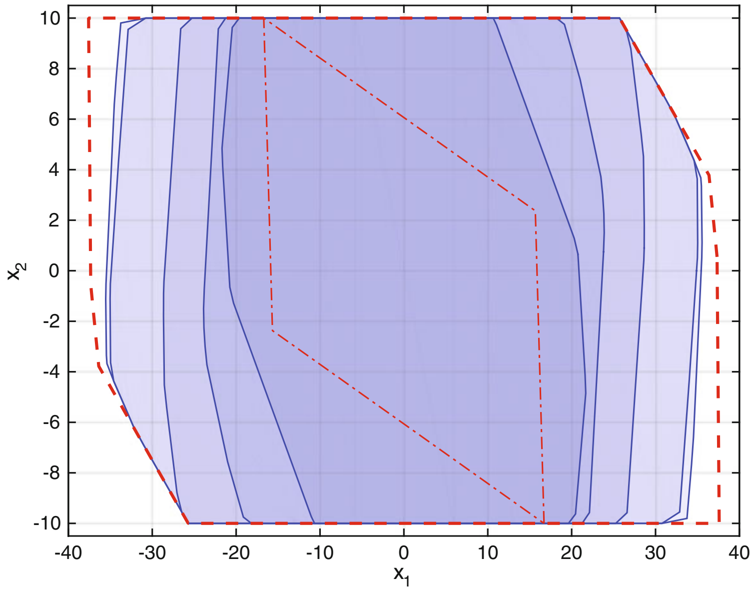

Fig. 5.7 The feasible sets, $\mathcal{F}_N$ for Algorithm 5.6 with $N = 1, 2, 3, 4, 5$ (*solid lines*), and the maximal robustly controlled invariant set (*dashed line*). The terminal set is indicated by the *dash-dotted lines*

Table 5.2 Number of variables, constraints and area of feasible region for the general complexity and low-complexity polytopic tube MPC strategies of Algorithms 5.6 and 5.7

N	General complexity tube			Low-complexity tube		
	# variables	# constraints	Area of $\mathcal{F}_N$	# variables	# constraints	Area of $\mathcal{F}_N$
1	17	56	989	9	24	747
2	26	84	1198	14	40	902
3	35	112	1407	19	56	1110
4	44	140	1415	24	72	1324
5				29	88	1345

5.6 Mixed Additive and Multiplicative Uncertainty

The results presented in Sects. 5.4 and 5.5 allow for the definition of polytopic tubes that contain the predicted state and control trajectories of an uncertain system for all future realizations of model uncertainty. Such tubes provide a systematic means of handling constraints and, crucially, the complexity of their cross sections can be controlled by the designer and does not depend on the length of prediction horizon. Sections 5.4 and 5.5 considered multiplicative model uncertainty only but the method can easily be extended to cater for the more general case of mixed multiplicative and additive uncertainty. This section considers the model (5.5) with dynamics $x_{k+1} = A_k x_k + B_k u_k + D_k w_k$ containing both multiplicative parametric uncertainty and unknown additive disturbances. Here (A_k, B_k, D_k) belong for all k to a compact polytopic parameter set $\tilde{\Omega}$ and w_k is confined for all k to a compact polytopic set $\mathcal{W}$ containing $w = 0$:

$$(A_k, B_k, D_k) = \sum_{j=1}^{m} q_k^{(j)} (A^{(j)}, B^{(j)}, D^{(j)}), \quad w_k = \sum_{l=1}^{q} r_k^{(l)} w^{(l)} \tag{5.105a}$$

$$q_k^{(j)}, r_k^{(l)} \geq 0, \quad \sum_{j=1}^{m} q_k^{(j)} = 1, \quad \sum_{l=1}^{q} r_k^{(l)} = 1 \tag{5.105b}$$

where the superscripts j and l are used to denote, respectively, multiplicative and additive uncertainty vertices.

The approach of Sect. 5.5 can be extended to the case of mixed model uncertainty by modifying the conditions (5.92a) and (5.96a) defining the evolution of tubes that bound the predicted states over the mode 1 and mode 2 prediction horizons, respectively. However we discuss here a more general approach that uses optimized controller dynamics to design the mode 2 prediction dynamics for the case of mixed uncertainty [21–23]. The use of general polytopic tubes in this context results in an approach that subsumes the method of Sect. 5.5 as a special case and allows for a terminal set larger than the maximal robustly positively invariant set under any given linear feedback law.

The optimization of the mode 2 prediction dynamics is aimed at maximizing the volume of an invariant terminal set, and we therefore introduce further degrees of freedom over the mode 1 prediction horizon for the purposes of improving performance and increasing the region of attraction. Hence the predicted control strategy is of the form

$$u_{i|k} = Kx_{i|k} + C_c\mathbf{c}_{i|k} + f_{i|k}, \tag{5.106a}$$

$$\mathbf{c}_{i+1|k} \in \text{Co}\{A_c^{(j)}\mathbf{c}_{i|k} + C_w^{(j)}w_{i|k},\ j = 1, \ldots, m\} \tag{5.106b}$$

with $f_{i|k} = 0$ for all $i \geq N$. At each time instant $k \geq 0$, $\mathbf{c}_k \doteq \mathbf{c}_{0|k} \in \mathbb{R}^{n_x}$ and $\mathbf{f}_k \doteq (f_{0|k}, \ldots, f_{N-1|k}) \in \mathbb{R}^{Nn_u}$ are variables in the online MPC optimization. We note that the disturbance affine term in (5.106b) provides feedback from the future disturbances which, although unknown when the predictions are optimized at time k, will be available to the controller at time $k + i$, and (5.106a, 5.106b) therefore constitutes a closed-loop prediction strategy.

In this setting, the optimized controller dynamics are introduced at every time step of the entire prediction horizon, with C_c, $A_c^{(j)}$, $C_w^{(j)}$ chosen so as to maximize the volume of a robustly invariant ellipsoidal terminal set. For $i \geq N$, we use the following description of the mode 2 prediction dynamics corresponding to (5.5) under the terminal control law $u_{i|k} = Kx_{i|k} + C_c\mathbf{c}_{i|k}$:

$$z_{i+1|k} = \Psi_{i|k}z_{i|k} + \tilde{D}_{i|k}w_{i|k}, \quad z_{N|k} = \begin{bmatrix} x_{N|k} \\ \mathbf{c}_{N|k} \end{bmatrix} \tag{5.107}$$

where

$$(\Psi_{i|k}, \tilde{D}_{i|k}) \in \text{Co}\big\{(\Psi^{(j)}, \tilde{D}^{(j)}),\ j = 1, \ldots, m\big\}$$

and, for $j = 1, \ldots, m$,

$$\Psi^{(j)} = \begin{bmatrix} \Phi^{(j)} & B^{(j)}C_c \\ 0 & A_c^{(j)} \end{bmatrix}, \quad \tilde{D}^{(j)} = \begin{bmatrix} D^{(j)} \\ C_w^{(j)} \end{bmatrix}.$$

Theorem 5.9 *The ellipsoidal set $\mathcal{E}_z = \{z : z^T P_z z \leq 1\}$ is robustly invariant for the dynamics (5.107) with the constraint (5.3) if and only if there exists a positive definite matrix P_z and a nonnegative scalar σ such that*

$$\begin{bmatrix} P_z & P_z\Psi^{(j)} & P_z\tilde{D}^{(j)}w^{(l)} \\ \star & \sigma P_z & 0 \\ \star & \star & 1-\sigma \end{bmatrix} \succeq 0, \quad j = 1, \ldots, m, \quad l = 1, \ldots, q. \tag{5.108}$$

Proof The conditions for invariance require $z_{i+1|k}^T P_z z_{i+1|k} \leq 1$ for all $z_{i|k}$ such that $z_{i|k}^T P_z z_{i|k} \leq 1$. From (5.107), this is equivalent to the conditions that, for all $z \in \mathbb{R}^{2n_x}$, and for $j = 1, \ldots, m$ and $l = 1, \ldots, q$,

$$1 - \left(\Psi^{(j)} z + \tilde{D}^{(j)} w^{(l)}\right)^T P_z \left(\Psi^{(j)} z_{k|k} + \tilde{D}^{(j)} w^{(l)}\right) \leq \sigma_{jl}(1 - z^T P_z z).$$

These conditions can be expressed in terms of σ, defined as the minimum of σ_{jl} over j and l, as

$$\begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} \sigma P_z - \Psi^{(j)^T} P_z \Psi^{(j)} & \Psi^{(j)^T} P_z \tilde{D}^{(j)} w^{(l)} \\ \star & 1 - \sigma - w^{(l)^T} \tilde{D}^{(j)^T} P_z \tilde{D}^{(j)} w^{(l)} \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \quad (5.109)$$

for all z, j and l, which can be shown, using Schur complements, to be equivalent to (5.108). From the linear dependence of (5.108) on $w^{(l)}$ for $l = 1, \ldots, q$, it follows that (5.108) is necessary and sufficient for invariance over the entire class of additive uncertainty. The same applies in respect of the multiplicative uncertainty class since (5.108) depends linearly on $(\Psi^{(j)}, \tilde{D}^{(j)})$ for $j = 1, \ldots, m$. This argument makes the implicit assumption that $\sigma P_z - \Psi^{(j)T} P_z \Psi^{(j)}$ is strictly positive definite for each j, but we note that, if this matrix is only positive semidefinite, a similar argument applies with the matrix inverse replaced by the relevant Moore–Penrose pseudoinverse [24]. □

Given the invariance property of Theorem 5.9, the constraints (5.3) can be imposed over the mode 2 prediction horizon by ensuring that $Fx + Gu \leq \mathbf{1}$ holds for all $z \in \mathcal{E}_z = \{z : z^T P_z z \leq 1\}$. This is equivalent to the conditions

$$\begin{bmatrix} H & \left[F + GK \;\; GC_c\right] P_z^{-1} \\ \star & P_z^{-1} \end{bmatrix} \succeq 0, \quad e_i^T H e_i \leq 1, \quad i = 1, \ldots, n_C, \qquad (5.110)$$

for some symmetric matrix H, where e_i is the ith column of the identity matrix. Using the convexification technique of Sect. 5.3.1, it is possible to express (5.108) and (5.110) in terms of the equivalent LMI conditions:

$$\begin{bmatrix} \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi^{(j)} Y + B^{(j)} \Gamma & \Phi^{(j)} X \\ \Xi^{(j)} + \Phi^{(j)} Y + B^{(j)} \Gamma & \Phi^{(j)} X \end{bmatrix} & \begin{bmatrix} D^{(j)} \\ D^{(j)} + \Gamma_w^{(j)} \end{bmatrix} \\ \star & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & 0 \\ \star & \star & 1 - \sigma \end{bmatrix} \succeq 0 \qquad (5.111)$$

for $j = 1, \ldots, m$ and (5.44b). Here the transformed variables X, Y, $\Xi^{(j)}$, Γ are as defined in (5.45) and (5.46b), and $\Gamma_w^{(j)}$ is an additional variable satisfying $C_w^{(j)} = \Gamma_w^{(j)} U^{-1}$ for $j = 1, \ldots, m$.

The parameters P_z, C_c and $A_c^{(j)}$, $C_w^{(j)}$, $j = 1, \ldots, m$ that maximize the volume of the x-subspace projection of $\mathcal{E}_z$ for the mode 2 prediction dynamics of (5.107) can be computed offline by solving a semidefinite program. In particular, the projection of $\mathcal{E}_z$ onto the x-subspace is given by $\{x : x^T Y^{-1} x \leq 1\}$, and the volume of this ellipsoidal set is maximized through the maximization of $\log\det(Y)$ subject to (5.111) and (5.44b).

As pointed out in Sect. 5.3.1, using the ellipsoid $\mathcal{E}_z$ to impose linear constraints on predicted trajectories implies a degree of conservativeness in the handling of constraints. Here instead we use the design of $\mathcal{E}_z$ simply as a means of optimizing the mode 2 prediction dynamics, and impose the constraints $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$ at all prediction times using polytopic tubes. This can be done by modifying the approach of Sect. 5.5 to account for the predicted values of the controller state $\mathbf{c}_{i|k}$ and the additive uncertainty $w_{i|k}$. For $i = 0, \ldots, N-1$ the predicted trajectories are governed by

$$z_{i+1|k} = \Psi_{i|k} z_{i|k} + \tilde{B}_{i|k} f_{i|k} + \tilde{D}_{i|k} w_{i|k}, \quad z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}$$
$$(\Psi_{i|k}, \tilde{B}_{i|k}, \tilde{D}_{i|k}) \in \mathrm{Co}\big\{(\Psi^{(j)}, \tilde{B}^{(j)}, \tilde{D}^{(j)}),\ j = 1, \ldots, m\big\}$$
$$w_{i|k} \in \mathrm{Co}\{w^{(l)},\ l = 1, \ldots, q\}$$

where

$$\tilde{B}_{i|k} = \begin{bmatrix} B_{i|k} \\ 0 \end{bmatrix}, \quad \tilde{B}^{(j)} = \begin{bmatrix} B^{(j)} \\ 0 \end{bmatrix}.$$

Define the polytopic set

$$\mathcal{Z}_{i|k} = \{z : Vz \leq \alpha_{i|k}\},$$

where $V \in \mathbb{R}^{n_V \times 2n_x}$ is to be designed offline and $\alpha_{i|k}$ for $i = 0, \ldots, N$ are variables in the online MPC optimization performed at each time step k. Then, by Lemma 5.6, the conditions $z_{i|k} \in \mathcal{Z}_{i|k}$, for $i = 1 \ldots, N$ are enforced by the following constraints, for some $H^{(j)} \geq 0$,

$$\alpha_{i+1|k} \geq H^{(j)} \alpha_{i|k} + V\tilde{B}^{(j)} f_{i|k} + V\tilde{D}^{(j)} w^{(l)}, \ \ i = 0, \ldots, N-1 \tag{5.112a}$$
$$H^{(j)} V = V\Psi^{(j)} \tag{5.112b}$$

for $j = 1, \ldots, m, l = 1, \ldots, q$. The constraints $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$ are likewise satisfied for all $i = 0, \ldots, N-1$ if, for some $H_c \geq 0$ the following constraints hold:

$$H_c \alpha_{i|k} + Gf_{i|k} \leq \mathbf{1}, \ \ i = 0, \ldots, N-1 \tag{5.113a}$$
$$H_c V = \big[F + GK \ \ GC_c\big]. \tag{5.113b}$$

Based on the discussion of terminal conditions in Sect. 5.5, we introduce the terminal conditions

$$H^{(j)} \alpha_{N|k} + V\tilde{D}^{(j)} w^{(l)} \leq \alpha_{N|k} \tag{5.114a}$$
$$H_c \alpha_{N|k} \leq \mathbf{1} \tag{5.114b}$$

for $j = 1, \ldots, m$, $l = 1, \ldots, q$. By Lemma 5.6, these conditions are sufficient to ensure $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$ for all $i \ge 0$ if $\alpha_{0|k}$ satisfies the initial condition

$$\alpha_{0|k} \ge V \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}. \tag{5.115}$$

As in Sect. 5.5, the matrix V is assumed to be chosen offline so that $\{z : Vz \le \mathbf{1}\}$ is λ-contractive, for some $\lambda \in [0, 1)$, under the mode 2 prediction dynamics (5.107). The nonnegative matrices H_c are computed offline by solving the linear programs

$$h_i^* = \arg\min_{h \in \mathbb{R}^{n_V}} \mathbf{1}^T h \;\text{ subject to }\; h^T V = (\begin{bmatrix} F + GK & GC_c \end{bmatrix})_i \text{ and } h \ge 0$$

and setting $(H_c)_i = h_i^{*T}$ for $i = 1, \ldots, n_C$, where $(H_c)_i$ and $([F + GK \;\; GC_c])_i$ denote the ith rows of H_c and $[F + GK \;\; GC_c]$. Likewise $H^{(j)}$, $j = 1, \ldots, m$, are computed by solving

$$h_i^{(j)*} = \arg\min_{h \in \mathbb{R}^{n_V}} \mathbf{1}^T h \;\text{ subject to }\; h^T V = V_i \Psi^{(j)} \text{ and } h \ge 0$$

and setting $H_i^{(j)} = {h_i^{(j)*}}^T$ for $i = 1, \ldots, n_V$ and $j = 1, \ldots, m$, where $H_i^{(j)}$ and V_i denote the ith rows of $H^{(j)}$ and V, respectively.

To construct a robust MPC algorithm using this formulation of constraints, we next consider the form of the MPC cost index. Here we consider a worst-case cost with respect to the model uncertainty. To evaluate this cost, the prediction dynamics are first expressed in compact form as

$$\xi_{i+1|k} = \Theta_{i|k}\xi_{i|k} + \hat{D}w_{i|k}, \quad \xi_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \\ \mathbf{f}_k \end{bmatrix} \tag{5.116}$$

with $\mathbf{f}_k = (f_{0|k}, \ldots, f_{N-1|k})$ and

$$(\Theta_{i|k}, \hat{D}_{i|k}) = \mathrm{Co}\big\{(\Theta^{(j)}, D^{(j)}),\ j = 1, \ldots, m\big\},$$
$$\Theta^{(j)} = \begin{bmatrix} \Phi^{(j)} & B^{(j)}C_c & B^{(j)}E \\ 0 & A_c^{(j)} & 0 \\ 0 & 0 & M \end{bmatrix}, \quad \hat{D}^{(j)} = \begin{bmatrix} D^{(j)} \\ C_w^{(j)} \\ 0 \end{bmatrix}$$

with the matrices E and M defined as in (2.26b). For the predicted cost defined by

$$\check{J}(x_k, \mathbf{c}_k, \mathbf{f}_k) = \max_{\substack{(A_{i|k}, B_{i|k}, D_{i|k}) \in \tilde{\Omega}, \\ w_{i|k},\ i=0,1,\ldots}} \sum_{i=0}^{\infty} \|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - \gamma^2 \|w_{i|k}\|^2 \tag{5.117}$$

the following result provides conditions allowing an upper bound on $\check{J}(x_k, \mathbf{c}_k, \mathbf{f}_k)$ to be determined.

Lemma 5.8 *For given γ, the predicted cost (5.117) is bounded from above as*

$$\check{J}(x_k, \mathbf{c}_k, \mathbf{f}_k) \leq \xi_{0|k}^T \check{W} \xi_{0|k}$$

for all $(x_k, \mathbf{c}_k, \mathbf{f}_k)$ if and only if $\check{W} \succ 0$ satisfies the LMIs, for $j = 1, \ldots, m$:

$$\check{W} - \begin{bmatrix} \Theta^{(j)T} & 0 \\ \hat{D}^{(j)T} & 1 \end{bmatrix} \check{W} \begin{bmatrix} \Theta^{(j)} & \hat{D}^{(j)} \\ 0 & 1 \end{bmatrix} \succeq \begin{bmatrix} \hat{Q} & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \tag{5.118}$$

where

$$\hat{Q} = \begin{bmatrix} Q + K^T R K & K^T R C_c & K^T R E \\ \star & C_c^T R C_c & C_c^T R E \\ \star & \star & E^T R E \end{bmatrix}.$$

Proof The bound on $\check{J}(x_k, \mathbf{c}_k, \mathbf{f}_k)$ is obtained by summing over all $i \geq 0$ the bound $\|\xi_{i|k}\|_{\check{W}}^2 - \|\xi_{i+1|k}\|_{\check{W}}^2 \geq \|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - \gamma^2 \|w_{i|k}\|^2$, which by (5.116) can be expressed as $\|\xi\|_{\check{W}}^2 - \|\Theta\xi + \hat{D}w\|_{\check{W}}^2 \geq \|\xi\|_{\hat{Q}}^2 - \gamma^2\|w\|^2$ or equivalently as

$$\begin{bmatrix} \xi \\ w \end{bmatrix}^T \begin{bmatrix} \check{W} - \Theta^T \check{W} \Theta & \Theta \check{W} \hat{D} \\ \star & \gamma^2 - \hat{D}^T \check{W} \hat{D} \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \geq 0$$

Since the disturbance set $\mathcal{W}$ contains $w = 0$ by assumption, the matrix appearing in this expression must be positive semidefinite, and by rearranging terms this condition is equivalent to

$$\check{W} - \begin{bmatrix} \Theta^T & 0 \\ \hat{D}^T & 1 \end{bmatrix} \check{W} \begin{bmatrix} \Theta & \hat{D} \\ 0 & 1 \end{bmatrix} \succeq \begin{bmatrix} \hat{Q} & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

Using Schur complements, this can be written as an LMI in Θ and $\hat{D}$, which by convexity is satisfied for all $(\Theta, \hat{D}) \in \mathrm{Co}\{(\Theta^{(j)}, \hat{D}^{(j)},\ j = 1, \ldots, m\}$ if and only (5.118) holds. □

A unique value of $\check{W}$ corresponding to a tight bound on $J(x_k, \mathbf{c}_k, \mathbf{f}_k)$ can be obtained by solving the semidefinite program:

$$\underset{\check{W}}{\text{minimize}}\ \mathrm{tr}(\check{W}) \ \text{ subject to (5.118).}$$

Algorithm 5.8 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\begin{aligned} &\underset{\substack{\mathbf{c}_k, \mathbf{f}_k \\ \alpha_{0|k},\ldots,\alpha_{N|k}}}{\text{minimize}} \;\; \|(x_k, \mathbf{c}_k, \mathbf{f}_k)\|^2_{\check{W}} \\ &\text{subject to } \;(5.112\text{a}), (5.113\text{a}), (5.114\text{a,b}), (5.115) \end{aligned} \tag{5.119}$$

(ii) Apply the control law $u_k = Kx_k + C_c\mathbf{c}_k^* + f^*_{0|k}$, where $\mathbf{f}_k^* = (f^*_{0|k}, \ldots, f^*_{N-1|k})$ and $\mathbf{c}_k^*$, $\mathbf{f}_k^*$ are the optimal values of $\mathbf{c}_k$ and $\mathbf{f}_k$ in (5.119). ◁

Theorem 5.10 *The optimization (5.119) is recursively feasible, and for all initial conditions x_0 in the feasible set*

$$\mathcal{F}_N = \{x_k : \exists(\mathbf{c}_k, \mathbf{f}_k) \text{ satisfying (5.112a), (5.113a), (5.114a, 5.114b), (5.115)}\},$$

the trajectories of the system (5.1)–(5.3) under the control law of Algorithm 5.8 satisfy the l_2 bound:

$$\sum_{k=0}^{\infty}(\|x_k\|_Q^2 + \|u_k\|_R^2) \le \gamma^2 \sum_{k=0}^{\infty} \|w\|_k^2 + \|(x_0, \mathbf{c}_0^*, \mathbf{f}_0^*)\|^2_{\check{W}}. \tag{5.120}$$

Proof Assume $x_k \in \mathcal{F}_N$ and consider the solution at time $k+1$ given by

$$\begin{aligned} \mathbf{c}_{k+1} &= A_{c,k}\mathbf{c}_k^* + C_{w,k}w_k \\ \mathbf{f}_{k+1} &= (f^*_{1|k}, \ldots, f^*_{N-1|k}, 0) \\ \alpha_{i|k+1} &= \alpha^*_{i+1|k}, \quad i = 0, \ldots, N-1, \\ (\alpha_{N|k+1})_l &= \max_{j \in \{1,\ldots,m\}} H_l^{(j)} \alpha^*_{N|k}, \quad l = 1, \ldots, n_V \end{aligned}$$

for some $(A_{c,k}, C_{w,k}) \in \text{Co}\{(A_c^{(1)}, C_w^{(1)}), \ldots, (A_c^{(m)}, C_w^{(m)})\}$. This choice of variables gives $\mathcal{Z}_{i|k+1} = \mathcal{Z}_{i+1|k}$ for $i = 0, \ldots, N-1$ and hence (5.112a), (5.113a) and (5.114a) are necessarily satisfied at $k+1$. Also the definition of $\alpha_{N|k+1}$ satisfies (5.114b) and $z_{1|k} \in \mathcal{Z}_{1|k}$ implies that (5.115) is satisfied by $(x_{k+1}, \mathbf{c}_{k+1})$, which demonstrates that (5.119) is recursively feasible.

From (5.108) and optimality of the solution of (5.119) at time $k+1$, it follows that

$$\begin{aligned} \|(x_{k+1}, \mathbf{c}^*_{k+1}, \mathbf{f}^*_{k+1})\|^2_{\check{W}} &\le \|(x_{k+1}, \mathbf{c}_{k+1}, \mathbf{f}_{k+1})\|^2_{\check{W}} \\ &\le \|(x_k, \mathbf{c}_k^*, \mathbf{f}_k^*)\|^2_{\check{W}} - \|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2\|w_k\|^2 \end{aligned}$$

Summing both sides of this inequality over all $i \ge 0$ yields the bound (5.120). □

To conclude this section, we note that, if the sequence $\{w_k,\ k = 0, 1, \ldots\}$ is square summable, then under the control law of Algorithm 5.8 the origin of the state-space will be asymptotically stable since in this case the bound (5.120) implies that the stage cost converges to zero. We also note that, as explained in Chap. 3, the definition of the MPC cost as an upper bound on the min–max cost (5.117) ensures that γ provides a bound on the induced l_2 norm from the disturbance to the closed-loop system state. In this sense, one may wish to choose γ to be small. However, the smaller the γ is, the larger the trace of $\check{W}$ will have to be in order that $\check{W}$ can satisfy (5.118). Thus a compromise between the tightness of the cost bound and the disturbance rejection ability of Algorithm 5.8 is needed.

Example 5.5 Consider the system (5.5) with parametric uncertainty and unknown disturbances contained in the polytopic sets defined in (5.105a, 5.105b). The parameters $A^{(j)}, B^{(j)}$ for $j = 1, 2, 3$ are as given in Example 5.1. Also $D^{(j)} = I$ for $j = 1, 2, 3$ and the disturbance set $\mathcal{W} \subset \mathbb{R}^2$ is given by

$$\mathcal{W} = \text{Co}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}\right\}.$$

The system is subject to the constraints of Example 5.1, namely the state constraints $-10 \le [0\ 1]x_k \le 10$ and input constraints $-5 \le u_k \le 5$.

The feedback gain K is again given by $K = [0.19\ 0.34]$, and optimizing the prediction dynamics subject to (5.109) and (5.110) yields $\sigma = 0.898$ and

$$A_c^{(1)} = \begin{bmatrix}-0.69 & 0.20\\ -0.28 & -0.12\end{bmatrix}, \quad A_c^{(2)} = \begin{bmatrix}-0.74 & -0.004\\ 0.25 & 0.02\end{bmatrix}, \quad A_c^{(3)} = \begin{bmatrix}-0.63 & -0.21\\ -0.06 & -0.26\end{bmatrix}$$

with $C_w^{(j)} = -I$ (to 2 decimal places) for $j = 1, 2, 3$ and

$$C_c = \begin{bmatrix}0.13 & -0.14\end{bmatrix}.$$

The matrix V defining the polytopic tube cross sections $\mathcal{Z}_{i|k}$ is chosen so that the set $\{z : Vz \le \mathbf{1}\}$ is λ-contractive, with $\lambda = 0.9$. Thus $\{z : Vz \le \mathbf{1}\}$ is defined as the maximal RPI set for the system $z_{k+1} = (1/\lambda)(\Psi z_k + \tilde{D}w_k)$, $(\Psi, \tilde{D}) \in \text{Co}\{(\Psi^{(j)}, \tilde{D}^{(j)}),\ j = 1, \ldots, m\}$, $w_k \in \mathcal{W}$, which yields a V with 22 rows. The corresponding terminal conditions (5.113) consist of 70 constraints and the terminal set is shown in Fig. 5.8. From Fig. 5.8, it can be seen that this terminal set contains and extends beyond the maximal RPI set under the linear feedback law $u = Kx$.

For comparison, if the terminal control law is chosen as $u = Kx$ and if the state tube cross sections are defined in the space of x (rather than z) in terms of the matrix V_x such that $\{x : V_x x \le \mathbf{1}\}$ is the maximal RPI set for $x_{k+1} = (1/\lambda)(\Phi x_k + Dw_k)$, $(\Phi, D) \in \text{Co}\{(\Phi^{(j)}, D^{(j)}),\ j = 1, \ldots, m\}$, $w_k \in \mathcal{W}$, then V_x has 8 rows, the terminal conditions are defined by defined by 28 constraints and the corresponding terminal set coincides with the maximal RPI set under $u = Kx$. The larger terminal set that is obtained with the optimized prediction dynamics enables the feasible set

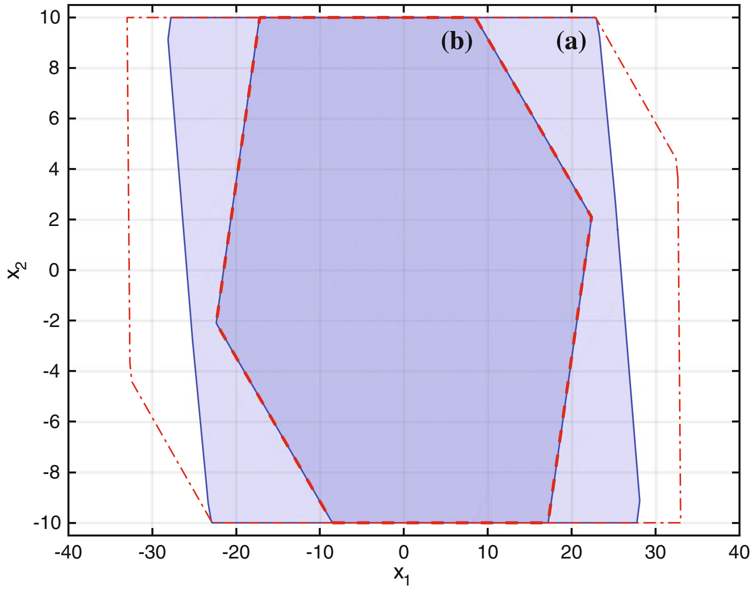

Fig. 5.8 The terminal sets for the case that the terminal controller is defined using **a** optimized prediction dynamics, and **b** a fixed linear controller (*solid lines*). Also shown are the maximal robustly controlled invariant set (*dash-dotted line*) and the maximal RPI set under the linear feedback law (*dashed line*)

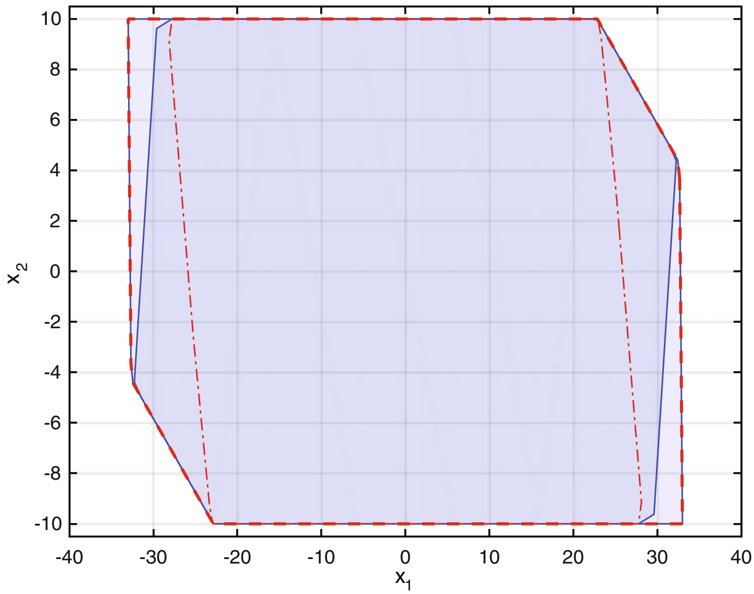

Fig. 5.9 The feasible initial condition sets $\mathcal{F}_N$ for Algorithm 5.8, for $N = 1, 2$ (*solid lines*). The terminal set (*dash-dotted line*) and the maximal robustly controlled invariant set (*dashed line*) are also shown

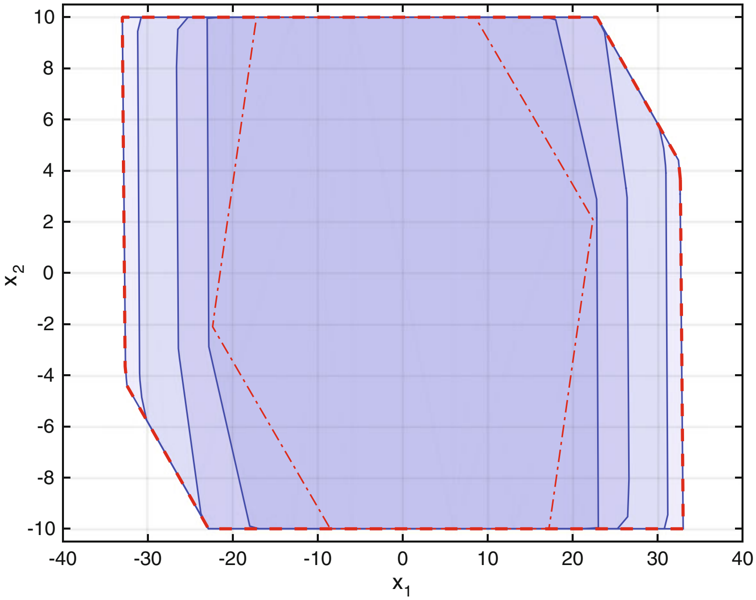

Fig. 5.10 The feasible initial condition sets $\mathcal{F}_N$ for Algorithm 5.8 with a fixed linear terminal controller, for $N = 1, 2, 3, 4$ (*solid lines*). The terminal set (*dash-dotted line*) and the maximal robustly controlled invariant set (*dashed line*) are also shown

for Algorithm 5.8 to cover the entire maximal robustly controlled invariant set for this system with a prediction horizon of just $N = 2$ (Fig. 5.9). On the other hand, with the fixed linear terminal feedback law $u = Kx$ and state tube cross sections defined in the space of x, a prediction horizon of $N = 4$ is needed in order that the feasible set coincides with the maximal robustly CPI set (Fig. 5.10).

However, for this example the approach based on optimized prediction dynamics with $N = 2$ requires a greater number of optimization variables and constraints than when a fixed terminal feedback gain is employed (Table 5.3). The extra degrees of

Table 5.3 Number of variables, constraints and area of feasible region for the MPC strategy of Algorithm 5.8 with terminal control law defined by optimized prediction dynamics and with terminal control law defined by fixed linear feedback

N	Optimized prediction dynamics			Fixed terminal feedback gain		
	Variables	Constraints	Area of $\mathcal{F}_N$	Variables	Constraints	Area of $\mathcal{F}_N$
1	47	162	1200	17	64	880
2	70	332	1255	26	92	1038
3				35	120	1202
4				44	148	1255

freedom associated with the optimized prediction dynamics translates into improved predicted performance. For example, the worst-case predicted cost, evaluated at the vertices of the maximal CPI set, is on average 3.1 % (at most 4.6 %) higher if a fixed linear terminal feedback gain is used than if the optimized prediction dynamics are employed. ◊

5.7 Exercises

1 A symmetric, real-valued $n \times n$ matrix P is positive definite ($P \succ 0$) if and only if $v^T P v > 0$ for all non-zero real vectors v. Use this property to prove the following statements.

(a) The linear matrix inequality $M(x) \succ 0$ is convex in $x = (x_1, \ldots, x_n)$ where

$$M(x) = M_0 + x_1 M_1 + \cdots + x_n M_n$$

for given symmetric matrices $M_0, \ldots, M_n$.

(b) The condition

$$\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \succ 0$$

for symmetric P and R is equivalent to the Schur complements

$$R \succ 0, \quad P - QR^{-1}Q^T \succ 0.$$

(c) If $P = S^{-1} \succ 0$, then the condition $S - ASA^T \succ 0$ is equivalent to $P - A^T PA \succ 0$.

2 Let $\mathcal{E}$ be the ellipsoidal set defined by $\mathcal{E} = \{x : x^T Px \le 1\}$, for some symmetric matrix $P \succ 0$.

(a) Show that $\mathcal{E} \subseteq \mathcal{X}$, where $\mathcal{X}$ is the polytope $\mathcal{X} = \{x : Vx \le \mathbf{1}\}$ for a given matrix $V \in \mathbb{R}^{n_V \times n_x}$, if and only if

$$V_i P^{-1} V_i^T \le 1, \quad i = 1, \ldots, n_V$$

where V_i for $i = 1, \ldots, n_V$ are the rows of V.

(b) Hence, show that $Fx + Gu \le \mathbf{1}$ holds for all $x \in \mathcal{E}$, where $u = Kx$, if and only if the following LMI conditions in variables S, Y hold

$$\begin{bmatrix} 1 & F_i S + G_i Y \\ (F_i S + G_i Y)^T & S \end{bmatrix} \succeq 0, \quad i = 1, \ldots, n_C$$

where F_i and G_i, $i = 1, \ldots, n_C$ are the rows of F and G, and where $(S, Y) = (P^{-1}, KP^{-1})$.

3 A system is described by the model $x_{k+1} = A_k x_k + B_k u_k$ with

$$\begin{aligned}(A_k, B_k) &\in \mathrm{Co}\big\{(A^{(1)}, B^{(1)}), (A^{(2)}, B^{(2)})\big\}\\ &= \mathrm{Co}\left\{\left(\begin{bmatrix} 0.7 & -0.6\\ -0.7 & -1.8\end{bmatrix}, \begin{bmatrix} 0.3\\ -0.5\end{bmatrix}\right), \left(\begin{bmatrix} 0.5 & -0.8\\ -0.6 & -1.8\end{bmatrix}, \begin{bmatrix} 0.1\\ -0.4\end{bmatrix}\right)\right\}\end{aligned}$$

at each time step $k = 0, 1, \ldots$. The control input is subject to the constraints $-1 \le u_k \le 1$ for all k. A robust MPC law $u_k = K_k x_k$ is to be designed for this system with the aim of minimizing, at each time k, a quadratic bound on the worst-case predicted cost:

$$\check{J}(x_k, K_k) = \max_{\{(A_k,B_k),(A_{k+1},B_{k+1}),\ldots\}} \sum_{i=0}^{\infty} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big)$$

subject to $-1 \le u_{i|k} \le 1$, $i = 0, 1, \ldots$.

(a) Show that the worst-case cost satisfies the bound $\check{J}(x_k, K_k) \le \gamma_k$ if

$$\begin{aligned} &P - (A^{(j)} + B^{(j)}K_k)^T P (A^{(j)} + B^{(j)}K_k) \succeq \gamma_k^{-1}(Q + K_k^T R K_k), && j = 1, 2\\ &(F_i + G_i K_k) P^{-1} (F_i + G_i K_k)^T \le 1, && i = 1, \ldots, n_C\\ &x_k^T P x_k \le 1 \end{aligned}$$

for some matrix $P = P^T \succ 0$, where F_i, G_i for $i = 1, \ldots, n_C$ are the rows of F and G.

(b) Suggest a suitable transformation of optimization variables to enable the value of K_k that minimizes γ_k subject to the constraints of (a) to be computed using semidefinite programming. Hence, verify that for $x_0 = (4, -1)$ this gives $K_0 = \begin{bmatrix}-0.962 & -3.678\end{bmatrix}$ and $\gamma_0 = 152.4$.

(c) Explain why a better approximation of the worst-case cost is given by the bound $J(x_k, K_k) \le x_k^T \Theta_k^* x_k$, where

$$\begin{aligned}\Theta_k^* = \arg\min_{\Theta}\ & x_k^T \Theta x_k \ \text{ subject to}\\ & \Theta - (A^{(j)} + B^{(j)}K_k)^T \Theta (A^{(j)} + B^{(j)}K_k) \succeq Q + K_k^T R K_k, \quad j = 1, 2\end{aligned}$$

Hence, verify that for $x_0 = (4, -1)$ and $K_0 = \begin{bmatrix}-0.962 & -3.678\end{bmatrix}$ the worst-case cost satisfies the upper bound $J(x_0, K_0) \le 69.0$.

(d) Comment on the suggestion that a better control strategy could be constructed by choosing K_k so as to minimize, at each time k, the value of $x_k^T \Theta x_k$ subject to

$$\begin{aligned} &\Theta - (A^{(j)} + B^{(j)}K_k)^T \Theta (A^{(j)} + B^{(j)}K_k) \succeq Q + K_k^T R K_k, && j = 1, 2\\ &P - (A^{(j)} + B^{(j)}K_k)^T P (A^{(j)} + B^{(j)}K_k) \succeq 0, && j = 1, 2 \end{aligned}$$

$$(F_i + G_i K_k)P^{-1}(F_i + G_i K_k)^T \leq 1, \qquad i = 1, \ldots, n_C$$
$$x_k^T P x_k \leq 1.$$

4 (a) Let $\mathcal{E} = \{x : x^T P x \leq 1\}$ for given $P \succ 0$ and let the vector $x \in \mathbb{R}^n$ be partitioned according to $x = (u, v)$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^{n-m}$. Show that the u-subspace projection of $\mathcal{E}$ (i.e. the set $\{u : \exists v \text{ such that } (u, v) \in \mathcal{E}\}$) is given by

$$\mathcal{E}_u = \{u : u^T P_u u \leq 1\}$$

where

$$P_u^{-1} = \begin{bmatrix} I_m & 0 \end{bmatrix} P^{-1} \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

and $\begin{bmatrix} I_m & 0 \end{bmatrix} x = u$.

(b) Using the system model and constraints of Question 3, calculate the robustly invariant ellipsoidal set $\mathcal{E}_z$ for the uncertain dynamics

$$z_{i+1|k} \in \mathrm{Co}\{\Psi^{(1)} z_{i|k}, \Psi^{(2)} z_{i|k}\}$$

and constraints $-1 \leq \begin{bmatrix} K & E \end{bmatrix} z_{i|k} \leq 1$, with $N = 12$ (i.e. $z_{i|k} \in \mathbb{R}^{14}$) and

$$\Psi^{(j)} = \begin{bmatrix} A^{(j)} + B^{(j)} K & B^{(j)} E \\ 0 & M \end{bmatrix}, \quad j = 1, 2,$$

$$K = \begin{bmatrix} -1.078 & -3.523 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

such that the area of its projection onto the state space of the system is maximized. Verify that the maximum area projection is given by $\{x : x^T P_x x \leq 1\}$ with $\det(P_x) = 0.783$.

(c) For the system in Question 4, with nominal model parameters defined by $(A^{(0)}, B^{(0)}) = \frac{1}{2}\big((A^{(1)}, B^{(1)}) + (A^{(2)}, B^{(2)})\big)$, the unconstrained optimal feedback gain for the nominal cost with weights $Q = I$, $R = 1$ is $K = \begin{bmatrix} -1.078 & -3.523 \end{bmatrix}$ and the corresponding solution of the Riccati equation (2.9) is

$$W_x = \begin{bmatrix} 2.691 & 3.668 \\ 3.668 & 21.94 \end{bmatrix}.$$

Taking $N = 12$, determine the matrix W in the expression for the nominal predicted cost:

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty}\bigl(\|x_{i|k}\|^2 + u_{i|k}^2\bigr) = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}^T W \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix},$$

where the nominal predicted state and control trajectories evolve according to

$$x_{i+1|k} = A^{(0)} x_{i|k} + B^{(0)} u_{i|k}, \quad u_{i|k} = K x_{i|k} + c_{i|k},$$

with $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ and $c_{i|k} = 0$ for $i \geq N$.

Show that the minimum value of this cost at $k = 0$ with $x_0 = (4, -1)$ subject to the constraint $(x_0, \mathbf{c}_0) \in \mathcal{E}_z$, where $\mathcal{E}_z$ is the robustly invariant ellipsoid determined in part (b), is $J^*(x_0) = 39.9$.

5 It is suggested that the value of the predicted cost in Question 4 could be reduced using a univariate search:

$$\alpha_k^* = \min_{\alpha_k \in [0,1]} \alpha_k \ \text{ subject to} - 1 \leq K x_k + \alpha_k \mathbf{c}_k^* \leq 1,$$
$$\Psi^{(j)} \begin{bmatrix} x_k \\ \alpha_k \mathbf{c}_k^* \end{bmatrix} \in \mathcal{E}_z, \quad j = 1, 2$$

where $\mathbf{c}_k^*$ is the solution of the minimization in Question 4(c) at time k.

(a) Explain the purpose of each the constraints in this line search.
(b) For the system of Question 4, find the smallest value of σ for which there exists $\Theta \succ 0$ satisfying

$$\begin{bmatrix} \Theta - I & (A^{(j)} + B^{(j)} K)^T \Theta & 0 \\ \star & \Theta & \Theta B^{(j)} \\ \star & \star & \sigma^2 I_{n_u} \end{bmatrix} \succeq 0, \quad j = 1, 2$$

What does this imply about the state of the closed-loop system under a control law of the form $u_k = K x_k + c_{0|k}$?
(c) Suppose that, at each time step, $k = 0, 1, \ldots$, the optimization of Question 4(c) and the line search in (a) is performed and the solution is used to define an MPC law $u_k = K x_k + \alpha_k^* \mathbf{c}_k^*$. Will the closed-loop system be stable?

6 (a) For the system of Question 3 with $K = \begin{bmatrix} -1.078 & -3.523 \end{bmatrix}$, solve (5.48) to determine the prediction dynamics that give the robustly invariant ellipsoid $\mathcal{E}_z$ with the largest area projection onto the model state space, and confirm that the solution gives

$$A_c^{(1)} = \begin{bmatrix} 0.835 & -1.539 \\ -0.035 & -0.612 \end{bmatrix}, \quad A_c^{(1)} = \begin{bmatrix} 0.692 & -1.113 \\ 0.026 & -0.821 \end{bmatrix}$$
$$C_c = \begin{bmatrix} -0.176 & -0.394 \end{bmatrix}.$$

(b) Compute the matrix $W_c \succ 0$ with smallest trace satisfying

$$W_c - A_c^{(j)\,T} W_c A_c^{(j)} \succeq B^{(0)\,T} W_x B^{(0)} + R, \quad j = 1, 2$$

Hence, verify that, with $x_0 = (4, -1)$, the minimum over $\mathbf{c}_0$ of the cost $J(x_0, \mathbf{c}_0) = \|x_0\|^2_{W_x} + \|\mathbf{c}_0\|^2_{W_c}$ subject to $(x_0, \mathbf{c}_0) \in \mathcal{E}_z$, where $\mathcal{E}_z$ is the ellipsoidal set determined in (a), is $J^*(x_0) = 44.7$.

(c) Explain how the inequality defining W_c in (b) ensures the stability of the closed-loop system under the control law $u_k = Kx_k + c^*_{0|k}$, where at each time $k = 0, 1, \ldots$, $c^*_{0|k}$ is the first element of the optimal $\mathbf{c}^*_k$ for the minimization of the cost $J(x_k, \mathbf{c}_k)$ subject to $(x_k, \mathbf{c}_k) \in \mathcal{E}_z$.

(d) Determine the maximum scaling σ such that σx_0, with $x_0 = (4, -1)$, lies in the feasible set for the minimization of $J(x_0, \mathbf{c}_0)$ in (b) (i.e. Algorithm 5.4). Compare this with the maximum value of σ such that σx_0 is feasible for the minimization in Question 4(c) (i.e. Algorithm 5.3) and that of Question 3(b) (i.e. Algorithm 5.1). What conclusions can be drawn from this comparison?

7 Explain why the problem of maximizing the volume of the low-complexity polytopic set, $\Pi(V, \alpha) \doteq \{x : |Vx| \le \alpha\}$, where $V \in \mathbb{R}^{n_x \times n_x}$ is an invertible matrix, is in general nonconvex when V and α are both considered to be optimization variables. Show that the volume maximization can be formulated as a convex problem if V is fixed.

8 This question considers how to construct a low-complexity polytopic set for the system and constraints of Question 3 under the control law $u = Kx$, $K = \begin{bmatrix} -1.078 & -3.523 \end{bmatrix}$.

(a) Show that if $V = W^{-1}$, where W is the (right) eigenvector matrix of $\Phi^{(0)} = A^{(0)} + B^{(0)}K$, i.e.

$$W = \begin{bmatrix} 0.982 & 0.851 \\ -0.187 & 0.525 \end{bmatrix}, \quad V = \begin{bmatrix} 0.778 & -1.262 \\ 0.277 & 1.456 \end{bmatrix},$$

then there necessarily exists a vector α such that the low-complexity polytope $\{x : |Vz| \le \alpha\}$ is invariant for the uncertain dynamics $x_{k+1} \in \mathrm{Co}\{\Phi^{(1)} x_k, \Phi^{(2)} x_k\}$ where $\Phi^{(j)} = A^{(j)} + B^{(j)}K$, $j = 1, 2$.

(b) Formulate and solve a convex optimization to determine α so that the volume of the set $\Pi(V, \alpha)$ is maximized, where V is fixed at the value specified in (a).

9 The maximal robustly invariant polytopic set for the system and constraints of Question 3 under $u = Kx$ is given by

$$\{x : Vx \le \mathbf{1}\}, \quad V = \begin{bmatrix} -1.078 & -3.523 \\ 1.078 & 3.523 \\ 0.172 & 2.622 \\ -0.172 & -2.622 \end{bmatrix}$$

(a) Determine nonnegative matrices $H^{(1)}$, $H^{(2)}$ and H_c satisfying

$$\begin{aligned} H^{(j)} V &= V \Phi^{(j)}, \quad j = 1, 2 \\ H_c V &= G K \end{aligned}$$

where $G = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, such that the sum of the elements in each row of each of $H^{(1)}$, $H^{(2)}$ and H_c is minimized.

(b) Show that, for any given mode 1 horizon N, if there exists a pair of sequences $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ and $\boldsymbol{\alpha}_k = (\alpha_{0|k}, \ldots, \alpha_{N|k})$ satisfying the constraints defined at time k by

$$\begin{aligned} V x_k &\leq \alpha_{0|k} \\ H^{(j)} \alpha_{i|k} + V B^{(j)} c_{i|k} &\leq \alpha_{i+1|k}, \quad j = 1, 2, \qquad i = 0, \ldots, N-1 \\ H_c \alpha_{i|k} + G c_{i|k} &\leq \mathbf{1}, \qquad i = 0, \ldots, N-1 \\ H^{(j)} \alpha_{N|k} &\leq \alpha_{N|k}, \quad j = 1, 2 \\ H_c \alpha_{N|k} &\leq \mathbf{1} \end{aligned}$$

then there must exist $\mathbf{c}_{k+1}$ and $\boldsymbol{\alpha}_{k+1}$ satisfying the corresponding constraints at time $k+1$ if $u_k = K x_k + c_{0|k}$.

(c) For a mode 1 horizon of $N = 8$ and $x_0 = (4, -1)$, determine the maximum scalar σ such that σx_0 is feasible for the constraints in part (b).

(d) At $k = 0$ with $x_0 = (4, -1)$, solve the MPC optimization:

$$\underset{s_{0|k}, \mathbf{c}_k, \boldsymbol{\alpha}_k}{\text{minimize}} \; J(s_{0|k}, \mathbf{c}_k)$$

subject to $V s_{0|k} \leq \alpha_{0|k}$ and the constraints of part (b) with $N = 8$, where the nominal cost is defined by

$$J(s_{0|k}, \mathbf{c}_k) = \|s_{0|k}\|_{W_x}^2 + \|\mathbf{c}_k\|_{W_c}^2.$$

Confirm that the optimal predicted cost for this initial condition is $J(s_{0|k}^*, \mathbf{c}_0^*) = 37.9$.

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Part III
Stochastic MPC

Chapter 6
Introduction to Stochastic MPC

Uncertainty forms an integral part of most control problems and earlier chapters discussed how MPC algorithms can be constructed in order to treat model uncertainty in a robust sense. One of the key features of robust MPC is that it requires constraints to be satisfied for all possible realizations of uncertainty. Thus each element of the set of values that can be assumed by an uncertain model parameter or disturbance input is treated with equal importance, and robust MPC does not discriminate between alternative realizations on the basis of their respective likelihood.

However, in practical applications it is often the case that some realizations of model uncertainty, for example parameter realizations that lie close to the nominal value of that parameter, are more likely than others, such as parameter realizations that lie on the boundary of the uncertainty set. In fact model uncertainty is often stochastic with a probability distribution that is known, either as a result of statistical analysis performed during model identification, or because of physical principles underlying the model. Clearly the distribution of model uncertainty is useful information that should be taken into account in the design of MPC algorithms.

An obvious way to address this is through the definition of the cost. Thus, rather than being defined as a nominal or worst-case value, the MPC cost can be chosen to be the expected value, over the distribution of uncertainty, of the usual quadratic predicted cost. An additional and often more crucial use of distribution information corresponds to the case in which some or all of the system constraints are probabilistic in nature. In this case, constraint violations are permitted provided the frequency of constraint violations (or more generally the number of violations in a given time interval) is below a predefined limit.

An example of a control problem involving probabilistic constraints concerns the optimal allocation of resources for sustainable development. Consider, for example, investment in electricity generating technology, where it is important to minimize (among other objectives) the cost of energy for the consumer while meeting (among other constraints) limits on emissions of greenhouse gases. Taking the horizon of

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interest to be the 30 years separating generations, it does not make sense to expect the accumulated emissions of CO_2 over 30 years, say y, to be less than a given amount, say A, because of the various unknown factors associated with economic and technological development over a horizon of this length. In fact, given that such factors are stochastic in nature, it follows that y itself will be a random variable, so that a constraint that $y \leq A$ is meaningless. If, however, information is available on the probability distribution of y, then an aspirational constraint such as $\Pr\{y \leq A\} \geq p$, namely that the probability that y is less that a target bound A should exceed a given level p, is eminently sensible [1, 2].

Many other examples where probabilistic constraints are more natural than deterministic constraints can be found in diverse areas of engineering and related fields such as process control [3, 4], financial engineering [5, 6], electricity generation, distribution and pricing [7, 8], building climate control [9] and telecommunications network traffic control [10]. For the purposes of motivation, we briefly describe here another example taken from a problem that concerns the operation of wind turbines. Large wind turbines for electricity generation are typically designed for a given service life, which is typically around 20 years, but this lifespan may be compromised as a result of fatigue damage caused, for example, by the fore-aft movement of the tower. Controlling this movement so as to control the rate of accumulation of fatigue damage in the wind turbine forms one of the objectives of a supervisory controller for variable speed wind turbines (see e.g. [11]). However, in order to maximize electrical power output, it is possible to allow violations of the constraints on the tower oscillations, provided these do not happen more often than some pre-specified limit. Such violations occur as a result of fluctuations in wind speed, the variability of which can be modelled by suitable probability distributions (e.g. [12]). This approach therefore imposes constraints on stochastic variables, and the implied constraints are naturally stated in a probabilistic manner.

Exacting performance requirements often cause constrained variables to reach their limits and in this sense it is vital that the definition of the system constraints takes into account the stochastic nature of the given application. Even if information on the probability distribution of model uncertainty is available, it may of course be possible to act cautiously and enforce constraints robustly, namely to impose conditions that require constraints to be satisfied with certainty. But in cases where constraint violation is allowed (up to a specified probability) it should of course be evident that such a strategy is conservative, and will result in poorer performance and smaller regions of attraction.

It is the purpose of Stochastic MPC (SMPC) to address these issues. In particular, SMPC is concerned with the repetitive optimization of an appropriate predicted cost for systems with stochastic uncertainty. This optimization is to be performed subject to constraints, some of which are probabilistic in nature. Constraint satisfaction requires the determination of probability distributions for predicted variables, something which is relatively easy in the case of additive uncertainty only, given the linear dependence of predictions on disturbances. Stochastic uncertainty that appears multiplicatively in the system model, for example as a result of stochastic model parameters, is often more challenging. This is because the predicted future model states are random variables, and these are multiplied by stochastic model parameters to generate the successor states, compounding the problem of determining the distributions of predicted states over multiple prediction time steps.

As with classical and robust MPC, there is a concern here with recursive feasibility. In fact it is easy to see (as will be discussed in detail Chap. 7) that, except in certain special cases, strict recursive feasibility can only be guaranteed for the case that the uncertainty in predicted states and inputs has a finitely supported distribution. The reason for this is that whereas the current state may be such that a feasible predicted trajectory exists, uncertainty with unbounded support could, albeit perhaps with low probability, result in a successor state for which it is impossible to guarantee the existence of a feasible trajectory. This particular difficulty has been a feature of earlier SMPC formulations, which almost exclusively considered uncertainty with Gaussian distributions. This choice was natural given the mathematical convenience of the Gaussian assumption. However, in addition to preventing the statement of recursive feasibility results, this assumption is often not consistent with practice, since for many physical systems the probability of an uncertainty realization exceeding an arbitrarily large threshold is zero rather than arbitrarily small. SMPC is also concerned with the definition of a suitable cost that enables the statement of stability.

This chapter introduces the SMPC problem formulation, describes earlier work in this area and discusses quadratic expected value costs. Probabilistic constraints are introduced here in the context of uncertain moving average models, for which recursive feasibility can be ensured even when the probability distribution of model uncertainty is not finitely supported. The discussion of probabilistic constraints for general classes of linear system and model uncertainty is deferred to Chap. 7, where the requirements for a guarantee of recursive feasibility and methods of handling general probability distributions are considered. Chapter 7 also extends the framework for analysing closed-loop stability to the case of expected value costs. Chapter 8 combines these techniques with tube-based methods of constraint handling in order to construct stochastic MPC algorithms with guaranteed closed-loop properties for general classes of model uncertainty and probabilistic constraints.

6.1 Problem Formulation

There are several ways to obtain mathematical descriptions of systems that are subject to stochastic uncertainty. Perhaps the most convenient is through black box identification of auto-regressive moving average (ARMA) relationships between the input variable u and output variable y:

$$\begin{aligned} y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n} \\ = b_0 u_{k-d} + \cdots + b_n u_{k-d-n} + e_k + c_1 e_{k-1} + \cdots + c_n e_{k-n}, \end{aligned} \tag{6.1}$$

where e denotes a zero-mean disturbance and the positive integer d accounts for the delay in the system. This type of model is general enough to include systems with multiple inputs and outputs since y, u, e can be vector-valued and a, b, c matrices of conformal dimensions. Descriptions of this kind (which include moving average (MA) models as a special case when the parameters $a_1, \ldots, a_n$ are identically equal to zero) have been used in diverse fields including, for example, process control and econometrics [13–15]. A useful by-product of black box identification is that it provides information on the statistical properties of the model parameters [16]. This could be obtained from a single experiment on the basis of information on the noise affecting the system output measurements or by repeated experiments capturing different realizations of the uncertain parameters.

To use the constraint handling machinery of invariant sets and state and control tubes, it is convenient to convert the ARMA model (6.1) into a state-space form. Given the linearity of (6.1), this can be expressed as

$$x_{k+1} = A_k x_k + B_k u_k + D w_k \tag{6.2}$$

where $x_k \in \mathbb{R}^{n_x}$ and $u_k \in \mathbb{R}^{n_u}$ are the state and control inputs at time k. In setting up the stochastic MPC problem, we make the assumption that the matrices A_k and B_k containing multiplicative model parameters and the additive disturbance input $w_k \in \mathbb{R}^{n_w}$ can be expressed in terms of a linear expansion over a known basis set:

$$(A_k, B_k, w_k) = (A^{(0)}, B^{(0)}, 0) + \sum_{j=1}^{\rho} (A^{(j)}, B^{(j)}, w^{(j)}) q_k^{(j)}. \tag{6.3a}$$

Here $q^{(j)}$ is a scalar random variable. The realization of $q^{(j)}$ at time k, denoted $q_k^{(j)}$, is unknown at time k but has a known probability distribution.[1] The vector $q_k = (q_k^{(1)}, \ldots, q_k^{(\rho)})$ may be time varying (in the sense that it has a different realization at each time instant k) but q_k is assumed to be identically distributed for each k.

We assume that q_k has a mean value of zero and covariance matrix equal to the identity matrix, namely

$$\mathbb{E}(q_k) = 0, \quad \mathbb{E}(q_k q_k^T) = I \tag{6.3b}$$

where $\mathbb{E}(\cdot)$ denotes the expectation operator. These assumptions do not imply any loss of generality because of the linearity of the dynamics (6.2) and the expansion (6.3a). In particular, the expected value of the additive disturbance is taken to be zero since a state translation can be used to account for any non-zero components. Thus a non-zero value of $\mathbb{E}(q_k)$ and a corresponding non-zero expected value

[1] Unless explicitly stated otherwise, all random variables that will be encountered in the discussion of stochastic MPC are functions of $q^{(j)}$ appearing in (6.3a) and every random event of interest is related in a straightforward way to the realizations of $q_k^{(j)}, j = 1, \ldots, \rho$ at time instant k. With this understanding, the underlying probability space is well defined and we are able to avoid cumbersome measure-theoretic notation.

of the disturbance, $\mathbb{E}(w_k) = w^{(0)}$, can be absorbed by transforming the state and disturbance variables of (6.2) according to $x_k - x^{(0)}$ and $w_k - w^{(0)}$, respectively, where $x^{(0)} = (I - A^{(0)})^{-1} D w^{(0)}$, and by replacing q_k with $q_k - \mathbb{E}(q_k)$ in (6.3a) and appropriately redefining $A^{(0)}$ and $B^{(0)}$.

Likewise, the assumption that $\mathbb{E}(q_k q_k^T) = I$ is justified by a linear transformation since (6.3a) can be expressed equivalently as

$$\begin{aligned}\operatorname{vec}(A_k, B_k, w_k) &= \operatorname{vec}(A_0, B_0, 0)\\ &\quad + \left[\operatorname{vec}(A^{(1)}, B^{(1)}, w^{(1)}) \cdots \operatorname{vec}(A^{(m)}, B^{(m)}, w^{(m)})\right] q_k\end{aligned}$$

where $\operatorname{vec}(\cdot)$ is a vectorization operation that rearranges the elements of a matrix into a column vector. Therefore, if the covariance matrix of q_k has the eigenvalue decomposition

$$\mathbb{E}(q_k q_k^T) = W_\Sigma \Lambda_\Sigma W_\Sigma^T$$

then it is possible to define triples $(\tilde{A}^{(j)}, \tilde{B}^{(j)}, \tilde{w}^{(j)})$ for $j = 1, \ldots, m$ and a vector $\tilde{q}$ by

$$\begin{aligned}\operatorname{vec}(\tilde{A}^{(j)}, \tilde{B}^{(j)}, \tilde{w}^{(j)}) &= \operatorname{vec}(A^{(j)}, B^{(j)}, w^{(j)}) W_\Sigma \Lambda_\Sigma^{1/2}\\ \tilde{q}_k &= \Lambda_\Sigma^{-1/2} W_\Sigma^T q_k.\end{aligned}$$

This transformation leaves the parameterization of (6.3a) unaffected because

$$\begin{aligned}\sum_{j=1}^{m} (\tilde{A}^{(j)}, \tilde{B}^{(j)}, \tilde{w}^{(j)}) \tilde{q}_k^{(j)} &= \left[\operatorname{vec}(A^{(1)}, B^{(1)}, w^{(1)}) \cdots \operatorname{vec}(A^{(m)}, B^{(m)}, w^{(m)})\right] \tilde{q}_k\\ &= \sum_{j=1}^{m} (A^{(j)}, B^{(j)}, w^{(j)}) q_k^{(j)},\end{aligned}$$

but at the same time the covariance matrix of the transformed vector of coefficients is given by $\mathbb{E}(\tilde{q}_k \tilde{q}_k^T) = I$.

We make the further assumption that q_k and q_i are statistically independent for $k \neq i$. This assumption is not necessarily restrictive in respect of the additive disturbance because a linear filter can be introduced into the state-space model (6.2) in order to generate temporally correlated disturbances if required. But the same approach cannot be used to conveniently introduce temporal correlation between multiplicative parameters in the model (6.2) since the dynamics are assumed to be linear. However, the assumption of independence of q_k and q_i is used here only to simplify the computation of predicted costs based on the expected value of sums of quadratic stage costs, and to simplify the analysis of stability based on this cost. The methods of handling constraints that are discussed here and in the following chapters do not rely on this assumption.

Like the nominal and robust MPC strategies considered in earlier chapters, the predicted performance cost of stochastic MPC is often taken to be a quadratic function of the degrees of freedom in predicted state and control trajectories. For example, for an N-step horizon we can define

$$\hat{J}(x_k, \mathbf{u}_k, \mathbf{q}_k) = \sum_{i=0}^{N-1} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big) + \|x_{N|k}\|_{W_T}^2$$

where $x_{i|k}$, $u_{i|k}$ are predicted values of x_{k+i}, u_{k+i} at time k with $x_k = x_{0|k}$ and $\mathbf{u}_k = \{u_{0|k}, \ldots, u_{N-1|k}\}$, where $\mathbf{q}_k = \{q_{0|k}, \ldots, q_{N-1|k}\}$ is a realization of the sequence $\{q_k, \ldots, q_{N-1}\}$ of uncertain model parameters, and where matrices $Q \succeq 0$ and $R \succ 0$ are cost weights with W_T a terminal weighting matrix. On account of the stochastic uncertainty in $\mathbf{q}_k$, the cost $\hat{J}(x_k, \mathbf{u}_k, \mathbf{q}_k)$ is stochastic and the cost index, $J(x_k, \mathbf{u}_k)$, of stochastic MPC must therefore be constructed under specific assumptions on $\mathbf{q}_k$. Thus it is possible to adopt the nominal cost

$$J(x_k, \mathbf{u}_k) \doteq \hat{J}(x_k, \mathbf{u}_k, 0),$$

or, if the model uncertainty is subject to bounds $q_k \in \mathcal{Q}$ for some compact set $\mathcal{Q}$, then the worst-case cost can be employed,

$$J(x_k, \mathbf{u}_k) \doteq \max_{\mathbf{q}_k \in \mathcal{Q} \times \cdots \times \mathcal{Q}} \hat{J}(x_k, \mathbf{u}_k, \mathbf{q}_k).$$

Given knowledge of the distribution of q_k, it is more common, however, to use a predicted cost that takes into account the stochastic nature of the problem through the expectation of a quadratic cost. Therefore we focus the discussion of stochastic MPC on an expected cost of the form

$$\begin{aligned} J(x_k, \mathbf{u}_k) &\doteq \mathbb{E}_k\big(\hat{J}(x_k, \mathbf{u}_k, \mathbf{q}_k)\big) \\ &= \sum_{i=0}^{N-1} \mathbb{E}_k\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big) + \mathbb{E}_k\big(\|x_{N|k}\|_{W_T}^2\big). \end{aligned} \tag{6.4}$$

Here, the notation $\mathbb{E}_k(\cdot)$ has been introduced[2] to indicate that the expectation is conditional on information available to the controller at time k, namely the current plant state x_k (for the case that x_k is measured directly), and is therefore dependent on the distribution of the model uncertainty sequence $\mathbf{q}_k$. Variations on this cost will also be discussed, such as quadratic costs based on probabilistic bounds on predicted variables or on combinations of their means and variances.

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[2]We use the simpler notation $\mathbb{E}(\cdot)$ for the expectation of a random variable that depends on a single realization of model uncertainty, e.g. $\mathbb{E}(A_k) = \mathbb{E}_k(A_k) = A^{(0)}$.

The aim of stochastic MPC is to obtain, through the repetitive minimization of a predicted cost, an approximation of the optimal control law, namely the optimal argument of (6.4) subject to constraints on the system states and/or control inputs. This is to be done while providing stability guarantees for all initial conditions in some region of state space. A precondition for such guarantees is that the expected value of the predicted stage cost in (6.4) converges to a finite limit as $i \to \infty$ under some control law. This in turn requires that the pair (A_k, B_k) is mean-square stabilizable [17], which is ensured by the following assumption.

Assumption 6.1 There exist matrices K and P such that $P = P^T \succ 0$ and

$$P - \mathbb{E}\Big((A_k + B_k K)^T P (A_k + B_k K)\Big) \succ 0. \tag{6.5}$$

It can be determined whether the system of (6.2) and (6.3a, 6.3b) satisfies Assumption 6.1 simply by checking the feasibility of a linear matrix inequality. Thus, using (6.3a, 6.3b) condition (6.5) can be expressed equivalently as

$$P - \sum_{j=0}^{m} (A^{(j)} + B^{(j)}K)^T P (A^{(j)} + B^{(j)}K) \succ 0.$$

Introducing convexifying transformations $S = P^{-1}$, and $Y = KP^{-1}$ and using Schur complements this condition can be written as

$$\begin{bmatrix} S & (A^{(0)}S + B^{(0)}Y)^T & (A^{(1)}S + B^{(1)}Y)^T & \cdots & (A^{(m)}S + B^{(m)}Y)^T \\ \star & S & 0 & \cdots & 0 \\ \star & \star & S & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & S \end{bmatrix} \succ 0 \tag{6.6}$$

(with $\star$ indicating a block of a symmetric matrix). Thus the conditions of Assumption 6.1 are satisfied by $P = S^{-1}$ and $K = YS^{-1}$ if and only if matrices $S = S^T \succ 0$ and Y exist satisfying (6.6).

For the case of multiplicative uncertainty alone and in the absence of constraints, Assumption 6.1 implies that the control law $u_k = Kx_k$ ensures that the variance of the state of (6.2) converges asymptotically to zero. This guarantees the existence of a predicted control sequence $\{u_{i|k},\ i = 0, 1, \ldots\}$ that causes the predicted state $x_{i|k}$ to converge (with probability 1) to zero as $i \to \infty$ [17]. Clearly, no feedback law can make the state converge identically to zero in the presence of persistent additive disturbances. In this case, Assumption 6.1 ensures, in the absence of constraints, that the expected value of the stage cost of (6.4) tends to a finite limit, which is discussed in Sect. 6.2.

Analogously to the formulations of classical MPC and robust MPC, a dual-mode prediction strategy can be used in stochastic MPC to define predicted state and control trajectories over an infinite prediction horizon while retaining only a finite number of free variables in their parameterization. This makes it possible to extend the cost of (6.4) over an infinite horizon by suitably choosing the weighting matrix W. We therefore assume that the predicted control law over the prediction horizon of mode 2 is defined by a predetermined feedback law: $u_{i|k} = Kx_{i|k}$ for all $i \geq N$.

Several different forms of state and input constraints have been proposed for stochastic MPC in the context of a model such as (6.2) and (6.3a, 6.3b). For example, constraints may be stated in terms of expected values as

$$\mathbb{E}_k(Fx_{i|k} + Gu_{i|k}) \leq \mathbf{1}, \quad i = 1, 2, \ldots \tag{6.7}$$

(e.g. [6, 10]). Alternatively, pointwise in time probabilistic constraints can be stated as

$$\Pr_k(Fx_{1|k} + Gu_{1|k} \leq \mathbf{1}) \geq p \tag{6.8}$$

for some specified probability p. An alternative form of probabilistic constraint can be imposed over an interval of T time steps:

$$\Pr_k(Fx_{i|k} + Gu_{i|k} \leq \mathbf{1},\ i = 1, \ldots, T) \geq p \tag{6.9}$$

for some given probability p and horizon T. In each case, the variable $Fx + Gu$ may be vector-valued; thus for example (6.8) requires that the probability of any element of $Fx_{k+1|k} + Gu_{k+1|k}$ exceeding a threshold of 1 should be less than $1 - p$.

An equivalent statement of (6.9) is that the expected number of times that any element of $Fx + Gu$ exceeds 1 over an interval of T predicted time steps should be less than $(1-p)T$. Clearly, it is possible to construct constraint sets involving various different probabilities and probabilistic conditions by combining constraints of the form of (6.8) and (6.9). This also applies to the special case of $p = 1$, thus allowing for problems that involve a mixture of probabilistic ($p < 1$) and robust ($p = 1$) constraints.

The notation $\Pr_k(\mathcal{A})$ in (6.8) and (6.9) refers to the probability of an event $\mathcal{A}$ that depends[3] on the sequence $\mathbf{q}_k = \{q_k, \ldots, q_{k+N-1}\}$, given that the initial prediction model state is x_k. Hence, $\Pr_k(Fx_{1|k} + Gu_{1|k} \leq \mathbf{1})$ is the probability that the one-step ahead predicted state and input satisfy $Fx_{1|k} + Gu_{1|k} \leq \mathbf{1}$, which is therefore a function of x_k and the probability distribution of the random variable q_k. Similarly, $\Pr_k(Fx_{i|k} + Gu_{i|k} \leq \mathbf{1})$ for $i \geq 1$ depends on x_k and on the probability distribution of $\{q_k, \ldots, q_{k+i-1}\}$. Our treatment of stochastic MPC concentrates on problem formulations for which recursive feasibility can be guaranteed, and we consider mainly the constraints of (6.8) and (6.9). A detailed discussion of recursive feasibility is given in Sect. 7.1.

[3] We also use $\Pr(\mathcal{A})$ for the probability of an event $\mathcal{A}$ that depends on a single realization of model uncertainty when it is obvious from the context which random variable $\mathcal{A}$ depends on, e.g. $\Pr(w_k \leq 0) = \Pr_k(w_k \leq 0) = \Pr(\sum_{j=1}^{m} w^{(j)} q_k^{(j)} \leq 0)$.

6.2 Predicted Cost and Unconstrained Optimal Control Law

Early stochastic MPC strategies, such as those discussed in Sect. 6.4, ensure closed-loop stability by imposing equality terminal constraints on the predicted input and output trajectories of the plant model. In the context of a moving average model, this can be done simply by setting the predicted input equal to a constant value at the end of the mode 1 prediction horizon. After a subsequent interval of n prediction time steps, where n is the order of the model, the expected value of the predicted output necessarily reaches its desired steady-state value. However equality terminal constraints can be overly stringent and may result in limited regions of attraction. A way to avoid this is to replace equality with inequality terminal constraints (as discussed in Chap. 2) and in this setting, recursive feasibility and closed-loop stability can be ensured through the use of invariant terminal sets defined in the plant model state space.

The construction of suitable terminal sets and the handling of constraints are key ingredients in the development of a stochastic MPC strategy, and will be considered in more detail in Chaps. 7 and 8. Before discussing these, however, we turn our attention to another two basic components of SMPC. The first of these concerns the definition and evaluation of a predicted cost, while the second concerns the unconstrained optimal feedback law that achieves the minimum of the predicted cost in the absence of constraints. Such unconstrained control laws are obvious candidates for use as terminal control laws.

We begin with the observation that the mean-square stability property of Assumption 6.1 implies that, when no constraints are present, the expected value of the state x_k of the model (6.2) and (6.3a, 6.3b) under the linear feedback law $u_k = Kx_k$ converges asymptotically to zero whenever K satisfies (6.5). However, the variance of x_k (and hence also the expected stage cost in (6.4)) converges to a non-zero value because of the presence of additive model uncertainty, as the following lemma shows.

Lemma 6.1 *[18] In the absence of constraints, the state of (6.2) and (6.3a, 6.3b) under* $u_k = Kx_k$ *satisfies the asymptotic conditions* $\lim_{k\to\infty} \mathbb{E}_0(x_k) = 0$ *and* $\lim_{k\to\infty} \mathbb{E}_0(x_k x_k^T) = \Theta$, *where* Θ *is the solution of*

$$\Theta - \mathbb{E}\Big((A_k + B_k K)\Theta(A_k + B_k K)^T\Big) = D\mathbb{E}(w_k w_k^T)D^T, \tag{6.10}$$

if and only if (6.5) holds for some $P \succ 0$.

Proof Since the system (6.2) is linear, its state can be decomposed for all $k = 0, 1, \ldots$ as $x_k = \zeta_k + \xi_k$, with

$$\zeta_{k+1} = \Phi_k \zeta_k, \qquad \zeta_0 = x_0 \tag{6.11a}$$

$$\xi_{k+1} = \Phi_k \xi_k + D w_k, \qquad \xi_0 = 0 \tag{6.11b}$$

where $\Phi_k = A_k + B_k K$ and A_k, B_k take values from the uncertainty class of (6.3a, 6.3b). Existence of $P \succ 0$ satisfying (6.5) is necessary and sufficient for mean-square stability of (6.11a). To prove sufficiency, let $Z_k \doteq \mathbb{E}_0(\zeta_k \zeta_k^T)$, then, since ζ_k and Φ_k are by assumption independent, (6.11a) gives

$$\operatorname{tr}(PZ_{k+1}) = \mathbb{E}\Big(\operatorname{tr}(P\Phi_k Z_k \Phi_k^T)\Big) = \operatorname{tr}\Big(\mathbb{E}(\Phi_k^T P \Phi_k) Z_k\Big).$$

Hence (6.5) and $Z_k \succeq 0$ imply that $\operatorname{tr}(PZ_{k+1}) < \operatorname{tr}(PZ_k)$ whenever $Z_k \neq 0$, and since $P \succ 0$ implies $\operatorname{tr}(PZ_k) > 0$ for all $Z_k \neq 0$, it follows that

$$\lim_{k\to\infty} Z_k = 0, \tag{6.12}$$

which implies that (6.11a) is mean-square stable. To show the necessity of (6.5), let $P_k \doteq \sum_{i=0}^k S_i$ where $S_{k+1} = \mathbb{E}(\Phi_k S_k \Phi_k)$ for arbitrary $S_0 \succ 0$. Then $P_{k+1} = \mathbb{E}(\Phi_k P_k \Phi_k^T) + S_0$ and the mean-square stability of (6.11a) implies that P_k converges to a finite limit as $k \to \infty$. Defining this limit as P, we can conclude from $S_0 \succ 0$ that $P \succ 0$ exists satisfying (6.5) whenever (6.11a) is mean-square stable.

From the mean-square stability property (6.12), it follows that $\zeta_k \to 0$ as $k \to \infty$ with probability 1 [17]. On the other hand, (6.11b) implies $\mathbb{E}_0(\xi_k) = 0$ for all k, and since $x_k = \zeta_k + \xi_k$ we can therefore conclude that $\lim_{k\to\infty} \mathbb{E}_0(x_k) = 0$.

Using (6.11b) and the zero-mean property of ξ_k, and noting that ξ_k and Φ_k are independent by assumption, we obtain

$$\mathbb{E}_0(\xi_{k+1}\xi_{k+1}^T) = \mathbb{E}\Big(\Phi_k \mathbb{E}_0(\xi_k \xi_k^T)\Phi_k^T\Big) + D\mathbb{E}(w_k w_k^T)D^T.$$

Combining this relationship with (6.10) and defining $\hat{\Theta}_k \doteq \mathbb{E}_0(\xi_k\xi_k^T) - \Theta$, it follows that

$$\hat{\Theta}_{k+1} = \mathbb{E}(\Phi_k \hat{\Theta}_k \Phi_k^T).$$

Therefore, $\lim_{k\to\infty} \hat{\Theta}_k = 0$ if and only if (6.11a) is mean-square stable, in which case we have $\lim_{k\to\infty} \mathbb{E}(\xi_k \xi_k^T) = \Theta$. This completes the proof because it then follows that $\lim_{k\to\infty} \mathbb{E}(x_k x_k^T) = \Theta$ since $\zeta_k \to 0$ with probability 1 as $k \to \infty$. $\square$

Note that the parameterization of model uncertainty in (6.3a, 6.3b) allows the expectation in (6.10) to be evaluated explicitly. Hence (6.10) is equivalent to a set of linear conditions on the elements of Θ:

$$\Theta - \sum_{j=0}^{m} (A^{(j)} + B^{(j)}K)\Theta(A^{(j)} + B^{(j)}K)^T = D\sum_{j=1}^{m} w^{(j)} w^{(j)\,T} D^T,$$

which can be shown to yield a unique positive definite solution for Θ whenever Assumption 6.1 is satisfied.

Many stochastic control problems are formulated in terms of an expected infinite horizon quadratic performance index of the form

$$\sum_{i=0}^{\infty} \mathbb{E}_0\big(\|x_k\|_Q^2 + \|u_k\|_R^2\big). \tag{6.13}$$

However, in the context of stochastic MPC two difficulties with this cost are immediately apparent. First, the cost is evaluated over an infinite sequence of control inputs, and in order to formulate an objective that can be minimized numerically subject to input and state constraints, this infinite control sequence must therefore be parameterized in terms of a finite number of optimization variables. Second, a consequence of Lemma 6.1 is that the cost (6.13) is in general infinite for the system (6.2) and (6.3a, 6.3b) under a linear feedback law. Moreover, in the absence of constraints the optimal controller is a linear-state feedback law (as we show later in this section), and therefore the minimum value of cost (6.13) is in general also infinite.

The first of these issues can be handled by introducing a dual-mode prediction scheme as was done for nominal and robust MPC in Sects. 2.3 and 3.1. Therefore we define the predicted control sequence at time k as

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \quad i = 0, 1, \ldots \tag{6.14}$$

where $\{c_{0|k}, \ldots, c_{N-1|k}\}$ are decision variables at time k and $c_{i|k} = 0$ for all prediction times $i \geq N$. The gain K is assumed to satisfy the mean-square stability condition (6.5), and ideally K should be chosen as the optimal feedback gain in the absence of constraints. To tackle the second issue, we ensure that the minimum value of the predicted cost associated with this predicted input sequence is finite by subtracting a constant from each stage of (6.13). Lemma 6.1 implies that the expected value of the stage cost for the system (6.2) and (6.3a, 6.3b) under $u_{i|k} = Kx_{i|k}$ converges to a steady-state value, which we denote as l_{ss}:

$$l_{ss} \doteq \lim_{i\to\infty} \mathbb{E}_k\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big) = \mathrm{tr}\Big(\Theta(Q + K^T RK)\Big).$$

We therefore define the predicted cost as

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \mathbb{E}_k\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}\big) \tag{6.15}$$

where $\mathbf{c}_k \in \mathbb{R}^{Nn_u}$ is defined by $\mathbf{c}_k \doteq (c_{0|k}, \ldots, c_{N-1|k})$.

From its definition in (6.15), it is clear that the cost $J(x_k, \mathbf{c}_k)$ is a quadratic function of the optimization variable $\mathbf{c}_k$. The most convenient way to compute this cost function is to express the predicted dynamics using the lifted autonomous formulation of Sect. 2.7. For the case of the model (6.2), this has the following form:

$$z_{i+1|k} = \Psi_{k+i} z_{i|k} + \bar{D} w_{k+i}, \quad z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \tag{6.16}$$

with state variable $z_{i|k} \in \mathbb{R}^{n_z}$, $n_z = n_x + Nn_u$. Here

$$\Psi_k = \begin{bmatrix} \Phi_k & B_k E \\ 0 & M \end{bmatrix}, \quad \Phi_k = A_k + B_k K, \quad \bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}$$

and the matrices E and M are defined as in (2.26b) so that $E\mathbf{c}_k = c_{0|k}$ and $M\mathbf{c}_k = (c_{1|k}, \ldots, c_{N-1|k}, 0)$. This autonomous formulation is the basis of the following result for evaluating the predicted cost.

Theorem 6.1 *[18] The predicted cost of (6.15), computed for the model (6.2) under the control law (6.14) is given by*

$$J(x_k, \mathbf{c}_k) = \begin{bmatrix} z_k \\ 1 \end{bmatrix}^T \begin{bmatrix} W_z & w_{z1} \\ w_{z1}^T & w_1 \end{bmatrix} \begin{bmatrix} z_k \\ 1 \end{bmatrix}, \tag{6.17}$$

where $W_z = W_z^T \in \mathbb{R}^{n_z \times n_z}$, $w_{z1} \in \mathbb{R}^{n_z}$ and $w_1 \in \mathbb{R}$ are defined by

$$W_z - \mathbb{E}(\Psi_k^T W_z \Psi_k) = \hat{Q} \tag{6.18a}$$

$$w_{z1}^T\Big(I - \mathbb{E}(\Psi^{(0)})\Big) = \mathbb{E}\big(w_k^T \bar{D}^T W_z \Psi_k\big) \tag{6.18b}$$

$$w_1 = -\mathrm{tr}(\Theta W_x) \tag{6.18c}$$

with $W_x = \begin{bmatrix} I_{n_x} & 0 \end{bmatrix} W_z \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix}$ and $\hat{Q} = \begin{bmatrix} Q + K^T RK & K^T RE \\ E^T RK & E^T RE \end{bmatrix}$.

Proof Let $V_{i|k} \doteq \|z_{i|k}\|^2_{W_z} + 2w_{z1}^T z_{i|k} + w_1$ for all $i \geq 0$. Then, since $z_{i|k}$ is by assumption independent of (Ψ_{k+i}, w_{k+i}), (6.16) implies

$$\begin{aligned} \mathbb{E}_k(V_{i|k}) - \mathbb{E}_k(V_{i+1|k}) = {} & \mathbb{E}_k\Big(z_{i|k}^T\big(W_z - \mathbb{E}(\Psi_{k+i}^T W_z \Psi_{k+i})\big)z_{i|k}\Big) \\ & + 2\Big(w_{z1}^T\big(I - \mathbb{E}(\Psi_{k+i})\big) - \mathbb{E}\big(w_{k+i}^T \bar{D}^T W_z \Psi_{k+i}\big)\Big)\mathbb{E}_k(z_{i|k}) \\ & - \mathbb{E}\left(w_{k+i}^T D^T W_x D w_{k+i}\right). \end{aligned}$$

From (6.18a, 6.18b), the sum of the first two terms on the RHS of this equation is equal to $\mathbb{E}_k(z_{i|k}^T \hat{Q} z_{i|k})$. Furthermore the last term is equal to $-l_{ss}$ since post-multiplying (6.10) by W_x and extracting the trace gives

$$\begin{aligned} \mathrm{tr}\Big(\Theta W_x - \mathbb{E}(\Phi_k \Theta \Phi_k^T) W_x\Big) &= \mathrm{tr}\Big(\Theta\big(W_x - \mathbb{E}(\Phi_k^T W_x \Phi_k)\big)\Big) \\ &= \mathrm{tr}\big(D\mathbb{E}(w_k w_k^T) D^T W_x\big) = \mathbb{E}(w_k^T D^T W_x D w_k), \end{aligned}$$

and hence, noting that (6.18a) implies $W_x - \mathbb{E}(\Phi_k^T W_x \Phi_k) = Q + K^T RK$, we have

$$\mathbb{E}(w_k^T D^T W_x D w_k) = \mathrm{tr}\big(\Theta(Q + K^T RK)\big) = l_{ss}.$$

Therefore $\mathbb{E}_k(V_{i|k}) - \mathbb{E}_k(V_{i+1|k}) = \mathbb{E}_k(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) - l_{ss}$, and by summing both sides of this equation over $i = 0, 1, \ldots$, we can conclude that

$$V_{0|k} - \lim_{i\to\infty} \mathbb{E}_k(V_{i|k}) = \sum_{i=0}^{\infty} \mathbb{E}_k\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}\big). \tag{6.19}$$

Finally, we note that (6.18c) ensures that $\lim_{i\to\infty} \mathbb{E}_k(V_{i|k}) = 0$, since the definition of $V_{i|k}$ implies

$$\mathbb{E}_k(V_{i|k}) = \mathbb{E}_k(z_{i|k}^T W_z z_{i|k}) + 2w_{z1}^T \mathbb{E}_k(z_{i|k}) - \operatorname{tr}(\Theta W_x)$$

where, by Lemma 6.1, $\lim_{i\to\infty} \mathbb{E}_k(z_{i|k}) = 0$ and $\lim_{i\to\infty} \mathbb{E}_k(z_{i|k}^T W_z z_{i|k}) = \operatorname{tr}(\Theta W_x)$. From (6.19) it then follows that $V_{0|k} = J(x_k, \mathbf{c}_k)$. □

In the absence of constraints, the expression for the predicted cost provided by Theorem 6.1 allows the optimal value of $\mathbf{c}_k$ to be determined analytically for any horizon N. Using a similar argument to that of Sect. 2.7.2, it is possible to deduce from this the unconstrained optimal value of the linear feedback gain K. First, we partition W_z and w_{z1} into blocks conformal with the dimensions of x_k and $\mathbf{c}_k$:

$$W_z = \begin{bmatrix} W_x & W_{xc} \\ W_{cx} & W_c \end{bmatrix}, \quad w_{z1} = \begin{bmatrix} w_{x1} \\ w_{c1} \end{bmatrix}.$$

Next note that mean-square stability of the dynamics $x_{k+1} = \Phi_k x_k$ in Assumption 6.1 implies that $z_{k+1} = \Psi_k z_k$ is also mean-square stable. From (6.18a), it follows that $W_z \succ 0$, and this in turn implies that W_c is positive definite. Therefore the $\mathbf{c}_k$ that achieves the minimum of (6.17) in the absence of constraints is given by

$$\arg\min_{\mathbf{c}_k} J(x_k, \mathbf{c}_k) = -W_c^{-1} W_{cx} x_k - W_c^{-1} w_{c1}. \tag{6.20}$$

The constant term $-W_c^{-1} w_{c1}$ appearing in this expression indicates that the optimal control law is in general an affine rather than a linear function of the model state. In fact, w_{c1} can be determined from (6.18b) using

$$w_{c1}^T = \Big(\mathbb{E}(w_k^T D^T W_x B_k) + w_{x1}^T B^{(0)}\Big)\big[I_{n_u} \ \cdots \ I_{n_u}\big] \tag{6.21a}$$

$$w_{x1}^T = \mathbb{E}(w_k^T D^T W_x \Phi_k)(I - \Phi^{(0)})^{-1}. \tag{6.21b}$$

These expressions imply that the second term on the RHS of (6.20) applies a constant perturbation to the predicted control law (6.14), which is independent of the prediction time step and independent of the horizon N.

From (6.21a, 6.21b), it can be seen that w_{c1} is non-zero unless the additive disturbance term w_k and the multiplicative uncertainty in the model parameters A_k, B_k are statistically uncorrelated. However, even in the more general case in which the optimal control law for the cost (6.13) is affine rather than linear-state feedback, it is still possible to determine the unconstrained optimal linear feedback gain since this corresponds to the case in which the minimizing argument of (6.17) is independent of x_k. From (6.20), this requires $W_{cx} = 0$, and from (6.18a) we therefore require that K satisfies

$$K = -\big(R + \mathbb{E}(B_k^T W_x B_k)\big)^{-1} \mathbb{E}(B_k^T W_x A_k) \tag{6.22}$$

where W_x is the solution of

$$W_x - \mathbb{E}\big((A_k + B_k K)^T W_x (A_k + B_k K)\big) = Q + K^T R K. \tag{6.23}$$

The corresponding solution for W_c can be determined from (6.18a) as

$$W_c = \operatorname{diag}\big\{R + \mathbb{E}(B_k^T W_x B_k), \ldots, R + \mathbb{E}(B_k^T W_x B_k)\big\}. \tag{6.24}$$

The preceding results concerning the unconstrained optimal linear feedback gain and the predicted cost are summarized as follows.

Corollary 6.1 *The linear feedback law that minimizes the performance index (6.13) for the dynamics of (6.2) and (6.3) is $u_k = Kx_k$ where K is given by (6.22) and (6.23). If the predicted control sequence (6.14) is defined in terms of this gain K, then the predicted cost of (6.15) is given by*

$$J(x_k, \mathbf{c}_k) = x_k^T W_x x_k + \mathbf{c}_k^T W_c \mathbf{c}_k + 2w_{x1}^T x_k + 2w_{c1}^T \mathbf{c}_k + w_1$$

where W_x, W_c and w_{x1}, w_{c1} are given by (6.23), (6.24) and (6.21a, 6.21b).

We conclude this section by considering how to compute K and W_x satisfying (6.22) and (6.23). It is possible to derive a set of algebraic conditions on W_x by using (6.22) to eliminate K from (6.23) and evaluating expectations using the parameterization of model uncertainty in (6.3a, 6.3b). However, the resulting algebraic Riccati equation is nonlinear and multivariate, and a computationally more convenient approach is to consider (W_x, K) as an extremal point of the feasible set of a particular LMI, as we now briefly discuss. For any given mean-square stabilizing $\tilde{K}$, let $W_x' \succ 0$ satisfy the equality

$$W_x' - \mathbb{E}\big((A_k + B_k \tilde{K})^T W_x' (A_k + B_k \tilde{K})\big) = Q + \tilde{K}^T R \tilde{K}$$

and let $\tilde{W}_x \succ 0$ satisfy the inequality

$$\tilde{W}_x - \mathbb{E}\big((A_k + B_k \tilde{K})^T \tilde{W}_x (A_k + B_k \tilde{K})\big) \succeq Q + \tilde{K}^T R \tilde{K} \tag{6.25}$$

Then, subtracting (6.23) we have

$$(\tilde{W}_x - W'_x) - \mathbb{E}\big((A_k + B_k\tilde{K})^T(\tilde{W}_x - W'_x)(A_k + B_k\tilde{K})\big) \succeq 0,$$

from which it follows that $\tilde{W}_x \succeq W'_x$ since $A_k + B_k\tilde{K}$ is mean-square stable by assumption. But $\tilde{W}_x \succeq W'_x$ implies $\mathrm{tr}(\tilde{W}_x) \geq \mathrm{tr}(W'_x)$ and W'_x is therefore equal to the solution of the problem of minimizing $\mathrm{tr}(\tilde{W}_x)$ over $\tilde{W}_x$ subject to (6.25) for a given fixed value of $\tilde{K}$. Since K in (6.22) and (6.23) is the feedback gain that minimizes the cost (6.15), the value of $\mathrm{tr}(W_x)$ satisfying (6.23) is equal to the minimum of $\mathrm{tr}(\tilde{W}_x)$ over the set of all $\tilde{W}_x$ satisfying (6.25) for variable $\tilde{K}$. In other words, the pair (W_x, K) satisfying (6.22) and (6.23) is the optimal argument of the problem

$$\underset{\tilde{W}_x,\tilde{K}}{\mathrm{minimize}} \quad \mathrm{tr}(\tilde{W}_x) \quad \text{subject to (6.25).}$$

This problem is nonconvex, but introducing transformed variables $S = \tilde{W}_x^{-1}$ and $Y = \tilde{K}\tilde{W}_x^{-1}$, (6.25) can be rewritten as

$$S - \mathbb{E}\big((A_kS + B_kY)^T S^{-1}(A_kS + B_kY)\big) \succeq SQS + Y^TRY.$$

Using the model parameterization (6.3a, 6.3b), this inequality can be expressed as the following LMI in $S \succ 0$, and Y:

$$\begin{bmatrix} S & (A^{(0)}S + B^{(0)}Y)^T & \cdots & (A^{(m)}S + B^{(m)}Y)^T & \begin{bmatrix} SQ^{1/2} & Y^TR^{1/2}\end{bmatrix} \\ \star & S & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \cdots & S & 0 \\ \star & \star & \cdots & \star & I \end{bmatrix} \succeq 0 \tag{6.26}$$

Thus W_x and K satisfying (6.22) and (6.23) can be computed by solving the semi-definite program:

$$(W_x, S, Y) = \arg\min_{W_x,S,Y} \quad \mathrm{tr}(W_x) \quad \text{subject to (6.26) and} \begin{bmatrix} W_x & I \\ I & S \end{bmatrix} \succeq 0$$

and setting $K = YW_x$.

6.3 Mean-Variance Predicted Cost

The cost considered in Sect. 6.2 gives a measure of the second moments of predicted states and control inputs. Thus (6.15) can be written for $\kappa = 1$ as

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \bigl(\|x_{i|k}^{(0)}\|_Q^2 + \|u_{i|k}^{(0)}\|_R^2\bigr)$$
$$+ \kappa^2 \sum_{i=0}^{\infty} \mathbb{E}_k \bigl(\|x_{i|k} - x_{i|k}^{(0)}\|_Q^2 + \|u_{i|k} - u_{i|k}^{(0)}\|_R^2 - l_{ss}\bigr) \tag{6.27}$$

where $x_{i|k}^{(0)} = \mathbb{E}(x_{i|k})$ and $u_{i|k}^{(0)} = \mathbb{E}(u_{i|k})$ denote the nominal values of the predicted states and control inputs and $l_{ss} = \operatorname{tr}\bigl(\Theta(Q + K^T RK)\bigr)$. These nominal sequences are governed by the nominal prediction dynamics:

$$x_{i+1|k}^{(0)} = A^{(0)} x_{i|k}^{(0)} + B^{(0)} u_{i|k}^{(0)}. \tag{6.28}$$

Expressed this way, the cost $J(x_k, \mathbf{c}_k)$ can be seen to evaluate a particular linear mix of two cost indices: one based on the mean and the other on the variance of predicted states and inputs.

In applications such as the sustainable development problem considered in Sect. 6.5 and those involving, for example, portfolio selection [19, 20], a typical control objective is to minimize a cost based on probabilistic bounds within which the future predicted states and control inputs will lie. In this case different affine mixes, corresponding to $\kappa \neq 1$ in (6.27), may be appropriate. For example, the cost proposed in [21] is concerned with minimizing the width of the probabilistic band defined by

$$\begin{aligned} \Pr_k(y_{i|k} \leq \bar{y}_{k+i}) &\geq p \\ \Pr_k(y_{i|k} \geq \underline{y}_{k+i}) &\geq p \end{aligned} \tag{6.29}$$

Then, under the assumption that $y_{i|k}$ is normally distributed, an appropriate stage cost would be

$$\frac{1}{2}(\underline{y}_{k+i}^2 + \bar{y}_{k+i}^2) = (y_{i|k}^{(0)})^2 + \kappa^2 \mathbb{E}_k \bigl((y_{i|k} - y_{i|k}^{(0)})^2\bigr) \tag{6.30}$$

where κ is the argument of the standard cumulative normal distribution function corresponding to probability p, i.e. $\Pr(X \leq \kappa) = p$ for a normally distributed random variable X with zero mean and unit variance.

For such applications, the cost of (6.27) with $\kappa^2 \neq 1$ is more appropriate and results in a predictive control strategy known as mean-variance SMPC [22]. Note that this cost can also be written as

$$J(x_k, \mathbf{c}_k) = (1 - \kappa^2) \sum_{i=0}^{\infty} \bigl(\|x_{i|k}^{(0)}\|_Q^2 + \|u_{i|k}^{(0)}\|_R^2\bigr)$$
$$+ \kappa^2 \sum_{i=0}^{\infty} \mathbb{E}_k \bigl(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}\bigr). \tag{6.31}$$

From this expression it can be seen that $J(x_k, \mathbf{c}_k)$ reduces to the expected quadratic cost of (6.15) for $\kappa = 1$, gives the nominal cost for $\kappa = 0$ and tends to the minimum variance cost as $\kappa \to \infty$.

The cost of (6.31), when evaluated for the model (6.2) and (6.3a, 6.3b) with the predicted control law (6.14), is a quadratic function of the degrees of freedom, $\mathbf{c}_k$. Using Theorem 2.10 to compute the nominal cost in the first term on the RHS of (6.31) and using the conditions of Theorem 6.1 to compute the second term, this function is given by

$$J(x_k, \mathbf{c}_k) = \begin{bmatrix} z_k \\ 1 \end{bmatrix}^T \begin{bmatrix} W_z & w_{z1} \\ w_{z1}^T & w_1 \end{bmatrix} \begin{bmatrix} z_k \\ 1 \end{bmatrix}$$

where $z_k = (x_k, \mathbf{c}_k)$ and W_z, w_{z1}, w_1 are defined by

$$W_z = (1 - \kappa^2)\bar{W}_z + \kappa^2 \hat{W}_z \qquad \begin{cases} \bar{W}_z - \Psi^{(0)\,T} \bar{W}_z \Psi^{(0)} = \hat{Q} \\ \hat{W}_z - \mathbb{E}(\Psi_k^T \hat{W}_z \Psi_k) = \hat{Q} \end{cases} \tag{6.32a}$$

and

$$w_{z1}^T (I - \Psi^{(0)}) = \mathbb{E}\Big(w_k^T \begin{bmatrix} D^T & 0 \end{bmatrix} \hat{W}_z \Psi_k\Big) \tag{6.32b}$$

$$w_1 = -\mathrm{tr}(\Theta \hat{W}_x), \tag{6.32c}$$

with $\hat{W}_x$ and $\Psi^{(0)}$ defined as

$$\hat{W}_x = \begin{bmatrix} I_{n_x} & 0 \end{bmatrix} \hat{W}_z \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix}, \qquad \Psi^{(0)} = \begin{bmatrix} \Phi^{(0)} & B^{(0)} E \\ 0 & M \end{bmatrix}.$$

The unconstrained optimal linear feedback gain can be determined by an analysis similar to that of Sect. 6.2, but in this case the gain K is given by the solution of a pair of coupled algebraic Riccati equations:

$$\bar{W}_x - (A^{(0)} + B^{(0)} K)^T \bar{W}_x (A^{(0)} + B^{(0)} K) = Q + K^T R K \tag{6.33a}$$

$$\hat{W}_x - \mathbb{E}\big((A_k + B_k K)^T \hat{W}_x (A_k + B_k K)\big) = Q + K^T R K. \tag{6.33b}$$

with

$$K = -\big(R + \mathbb{E}(B_k^T \hat{W}_x B_k)\big)^{-1} \big((1 - \kappa^2) B^{(0)\,T} \bar{W}_x A^{(0)} + \kappa^2 \mathbb{E}(B_k^T \hat{W}_x A_k)\big). \tag{6.33c}$$

The details of the derivation of this expression and an iterative method of computing the solution K can be found in [23].

6.4 Early Stochastic MPC Algorithms

Optimization methods for problems involving probabilistic constraints have been used in applications of Operations Research since the 1950s (see e.g. [24]). In the context of predictive control, one of the first proposals to use dynamic models containing stochastic uncertainty was reported in [25], where systems with constant but unknown parameters were considered. In this strategy, unknown model parameters are identified online and used to update a control law, and for this reason it is known as self-tuning control. The aim of the approach, however, is to minimize the expected value of a predicted cost that is computed using a dynamic model, and the optimal predicted control input is implemented as a receding horizon control law. Therefore self-tuning control can be viewed as a type of stochastic MPC strategy. This section briefly reviews self-tuning control strategies before considering the formulation of probabilistic constraints for moving average models.

6.4.1 Auto-Regressive Moving Average Models

In its simplest form, self-tuning control is based on a single-input single-output (SISO) ARMA model, as defined in (6.1). In [25], the model parameters a_i, b_i are assumed to be unknown constants and the additive noise process, $\{e_0, e_1, \ldots\}$, is assumed to be an independent and identically distributed (i.i.d.) sequence in which e_k is normally distributed with zero mean. In compact form, using z-transform operators, the system dynamics may be written as

$$A(z)y_k = B(z)u_{k-d} + C(z)e_k \tag{6.34}$$

where

$$\begin{aligned}A(z) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n}\\ B(z) &= b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}\\ C(z) &= 1 + c_1 z^{-1} + \cdots + c_n z^{-n}\end{aligned}$$

with z^{-1} representing the backward shift operator. The system is assumed to be unconstrained, and hence no constraints act on y_k and u_k.

The predicted cost is taken to be

$$J(x_k, \mathbf{u}_k) = \mathbb{E}_k(y_{d|k}^2) \tag{6.35}$$

where $\mathbf{u}_k = (u_{0|k}, \ldots, u_{d-1|k})$ and $y_{i|k}$ is the predicted value of the output y_{k+i} at time k. Because of this objective the strategy is sometimes described as a minimum variance (MV) control law. We note that on account of the delay d of the model (6.34), the d steps-ahead output, $y_{d|k}$, is the first output variable that can be influenced by the control input $u_{0|k}$.

In the absence of constraints, the minimization of the cost (6.35) can be performed analytically. To see this, define polynomials $F(z)$ and $G(z)$ by

$$z^d C(z) = A(z)F(z) + G(z), \tag{6.36}$$

where

$$F(z) = z^d + f_1 z^{d-1} + \cdots + f_d$$
$$G(z) = g_1 z^{-1} + g_2 z^{-2} + \cdots + g_n z^{-n}.$$

This identity allows the dependence of $y_{d|k}$ on the additive disturbance sequence to be split into terms that depend on the values of $e_{k+d}, \ldots, e_k$, which are unknown at time k, and terms involving only the past values $e_{k-1}, e_{k-2}, \ldots$, which have already been realized at time k and can therefore be determined from the observed values of the control input and system output. In particular, from the model (6.34) and the identity (6.36) we obtain

$$\begin{aligned} y_{k+d} &= \frac{B(z)}{A(z)} u_k + \frac{z^d C(z)}{A(z)} e_k \\ &= \left[\frac{B(z)}{A(z)} u_k + \frac{G(z)}{A(z)} e_k\right] + F(z) e_k \end{aligned} \tag{6.37}$$

where the square-bracketed quantity on the right-hand side of (6.37) is known at time k, whereas the last term depends on the unknown noise sequence $e_{k+d}, \ldots, e_k$, which is a random variable at time k. Given the i.i.d. and zero-mean assumptions on e_k, it follows that the control law that minimizes the cost of (6.35) necessarily satisfies

$$\frac{B(z)}{A(z)} u_k + \frac{G(z)}{A(z)} e_k = 0. \tag{6.38}$$

The dependence of e_k on present and past outputs and past inputs can be deduced from (6.34), which implies

$$\begin{aligned} e_k &= \frac{A(z)}{C(z)} y_k - \frac{B(z)}{C(z)} u_{k-d} \\ &= \frac{A(z)}{C(z)} y_k - \frac{B(z)}{z^d C(z)} u_k. \end{aligned}$$

In conjunction with (6.38), this results in the condition

$$z^d C(z)B(z)u_k - G(z)B(z)u_k + z^d G(z)A(z)y_k = 0. \tag{6.39}$$

Using the identity (6.36) to replace $z^d C(z)$ by $A(z)F(z) + G(z)$ then leads to the optimal control solution

$$u_k = -\frac{z^d G(z)}{B(z)F(z)} y_k. \tag{6.40}$$

Substituting the control law (6.40) into the system model (6.34) gives the closed-loop characteristic equation

$$\frac{A(z)F(z) + G(z)}{F(z)} y_k = \frac{C(z)}{F(z)} y_{k+d} = 0, \tag{6.41}$$

which therefore identifies the roots of $C(z)$ as the closed-loop poles of the prediction dynamics. Hence a necessary condition for this strategy to stabilize the system (6.34) is that every root of $C(z)$ should lie within the unit circle centred at the origin. In addition, by considering (6.40) it is easy to show that, for internal stability, the roots of $B(z)$ must also lie inside the unit circle centred at the origin. Therefore this strategy is restricted to minimum phase plants.

The restriction to minimum phase systems was subsequently removed by the generalized minimum variance (GMV) strategy [26], which modifies the predicted cost so as to include a term that penalizes control activity. However, neither the MV nor the GMV control strategy can provide an *a priori* guarantee of stability for the case of unknown model parameters (although stability can be checked *a posteriori*, namely after the specification of the problem parameters and the derivation of the optimal control law). Moreover, these approaches do not take into account constraints on outputs and control inputs.

Constraints in stochastic MPC in the context of ARMA models were not introduced until much later. For example, [27] considered SISO systems described by the model (6.34), but removed the assumption of independent additive disturbances and imposed hard and probabilistic constraints of the form

$$\underline{u} \le u_k \le \bar{u} \tag{6.42}$$

$$\Pr_k(\underline{y}_{k+i} \le y_{i|k} \le \bar{y}_{k+i}) \ge p. \tag{6.43}$$

Without the assumption of i.i.d. disturbances, the minimization subject to these constraints of the predicted cost, which in [27] was defined so as to penalize control increments

$$J(\mathbf{u}_k) = (u_{0|k} - u_{k-1})^2 + \sum_{i=1}^{N-1} (u_{i|k} - u_{i-1|k})^2, \tag{6.44}$$

becomes a computationally demanding problem that has to be solved online, and for which there is in general no guarantee of convergence to the global optimum. In addition, the use of a finite horizon cost precludes the possibility of providing *a priori* stability guarantees. Furthermore the presence in the system model of random variables with distributions that are not finitely supported makes it impossible to ensure the recursive feasibility of this approach.

6.4.2 Moving Average Models

Constraints of the form (6.7) and (6.9) present two major difficulties for stochastic MPC algorithms: how to impose these constraints on predicted control sequences given the distribution of model uncertainty, and how to ensure that the optimization problem to be solved online is recursively feasible. Both of these difficulties are encountered with the ARMA model (6.34) with stochastic coefficients (and likewise with the state-space model of (6.2) and (6.3a, 6.3b)) since the future evolution of the model state and output trajectories depends on earlier realizations of the model uncertainty. In particular, if the uncertainty does not have bounded support, then clearly it is not possible in general to ensure that constraints will be satisfied at future time instants.

However it is relatively straightforward to obtain a guarantee of recursive feasibility for a stochastic MPC strategy based on a moving average (MA) model. This form of model arises when the system dynamics are described by a discrete-time finite impulse response model of the form

$$y_k = \sum_{j=1}^{n} H_j u_{k-j} + d_k \tag{6.45}$$

where $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ are the control input and system output, $d \in \mathbb{R}^{n_y}$ is a stochastic additive disturbance, and $H_1, \ldots, H_n$ are the elements of the uncertain system impulse response after truncation to n terms. The probability distributions of d and H_j, $j = 1, \ldots, n$ are assumed to be known.

Knowledge of the distributions of uncertain model parameters makes it possible to transform probabilistic constraints into deterministic constraints on predicted input sequences. For the case of the model (6.45) with normally distributed model parameters, this is easy to do for the constraints of (6.8) because the output y_k in (6.45) depends linearly on the model parameters, and hence $y_{i|k}$ is also a normally distributed random variable. In this case, it is possible to invoke the chance-constrained optimization framework [24, 28] to convert probabilistic system constraints of the form of (6.8) into second-order cone (SOC) constraints as was done for example in [3].

Thus let $y_{i|k,l}$ denote the lth element of the predicted value at time k of the i steps-ahead output vector y_{k+i}. Then from (6.45), it follows that

$$y_{i|k,l} = h_l^T \begin{bmatrix} u_i^f \\ u_i^p \end{bmatrix} + d_{i,l}$$

where $u_i^f = (u_{i-1|k}, \ldots, u_{0|k})$ and $u_i^p = (u_{k-1}, \ldots, u_{k+i-n})$ denote, respectively, the predicted future and the past components of the input sequence, also $h_l^T = e_l^T \left[H_1 \ \ldots \ H_n\right]$, with e_l denoting the lth column of the identity matrix, and $d_{i,l}$ is the lth element of d_{k+i}. Then, under the assumption that $(h_l, d_{i,l})$ is normally distributed with mean $(\bar{h}, 0)$ and covariance matrix Σ, the probabilistic constraint

$$\Pr_k(y_{i|k,l} \le a_l) \ge p \tag{6.46}$$

can be written as a deterministic second-order cone constraint of the form

$$a_l - \bar{h}^T \begin{bmatrix} u_i^f \\ u_i^p \end{bmatrix} \ge \kappa \left\| (u_i^f, u_i^p, 1) \right\|_{\Sigma}. \tag{6.47}$$

The constant κ in (6.47) is defined by

$$\Pr(X \le \kappa) = p,$$

for a normally distributed scalar random variable X with mean 0 and variance 1. For p in the interval $[0.5, 1)$, the value of κ is nonnegative and, since the condition $a^T x - b \ge \kappa \|c + Dx\|$ is convex in x if $\kappa \ge 0$ for any given constants a, b, c, D, the constraint (6.47) is therefore convex in this case. Hence, for $p \ge 0.5$, the minimization of a convex quadratic predicted cost subject to (6.47) is a second-order cone programming problem (SOCP) that can be solved efficiently.

This is the approach used in [3], where a finite horizon cost is used to formulate a SMPC algorithm. However this formulation lacked a guarantee of closed-loop stability (on account of the finite horizon cost). In the presence of hard input constraints in addition to probabilistic output constraints, feasibility may also become problematic and requires constraint softening this context (see for example [29]).

A further disadvantage of the use of MA models is that they are limited to open-loop stable systems; the impulse responses of open-loop unstable systems cannot be truncated to give finite-order MA models. These aspects of closed-loop stability and feasibility, as well as the extension of the use of MA to the case of open-loop unstable systems have been addressed in [1, 2], where stochastic MPC is introduced as a tool for the assessment of sustainable development policies.

6.5 Application to a Sustainable Development Problem

Sustainable development addresses the problem of balancing the needs for economic, technological and industrial development of the current generation with those of future generations [30]. Human generations can be considered to be separated by about 30 years, and this defines a natural prediction horizon for assessing the likely effects of policy decisions on sustainable development. The treatment of the problem has been mostly discursive but it can be posed in a formal mathematical manner. Thus [31] considers the question of assessing policy in budget allocation between alternative technologies for electricity generation and develops a strongly stochastic model. This model describes the effect of adjusting a number of inputs ("instruments"), which form the elements of an input vector $u \in \mathbb{R}^{n_u}$, and which include for example measures of investment in combined cycle gas turbine technology and investment in renewable energy such as wind turbines, on a number of outputs ("indicators"). Amongst these indicators is included a measure of *benefit*, say y_1, related to the cost of the energy produced, and another, y_2 that measures *cost*, related to accumulated CO_2 emissions. These are measured at the end of the 30-year horizon and are therefore strongly stochastic on account of the vagaries of world economy. Clearly, it is desirable to maximize benefit while respecting constraints on cost, but since both y_1 and y_2 are random variables, a more appropriate stochastic optimization problem is given by

$$\begin{aligned} \underset{u, A_1}{\text{maximize}} \quad & A_1 \\ \text{subject to} \quad & \Pr(y_1 \geq A_1) \geq p_1 \\ & \Pr(y_2 \leq A_2) \geq p_2 \\ & \mathbf{1}^T u \leq b, \quad u \geq 0 \end{aligned} \tag{6.48}$$

which is given in terms of target bounds, A_1 and A_2, rather than directly in terms of the variables y_1, y_2 which, as stated earlier, are stochastic. The hard constraint in (6.48), that the sum of the inputs should not exceed a specified value b, expresses budgetary limitations.

It possible to introduce (6.48) into a SMPC framework by performing the optimization repetitively in a receding horizon manner, for example at the beginning of the kth year, for $k = 0, 1, \ldots$. Furthermore, rather than consider a single input adjustment and its effect on the output variables at the end of the prediction horizon, one can allow input adjustments to occur at each step of the prediction horizon and measure their effect on the output variables, again, over the entire horizon. This introduces an explicit time dependence into the input and output variables and requires the use of dynamic models. Accordingly, we denote the vector of budget adjustments and the vector of indicators in year k as u_k and $y_k \doteq (y_{k,1}, y_{k,2})$ respectively.

In [1], MA models of the form of (6.45) are used to describe the input–output dependence. Such models are convenient for the derivation of the parameter distributions from identification of experiments (performed using world economy models) since they avoid the difficulty of ARMA models in which stochastic model parameters multiply disturbance values which are also random variables. The limitation of MA models to open-loop stable systems can be overcome through the artifice of bicausality, according to which causal relationships such as given in (6.45) can be used to account for the stable dynamics, whereas unstable dynamics can be modelled using anti-causal regressions. As an illustration of this, consider a system described by a transfer function with a single pole which is unstable, namely $1/(1 - \lambda z^{-1})$

where $|\lambda| > 1$. This transfer function can also be written as $-\lambda^{-1}z/(1-\lambda^{-1}z)$ and, after suitable truncation, leads to the anti-causal regression

$$y_k = -\frac{1}{\lambda}\Big(u_k + \frac{1}{\lambda}u_{k+1} + \cdots + \Big(\frac{1}{\lambda}\Big)^{n-1} u_{k+n-1}\Big) \tag{6.49}$$

Superposition can be deployed to extend this treatment to the case of complex conjugate unstable poles and/or multiple poles.

The overall stochastic MPC strategy can be split into two phases, of which the first defines a suitable setpoint, and the second is concerned with tracking this setpoint. In the interests of maximizing the value of the objective steady state, Phase 1 considers a predicted input trajectory which assumes a constant steady-state value, u_{ss}, after N_u time steps. If N_u satisfies $N_u + n - 1 < N = 30$, then from (6.45) the outputs necessarily reach their steady-state values, $y_{ss,1}$ and $y_{ss,2}$, within the 30-year horizon. Hence, to maximize benefit, the setpoints for u and y_1 can be defined as

$$r \doteq \mathbb{E}(h_1^T)\mathbf{u}_{ss}$$

where $\mathbf{u}_{ss} = (u_{ss}, \ldots, u_{ss})$ and u_{ss} is defined, for given values of the probabilities $p_1, p_2 \in [0.5, 1)$, the threshold A_2 and the overall budget B for an N-step horizon, as the solution of the convex optimization:

$$\begin{aligned} \underset{u_{ss}, A_1}{\text{maximize}} \quad & A_1 \\ \text{subject to} \quad & \Pr(y_{ss,1} \geq A_1) \geq p_1 \\ & \Pr(y_{ss,2} \leq A_2) \geq p_2 \\ & \sum_{j=0}^{N-1} \rho^j \mathbf{1}^T u_{ss} \leq B, \quad u_{ss} \geq 0. \end{aligned} \tag{6.50}$$

The parameter ρ is a discounting factor chosen to lie in the interval $[0, 1)$, which is used in recognition of the fact that the value of expenditure decreases with the advance of time. The implication of the optimization (6.50) that defines this first phase of the algorithm is that the undershoot in the predicted steady-state benefit, $r - y_{ss,1}$, will with probability p_1 be no greater the threshold value $t_{ss} \doteq r - A_1$.

It is now possible to measure performance over the N-step prediction horizon using probabilistic thresholds:

$$\Pr_k(r - y_{i|k,1} \leq t_{i|k}) \geq p_1, \quad i = 1, \ldots, N.$$

The second phase of the algorithm aims at the minimization of these thresholds. However, given the stochastic nature of the problem and the desire to secure a monotonically decreasing property for the predicted cost, rather than penalize in the cost all $t_{i|k}$, only those that exceed t_{ss} will be taken into account. This suggests the following online implementation of stochastic MPC.

Algorithm 6.1 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{\substack{u_{0|k},\ldots,u_{N_u-1|k},u_{ss}\\ s_{i|k},\, t_{i|k},\, i=1,\ldots,N}}{\text{minimize}} \quad \sum_{i=1}^{N} s_{i|k}^2 \tag{6.51a}$$

subject to

$$s_{i|k} \geq t_{i|k} - t_{ss}, \quad s_{i|k} \geq 0 \tag{6.51b}$$

$$\Pr_k(r - y_{i|k,1} \geq t_{i|k}) \geq p_1 \tag{6.51c}$$

$$\Pr_k(y_{i|k,2} \leq A_2) \geq p_2 \tag{6.51d}$$

$$\sum_{j=l}^{N_u-1} \rho^{j-l}\mathbf{1}^T u_{j|k} + \Big(\frac{\rho^{N_u-l} - \rho^N}{1-\rho}\Big)\mathbf{1}^T u_{ss} \leq B \tag{6.51e}$$

$$u_{l|k} \geq 0, \quad u_{ss} \geq 0 \tag{6.51f}$$

for $i = 1, \ldots, N$ and $l = 0, \ldots, N_u - 1$.

(ii) Apply the control law $u_k = u^*_{0|k}$, where $(u^*_{0|k}, \ldots, u^*_{N_u-1|k}, u^*_{ss})$ is the minimizing input sequence in (6.51). ◁

For $l = 0$, (6.51e) applies the discounted budgetary constraint

$$\sum_{j=0}^{N-1} \rho^j \mathbf{1}^T u_{j|k} \leq B$$

to the predicted input sequence $(u_{0|k}, u_{1|k}, \ldots)$ with $u_{j|k} = u_{ss}$ for $j \geq N_u$. However (6.51e) is also applied for each $l = 1, \ldots, N_u - 1$ to ensure that the time-shifted input sequences, $(u_{l|k}, u_{l+1|k}, \ldots)$, with $u_{j|k} = u_{ss}$ for $j \geq N_u$, satisfy the corresponding discounted budgetary constraint:

$$\sum_{j=l}^{N-1+l} \rho^{j-l}\mathbf{1}^T u_{j|k} \leq B.$$

Given that the constraints of (6.50) are satisfied for the given t_{ss}, A_2 and B for some u_{ss}, this is all that is needed to ensure recursive feasibility of Algorithm 6.1. In addition, if $s^*_{i|k}$, $i = 1, \ldots, N$ is optimal for (6.51) at time k, then the sequence defined by $s_{i|k+1} = s^*_{i+1|k}$ for $i = 1, \ldots, N-1$ and $s_{N|k+1} = 0$ is by construction feasible at time $k+1$. Therefore the optimal objective of (6.51), denoted $J^*_k \doteq \sum_{i=1}^N s^{*\,2}_{i|k}$, necessarily satisfies

$$J^*_{k+1} \leq J^*_k - s^{*\,2}_{1|k} \tag{6.52}$$

for all k. Summing both sides of this inequality over all $k \geq 0$ gives

$$\sum_{k=0}^{\infty} s_{1|k}^{*\,2} \leq J_0^*$$

which implies that $s_{1|k}^* \rightarrow 0$ as $k \rightarrow \infty$. These results are summarized below.

Theorem 6.2 *For the closed-loop system (6.45) under the control law of Algorithm 6.1, the optimization (6.51) is feasible at all times $k = 0, 1, \ldots$ and $r - y_{1|k,1} \leq t_{ss}$ with probability p_1 as $k \rightarrow \infty$.*

Assuming the parameters of the model (6.45) to be normally distributed, the probabilistic constraints in (6.50), as well as those of (6.51c, 6.51d), can be converted into second-order cone constraints using the method by which (6.46) was transformed into the deterministic constraint (6.47). Therefore, the Phase 1 optimization (6.50) is convex and can be performed for example by solving a sequence of SOCPs. Similarly, the online optimization of Algorithm 6.2 can be expressed as a single SOCP problem.

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Chapter 7
Feasibility, Stability, Convergence and Markov Chains

This chapter considers the closed-loop properties of stochastic MPC strategies based on the predicted costs and probabilistic constraints formulated in Chap. 6. To make the analysis of closed-loop stability and performance possible, it must first be ensured that the MPC law is well-defined at all times and the most natural way to approach this is to ensure that the associated receding horizon optimization problem remains feasible whenever it is initially feasible. We therefore begin by discussing the conditions for recursive feasibility.

The requirement for future feasibility of probabilistic constraints induces constraints on the model state that must be satisfied for all realizations of model uncertainty. Although this introduces robust constraints into the problem, we show that these constraints provide the least restrictive means of ensuring recursive feasibility, and hence they are less restrictive than the conservative robust counterparts of the probabilistic constraints. This suggests a general framework for stochastic MPC that combines robust constraints for recursive feasibility with the probabilistic or expected value constraints of the control problem.

Closed-loop stability and convergence are discussed next in the context of a prototype stochastic MPC algorithm. We present an analysis of asymptotic mean-square bounds on the closed-loop state and control trajectories that are derived from bounds on the optimal value of the predicted cost. This is the basis of the stability analysis of the various stochastic MPC strategies considered in this chapter and in Chap. 8. We also briefly discuss an interesting and seldom-used alternative based on supermartingale convergence theory, which provides additional insight into the behaviour of the closed-loop system.

The conditions for recursive feasibility of pointwise-in-time probabilistic constraints require the predicted state and control trajectories to lie in a tube that ensures robust feasibility and satisfaction of the probabilistic constraints. Similar conditions apply to the case of probabilistic constraints that are imposed jointly at more than a single future time step, and we conclude the chapter by considering a strategy for

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this case that makes use of probabilistic bounds on the uncertain model parameters. The approach is first introduced using the concept of probabilistic invariance and then extended to a more general framework using Markov chains.

Throughout this chapter, we consider systems described by the uncertain model introduced in Chap. 6:

$$x_{k+1} = A_k x_k + B_k u_k + D w_k \tag{7.1a}$$

where A_k, B_k, w_k are functions of a stochastic parameter vector q_k which is unknown at time k. Thus $(A_k, B_k, w_k) = \big(A(q_k), B(q_k), w(q_k)\big)$ with

$$\big(A(q), B(q), w(q)\big) = (A^{(0)}, B^{(0)}, 0) + \sum_{j=1}^{m} (A^{(j)}, B^{(j)}, w^{(j)}) q^{(j)} \tag{7.1b}$$

and the probability distribution of $q_k = (q_k^{(1)}, \ldots, q_k^{(m)})$ is assumed to be known and to satisfy $\mathbb{E}(q_k) = 0$ and $\mathbb{E}(q_k q_k^T) = I$. Furthermore, q_k and q_i are assumed to be independent for all $k \neq i$.

7.1 Recursive Feasibility

This section examines conditions under which the online optimization of predicted performance in stochastic MPC can be guaranteed to remain feasible at all future sampling instants if it is initially feasible. Probabilistic or expectation constraints such as (6.7)–(6.9) are usually regarded as "soft" constraints since they are not required to hold for every possible realization of model uncertainty. However, in order that a problem is feasible, we require that all conditions of the problem are met, whether these are invoked for a predefined subset or for all model uncertainty realizations. Probabilistic or expectation constraints are, in general, only feasible if the system state belongs to a particular subset of state space, and the conditions for their feasibility thus impose additional constraints on states and control inputs.

Recursive feasibility of stochastic MPC algorithms can be handled in one of two ways. Either conditions to ensure robust feasibility are imposed as explicit constraints in the online optimization or the optimization is allowed to become infeasible whenever necessary. The latter approach typically includes a penalty on constraint violation in the MPC cost index [1, 2], or else directly minimizes a measure of the distance of the state from the feasible set whenever the problem is infeasible [3]. Without a guarantee of feasibility, however, it is generally impossible to make a definite statement about the degree to which the closed-loop system under a receding horizon controller satisfies constraints. Moreover, the closed-loop system may not satisfy the constraints of the problem, even if these are feasible at initial time.

In this section, we focus on the robust feasibility of stochastic MPC optimization problems subject to pointwise in time probabilistic constraints (including, by extension, mixtures of probabilistic and robust constraints that hold with probability 1). However we note that the same principles and analogous conditions for ensuring feasibility apply to other constraint formulations such as the expectation constraints and the joint probabilistic constraints of (6.7) and (6.9). To place this discussion in a general context, we consider constraints applied to an uncertain output variable:

$$\Pr_k(z_{0|k} \leq 0) \geq p \tag{7.2}$$

where $z_{i|k}$ is the predicted value at time k of the i-steps ahead output z_{k+i}, and where z_k is a function of the state x_k and control input u_k of (7.1a):

$$z_k \doteq f(x_k, u_k, v_k). \tag{7.3}$$

Here v is a random variable that is defined in terms of a stochastic model parameter $r_k = (r_k^{(1)}, \ldots, r_k^{(m)})$, which is unknown at time k but which has a known probability distribution, and a known set of vectors $\{v^{(1)}, \ldots, v^{(m)}\}$:

$$v_k = \sum_{j=1}^{m} v^{(j)} r_k^{(j)}. \tag{7.4}$$

In this set-up, r_k is not assumed to be independent of the stochastic parameter q_k appearing in (7.1a). Therefore the state constraints $\Pr_k(Fx_{1|k} \leq \mathbf{1}) \geq p$ are a special case of (7.2) with $z_k = f(x_k, u_k, v_k) = FAx_k + FBu_k + FDv_k - \mathbf{1}$ and $v_k = w_k$. Likewise, the mixed state and input constraints of (6.8) are included in (7.2) through a change of the definition of z_k:

$$z_k = f(x_k, u_k, u_{k+1}, v_k) = FA_k x_k + FB_k u_k + Gu_{k+1} + FDv_k - \mathbf{1}$$

and $v_k = w_k$.

Example 7.1 This example uses a simple system model to motivate the derivation of recursively feasible probabilistic constraints. The state x_k, control input u_k and output z_k at time k are governed by the first order dynamics:

$$\begin{aligned} x_{k+1} &= x_k + u_k + w_k \\ z_k &= x_k + v_k \end{aligned}$$

where, at time k, x_k is known and w_k, v_k are unknown discrete random variables with

$$\Pr(w_k = j) = \Pr(v_k = j) = \tfrac{1}{3}, \quad j = -1, 0, 1.$$

The system is subject to a probabilistic constraint, which is required to hold for all $k \geq 0$:

$$\Pr_k(z_{0|k} \leq 0) \geq \tfrac{1}{2}.$$

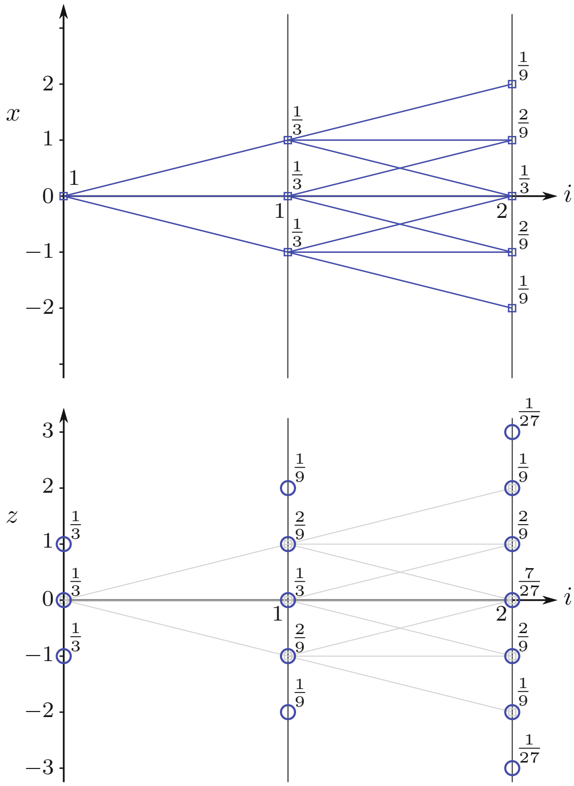

Fig. 7.1 The probability distributions of the predicted states, $\Pr_0(x_{i|0})$ (*upper plot*), and outputs, $\Pr_0(z_{i|0})$ (*lower plot*), of Example 7.1 for $i = 0, 1, 2$ with $u_{0|0} = u_{1|0} = 0$ and $x_0 = 0$

For the initial condition $x_0 = 0$, Fig. 7.1 shows the distributions of $x_{i|0}$ and $z_{i|0}$ that are obtained with the predicted input sequence $u_{0|0} = u_{1|0} = 0$. From this it can be seen that

$$\Pr_0(z_{0|0} \le 0) = \tfrac{2}{3}, \quad \Pr_0(z_{1|0} \le 0) = \tfrac{2}{3}, \quad \Pr_0(z_{2|0} \le 0) = \tfrac{17}{27},$$

which implies that $\Pr_0(z_{i|0} \le 0) \ge \frac{1}{2}$ holds for $i = 0, 1, 2$ (in fact it is easy to show that $\Pr_0(z_{i|0} \le 0) \ge \frac{1}{2}$ holds for all $i \ge 0$) if $u_{i|0} = 0$ for all $i \ge 0$.

However, somewhat counterintuitively, the existence of a predicted control sequence such that $\Pr_0(z_{i|0} \le 0) \ge \frac{1}{2}$ holds for given i does not ensure that the constraint $\Pr_k(z_{0|k} \le 0) \ge \frac{1}{2}$ will be feasible at time $k = i$. For example, if $u_0 = 0$, then at time $k = 1$ the condition $\Pr_1(z_{0|1} \le 0) \ge \frac{1}{2}$ may be violated since if $w_0 = 1$, then $x_1 = 1$ so that $z_1 = 1 + v_1$ and hence

$$\Pr_1(z_{0|1} \le 0) = \Pr(v_1 = -1) = \tfrac{1}{3}$$

in this case.

From $z_k = x_k + v_k$ and the probability distribution of v_k, it is easy to see that the condition $\Pr_k(z_{0|k} \le 0) \ge \frac{1}{2}$ is satisfied if and only if $x_k \le 0$. Given that $x_0 = 0$ and w_k lies in the interval $[-1, 1]$, we must therefore have $u_0 \le -1$ in order that $\Pr_1(z_{0|1} \le 0) \ge \frac{1}{2}$ is feasible. By this reasoning, a control law that ensures satisfaction of the probabilistic constraint for all k is given by

$$u_0 = -1, \quad u_k = -w_{k-1}, \quad k = 1, 2, \ldots$$

since this control strategy ensures $x_k \le 0$ for all k.

Returning to the problem of determining a feasible predicted control sequence at time $k = 0$, in order to ensure the future feasibility of the constraint $\Pr_i(z_{0|i} \le 0) \ge \frac{1}{2}$, the predicted control sequence must meet the condition $x_{i|0} \le 0$ for all possible realizations of the random sequence $\{w_0, \ldots, w_{i-1}\}$, for each $i > 0$. This can be formulated as the problem of determining $\{u_{0|0}, u_{1|0}, \ldots\}$ satisfying, for each $i > 0$,

$$\Pr_i\Bigl(\max_{w_0,\ldots,w_{i-1}\in[-1,1]} z_{i|0} \le 0\Bigr) \ge \tfrac{1}{2}$$

or equivalently

$$\Pr_i\Bigl(i + \sum_{j=0}^{i-1} u_{j|0} + v_i \le 0\Bigr) \ge \tfrac{1}{2}.$$

In order to impose the probabilistic constraint (7.2) in a way that guarantees recursive feasibility, it is therefore necessary to consider for each $i > 0$ the worst case realization of $\{w_0, \ldots, w_{i-1}\}$ whereas v_i can be treated as a stochastic variable. Note also that a robust reformulation of the constraint, i.e. $\Pr_k(z_{0|k} \le 0) = 1$, would have to consider worst case realizations of both w and v, and hence would require that $x_k \le -1$. $\diamondsuit$

Given the probability distribution of v, the function f defines a set of states for which there exists a control input such that the constraint (7.2) holds. Let

$$\mathcal{X} \doteq \bigl\{x : \exists u \ \text{ such that } \Pr\bigl(f(x, u, v) \le 0\bigr) \ge p\bigr\}, \tag{7.5}$$

then the future feasibility of (7.2) can be ensured by requiring that the predicted state sequence at time k satisfies $x_{i|k} \in \mathcal{X}$ for all possible realizations of the uncertainty sequence $\{q_k, \ldots, q_{k+i-1}\}$, for each $i > 0$. Denoting the support of q (namely the set of all possible realizations q_k) as $\mathcal{Q} \subseteq \mathbb{R}^m$, so that

$$\Pr(q_k \in \mathcal{Q}) = 1 \ \text{ and } \ \Pr(q_k \notin \mathcal{Q}) = 0, \tag{7.6}$$

the implied constraints on the control sequence predicted at time k are then

$$\Pr_k(z_{0|k} \le 0) \ge p \tag{7.7a}$$

$$\Pr_{k+i}\Bigl(\max_{q_k,\ldots,q_{k+i-1}\in\mathcal{Q}} z_{i|k} \le 0\Bigr) \ge p, \quad i = 1, 2, \ldots \tag{7.7b}$$

Thus the constraints on the i steps ahead output variable $z_{i|k}$ have been made robust with respect to $\{q_k, \ldots, q_{k+i-1}\}$ but remain stochastic with respect to v_{k+i}.

To verify these constraints are recursively feasible, suppose $\{u_{0|k}, u_{1|k}, \ldots\}$ satisfies (7.7a, 7.7b) for some k and consider the sequence defined at time $k+1$ by $u_{i|k+1} = u_{i+1|k}$ for $i = 0, 1, \ldots$ (or by $u_{i|k+1}(x) = u_{i+1|k}(x)$ if optimization is performed over feedback laws rather than open-loop control sequences). Then, for any $q_k \in \mathcal{Q}$, the condition (7.7b) with $i = 1$ implies $x_{k+1} \in \mathcal{X}$ and

$$\Pr_{k+1}(z_{0|k+1} \le 0) \ge p.$$

Likewise, the condition (7.7b), when invoked for $i = j + 1$, ensures that $x_{j|k+1} \in \mathcal{X}$ for all $\{q_k, \ldots, q_{k+j}\} \in \mathcal{Q} \times \cdots \times \mathcal{Q}$ and

$$\Pr_{k+1+j}\Bigl(\max_{q_{k+1},\ldots,q_{k+j}\in\mathcal{Q}} z_{j|k+1} \le 0\Bigr) \ge p, \quad j = 1, 2, \ldots$$

Hence there exists a predicted control sequence at time $k+1$ such that the conditions of (7.7a and 7.7b) hold with k replaced by $k+1$.

This argument demonstrates that the conditions of (7.7a and 7.7b) provide a recursively feasible set of constraints ensuring satisfaction of (7.2). It is important to note that these conditions are necessary as well as sufficient for recursive feasibility. In particular, if the predicted control sequence is optimized over arbitrary feedback laws with no restriction on the controller parameterization, then infeasibility of (7.7a, 7.7b) implies that, for some $i \ge 0$, it is not possible to satisfy $\Pr_{k+j}(z_{0|k+j} \le 0) \ge p$ for all $j = 0, \ldots, i$ under any control law.

A consequence of the constraints (7.7b) involving a maximization over a subset of the uncertain model parameters is that feasibility cannot generally be guaranteed for models containing random variables with infinite support. Thus if q has infinite support, then a predicted control sequence or feedback law satisfying (7.2) can only exist if the model state x is unobservable from the constrained output variable z, and furthermore the unbounded model uncertainty associated with q must only affect the components of x that are unobservable from z. In general, this rules out problems in which q is normally distributed since such models allow disturbances to have an arbitrarily large effect on the model state, albeit with vanishingly small probability.

In most applications the restriction to finitely supported model uncertainty is not limiting since control systems are rarely subject to unbounded uncertainty in practice. However it does affect both the modelling of disturbances and the numerical methods required to handle probabilistic constraints. On the other hand, the parameter v that appears directly in the constrained output z is treated as a stochastic variable and is not required to have finite support. One consequence of this is that problems based on moving average dynamic models, such as those considered in Chap. 6, are not restricted to finitely supported random variables since in this case all of the model uncertainty is contained in the output map.

To conclude this section, we discuss how the conditions of (7.2), consisting of an infinite number of constraints over an infinite prediction horizon, can be reduced to a finite number of constraints. As in the case of robust MPC, it is possible to impose the constraints (7.2) on predicted state and control trajectories through a finite set of conditions, which nevertheless are recursively feasible over an infinite horizon, by using a dual-mode prediction strategy and an appropriate terminal constraint. In this context the terminal constraint $x_{N|k} \in \mathcal{X}_T$ is required to hold for all realizations of the uncertain sequence $\{q_k, \ldots, q_{k+N-1}\}$ over the initial N-step prediction horizon. Moreover recursive feasibility requires that $\mathcal{X}_T$ is a robustly invariant subset of the feasible set $\mathcal{X}$ in (7.5) and hence the probabilistic constraint (7.2) must hold for all $x \in \mathcal{X}_T$ under the terminal feedback law.

Assuming a linear terminal control law, $u_k = Kx_k$, we therefore require that $\mathcal{X}_T$ satisfies, for all $x \in \mathcal{X}_T$, the conditions

$$\big(A(q) + B(q)K\big)x + Dw(q) \in \mathcal{X}_T \quad \forall q \in \mathcal{Q} \tag{7.8a}$$

and

$$\Pr\big(f(x, Kx, v) \le 0\big) \ge p. \tag{7.8b}$$

If (7.8b) can be invoked through an equivalent algebraic condition, then the maximal robustly positively invariant set satisfying (7.8a, 7.8b) (or a convex inner approximation of this set—see e.g. [4]) can be determined by a conceptually straightforward extension of Theorem 3.1, as we now briefly discuss. Defining the sequence of sets $\{\mathcal{S}_0, \mathcal{S}_1, \ldots\}$ by

$$\mathcal{S}_0 = \big\{x : \Pr\big(f(x, Kx, v) \le 0\big) \ge p\big\}$$

and, for $k = 1, 2, \ldots$

$$\mathcal{S}_k = \big\{x : \big(A(q) + B(q)K\big)x + Dw(q) \in \mathcal{S}_{k-1} \;\; \forall q \in \mathcal{Q}\big\},$$

the MRPI set is given by

$$\mathcal{X}^{\text{MRPI}} \doteq \bigcap_{k=0}^{\infty} \mathcal{S}_k = \bigcap_{k=0}^{\nu} \mathcal{S}_k$$

where ν satisfies $\bigcap_{k=0}^{\nu} \mathcal{S}_k \subseteq \mathcal{S}_{\nu+1}$. In many cases of practical interest (see e.g. [5]), it is not easy to determine an algebraic equivalent of (7.8b) and it may therefore be necessary to resort to a conservative approximation, for example based on random sampling methods.

7.2 Prototype SMPC Algorithm: Stability and Convergence

This section proposes a general formulation of stochastic MPC for the system (7.1a, 7.1b). The algorithms presented here are conceptual in the sense that we do not consider how to solve the implied online MPC optimization problem (this is discussed in detail in later sections of this chapter and in Chap. 8). Instead, the focus of this section is on analysing closed-loop behaviour using the optimal value of the predicted cost. We show that the closed-loop system inherits a quadratic stability property when the MPC objective function is a quadratic predicted cost.

Two alternatives are considered for the MPC cost: the expected value predicted cost of Sect. 6.2 and the nominal predicted cost of Sect. 3.3. The system is subject to the pointwise-in-time probabilistic constraints of (6.8) and the constraints of the MPC optimization are constructed to ensure recursive feasibility as described in Sect. 7.1. Using a dual-mode prediction scheme, the predicted control sequence at time k is parameterized as

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \quad i = 0, 1, \dots \tag{7.9}$$

where $\mathbf{c}_k \doteq (c_{0|k}, \dots, c_{N-1|k})$ is a vector of optimization variables at time k and $c_{i|k} = 0$ for all prediction times $i \geq N$, and where K satisfies the mean-square stability condition (6.5). It is assumed that a terminal set $\mathcal{X}_T$ is known, where $\mathcal{X}_T$ satisfies the condition (7.8) for robust invariance and the probabilistic constraint

$$\Pr\Big(\tilde{F}\big(\Phi(q)x + Dw(q)\big) \leq \mathbf{1}\Big) \geq p$$

for all $x \in \mathcal{X}_T$, where $\tilde{F} = F + GK$ and $\Phi(q) = A(q) + B(q)K$.

7.2.1 Expectation Cost

Define the predicted cost at time k as the expectation cost (6.15) of Sect. 6.2:

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \mathbb{E}_k\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}\big) \tag{7.10}$$

where $l_{ss} = \operatorname{tr}\big(\Theta(Q + K^T RK)\big)$ and Θ is the solution of (6.10), and let K be the optimal linear feedback gain for this cost given by (6.22) and (6.23). Then, by Corollary 6.1, $J(x, \mathbf{c})$ has the quadratic form:

$$J(x, \mathbf{c}) = x^T W_x x + \mathbf{c}^T W_c \mathbf{c} + 2w_{x1}^T x + 2w_{c1}^T \mathbf{c} + w_1$$

where $W_x \succ 0$, $W_c \succ 0$ and w_{x1}, w_{c1} are given by (6.23), (6.24) and (6.21a), (6.21b). A recursively feasible MPC algorithm that minimizes this cost subject to the constraint $\Pr_k(Fx_{1|k} + Gu_{1|k} \leq \mathbf{1}) \geq p$ can be stated as follows.

Algorithm 7.1 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{\mathbf{c}_k}{\text{minimize}} \quad J(x_k, \mathbf{c}_k) \tag{7.11a}$$

subject to

$$\Pr_k\big(\tilde{F}x_{1|k} + Gc_{1|k} \leq \mathbf{1}\big) \geq p \tag{7.11b}$$

$$\Pr_{k+i}\Bigg(\max_{q_k,\ldots,q_{k+i-1}\in\mathcal{Q}} \tilde{F}x_{i+1|k} + Gc_{i+1|k} \leq \mathbf{1}\Bigg) \geq p, \quad i = 1, \ldots, N-2 \tag{7.11c}$$

$$\Pr_{k+N-1}\Bigg(\max_{q_k,\ldots,q_{k+N-2}\in\mathcal{Q}} \tilde{F}x_{N|k} \leq \mathbf{1}\Bigg) \geq p \tag{7.11d}$$

$$\text{and } x_{N|k} \in \mathcal{X}_T \quad \text{for all} \quad \{q_k, \ldots, q_{k+N-1}\} \in \mathcal{Q} \times \cdots \times \mathcal{Q} \tag{7.11e}$$

(ii) Apply the control law $u_k = Kx_k + c^*_{0|k}$, where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N-1|k})$ is the optimal argument of (7.11). ◁

Theorem 7.1 *For the system (7.1a, 7.1b) under the control law of Algorithm 7.1, if the optimization (7.11) is feasible at $k = 0$, then it remains feasible for all $k > 0$. Also the closed-loop system satisfies the quadratic stability condition*

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r} \mathbb{E}_0\big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) \leq l_{ss} \tag{7.12}$$

and $\Pr_k(Fx_{1|k} + Gu_{1|k} \leq \mathbf{1}) \geq p$ *holds for all* $k > 0$.

Proof The argument of Sect. 7.1 shows that the constraints (7.11b–7.11e) are recursively feasible since if the optimization of (7.11) is feasible at time k, then

$$\mathbf{c}_{k+1} = (c^*_{1|k}, \ldots, c^*_{N-1|k}, 0)$$

necessarily satisfies (7.11b–7.11e) at time $k+1$. Feasibility of (7.11b), therefore, implies satisfaction of the probabilistic constraint $\Pr_k(Fx_{1|k} + Gu_{1|k} \leq \mathbf{1}) \geq p$ for all $k \geq 0$.

From the definition of the predicted cost, we have

$$\mathbb{E}_k\big(J(x_{k+1}, \mathbf{c}_{k+1})\big) \le J^*(x_k) - \big(\|x_k\|_Q^2 + \|u_k\|_R^2 - l_{ss}\big)$$

where $J^*(x_k) = J(x_k, \mathbf{c}_k^*)$, and since optimality at time $k+1$ implies, for any realization of $q_k \in \mathcal{Q}$, that $J^*(x_{k+1}) \le J(x_{k+1}, \mathbf{c}_{k+1})$ it follows that

$$\mathbb{E}_k\big(J^*(x_{k+1})\big) \le J^*(x_k) - \big(\|x_k\|_Q^2 + \|u_k\|_R^2 - l_{ss}\big). \tag{7.13}$$

Taking expectations conditional on x_0 of both sides of this inequality and noting that $\mathbb{E}_0\big(\mathbb{E}_k\big(J^*(x_{k+1})\big)\big) = \mathbb{E}_0\big(J^*(x_{k+1})\big)$, we obtain, for all $r > 0$:

$$\frac{1}{r}\sum_{k=0}^{r-1} \mathbb{E}_0\big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) \le l_{ss} + \frac{1}{r}\Big(J^*(x_0) - \mathbb{E}_0\big(J^*(x_r)\big)\Big).$$

Here $J^*(x_0)$ is by assumption finite whereas $J^*(x)$ has a finite lower bound since W_x and W_c are positive definite matrices; therefore, the second term on the RHS of this inequality vanishes as $r \to \infty$. □

The optimal value of the predicted cost $J(x, \mathbf{c})$ is not necessarily non-negative and the convergence analysis in (7.1) therefore relies on the existence of a finite lower bound on the predicted cost. A consequence of this is that an asymptotic value of the expected stage cost lower than l_{ss} may be achievable using an affine rather than linear state feedback law whenever the additive and multiplicative uncertainty in the model (7.1a, 7.1b) are correlated. From the expression for the unconstrained minimizer of (7.11) given by (6.20) and (6.21a) it can be seen that the control law of Algorithm 7.1 will in fact be equal to this affine feedback law if constraints are inactive, and this is the explanation for the inequality in the asymptotic bound of (7.12).

Two special cases are of particular interest in this convergence analysis. If the additive and multiplicative uncertain model parameters in (7.1b) are uncorrelated, then the discussion of Sect. 6.2 shows that l_{ss} is the minimum expected value of the stage cost that can be achieved under any control law. Therefore in this case the bound (7.12) implies that the control law of Algorithm 7.1 converges asymptotically to $u_k = Kx_k$, and it can then be shown that x_k converges with probability 1 to the minimal RPI set under this feedback law as $k \to \infty$.

A second special case is that in which the system (7.1a, 7.1b) is only affected by multiplicative uncertainty (i.e., $w_k = 0$ for all k). In this case, $l_{ss} = 0$ and it follows that $J^*(x)$ is a positive definite function of x. Therefore (7.13) implies that the closed-loop system is quadratically stable and hence $x_k \to 0$ as $k \to \infty$ with probability 1.

7.2.2 Mean-Variance Cost

A quadratic stability result also applies if the performance objective of Algorithm 7.1 is replaced by the mean-variance predicted cost (7.14) of Sect. 6.3:

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \big(\|x_{i|k}^{(0)}\|_Q^2 + \|u_{i|k}^{(0)}\|_R^2\big) + \kappa^2 \sum_{i=0}^{\infty} \mathbb{E}_k\big(\|x_{i|k} - x_{i|k}^{(0)}\|_Q^2 + \|u_{i|k} - u_{i|k}^{(0)}\|_R^2 - l_{ss}\big). \tag{7.14}$$

with $l_{ss} = \mathrm{tr}\big(\Theta(Q + K^T RK)\big)$ as before, and where κ is a given constant. However the optimal value of this predicted cost does not necessarily satisfy a condition such as (7.13), and as a result the available bounds on the mean-square value of the closed-loop system state are, in general, weaker than the bound given by Theorem 7.1. To analyse stability, we take an indirect approach similar to the analysis in Sect. 3.3 of robust stability of nominal MPC. This is based on the fact that the optimal linear feedback gain for the problem of minimizing (7.14) in the absence of constraints necessarily satisfies the condition (6.5) for mean-square stability, and it therefore induces a finite l_2 gain between the disturbance input and the closed-loop system state under an associated MPC law.

Assuming K to be the unconstrained optimal linear feedback gain defined by (6.33a–6.33c), the matrix W_z in (6.32a) is block-diagonal and the cost (7.14) can be expressed

$$J(x_k, \mathbf{c}_k) = x_k^T W_x x_k + \mathbf{c}_k^T W_c \mathbf{c}_k + 2w_{x1}^T x_k + 2w_{c1}^T \mathbf{c}_k + w_1.$$

Furthermore the structure of Ψ_k and $\Psi^{(0)}$ in (6.32a, 6.32b) implies that W_c and w_{c1} have the block structure:

$$W_c = \mathrm{diag}\{S, \ldots, S\}, \qquad w_{c1}^T = \big[v^T \cdots v^T\big]$$

where $S \in \mathbb{R}^{n_u \times n_u}$ and $v \in \mathbb{R}^{n_u}$, with $S \succ 0$. The statement of a stochastic MPC algorithm based on the minimization of this cost subject to the constraints of (7.11) is as follows.

Algorithm 7.2 At each time instant $k = 0, 1, \ldots$:

(i) Perform the optimization:

$$\underset{\mathbf{c}_k}{\text{minimize}} \quad \|\mathbf{c}_k\|_{W_c}^2 + 2w_{c1}^T \mathbf{c}_k \quad \text{subject to (7.11b–7.11e)} \tag{7.15}$$

(ii) Apply the control law $u_k = Kx_k + c_{0|k}^*$, where $\mathbf{c}_k^* = (c_{0|k}^*, \ldots, c_{N-1|k}^*)$ is the optimal argument of (7.15). ◁

Theorem 7.2 *For the control law of Algorithm 7.2 applied to the system (7.1a, 7.1b): if the optimization (7.15) is feasible at $k = 0$, then it is feasible for all $k > 0$, the closed-loop system satisfies the quadratic stability condition*

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r} \mathbb{E}_0\big(\|x_k\|^2\big) \le \gamma^2 \mathbb{E}\big(\|w_k\|^2\big) \tag{7.16}$$

for some finite scalar γ, and $\Pr_k(Fx_{1|k} + Gu_{1|k} \le \mathbf{1}) \ge p$ holds for all $k > 0$.

The proof of the bound in (7.16) relies on the following result.

Lemma 7.1 *There exist scalars β, γ and a matrix $P \succ 0$ such that, under the control law of Algorithm 7.2, the following bound holds*

$$\mathbb{E}_k\big(\|x_{k+1}\|_P^2\big) \le \|x_k\|_P^2 - \|x_k\|^2 + \beta^2\Big(\|c^*_{0|k}\|_S^2 + 2v^T c^*_{0|k}\Big) + \gamma^2\mathbb{E}\big(\|w_k\|^2\big). \tag{7.17}$$

Proof Since K satisfies (6.33b) with $\hat{W}_x \succ 0$, there exists $P \succ 0$ satisfying $P - \mathbb{E}(\Phi_k^T P \Phi_k) \succ I$, and this ensures that β, γ exist so that $H_1 \succ 0$, where

$$H_1 \doteq \begin{bmatrix} P - I & 0 & 0 \\ 0 & \beta^2 S & \beta^2 v \\ 0 & \beta^2 v^T & \gamma^2\mathbb{E}\big(\|w_k\|^2\big) \end{bmatrix} - \mathbb{E}\left(\begin{bmatrix} \Phi_k^T \\ B_k^T \\ w_k^T D^T \end{bmatrix} P \begin{bmatrix} \Phi_k & B_k & Dw_k \end{bmatrix}\right).$$

The bound in (7.17) is then obtained by pre- and post-multiplying H_1 by the vector $(x_k, c^*_{0|k}, 1)$ and using the system dynamics (7.1a). To show that $H_1 \succ 0$, suppose that $P - \mathbb{E}(\Phi_k^T P \Phi_k) \succeq (1+\epsilon)I$ for some $\epsilon > 0$, then using Schur complements we find that $H_1 \succ 0$ if $H_2 \succ 0$, where

$$\begin{aligned} H_2 \doteq &\begin{bmatrix} \beta^2 S & \beta^2 v \\ \beta^2 v^T & \gamma^2\mathbb{E}\big(\|w_k\|^2\big) \end{bmatrix} - \begin{bmatrix} \mathbb{E}(B_k^T P B_k) & \mathbb{E}(B_k^T P D w_k) \\ \mathbb{E}(w_k^T D^T P B_k) & \mathbb{E}\big(\|w_k\|_P^2\big) \end{bmatrix} \\ &+ \epsilon^{-1}\begin{bmatrix} \mathbb{E}(B_k^T P \Phi_k) \\ \mathbb{E}(w_k^T D^T P \Phi_k) \end{bmatrix} \begin{bmatrix} \mathbb{E}(\Phi_k^T P B_k) & \mathbb{E}(\Phi_k^T P D w_k) \end{bmatrix}. \end{aligned}$$

The bottom right element of H_2 has the lower bound

$$\Big[\gamma^2 - \bar{\lambda}(P)\Big(1 + \epsilon^{-1}\sum_{j=1}^{m} \|P^{1/2}\Phi^{(j)}\|^2\Big)\Big]\mathbb{E}\big(\|w_k\|^2\big)$$

and is therefore positive for all non-zero $\mathbb{E}(\|w_k\|^2)$ if γ is sufficiently large. Since $S \succ 0$ it follows that $H_2 \succ 0$ for any given P, $\epsilon > 0$ and $\mathbb{E}(\|w_k\|^2)$ whenever the values of β and γ are sufficiently large. □

We can now give the proof of Theorem 7.2.

Proof (*of Theorem* 7.2) The constraints of (7.15) are identical to those of the MPC optimization in Algorithm 7.1. Therefore recursive feasibility and satisfaction of the probabilistic constraint therefore follow from feasibility of

$$\mathbf{c}_{k+1} = (c^*_{1|k}, \ldots, c^*_{N-1|k}, 0)$$

in (7.15) at time $k+1$. Let $V^*(x_k)$ denote the optimal value of the objective of (7.15) at time k, then the optimality of the MPC optimization at $k+1$ implies that $V^*(x_{k+1}) \le \|\mathbf{c}_{k+1}\|^2_{W_c} + 2w_{c1}^T \mathbf{c}_k$ for every realization $q_k \in \mathcal{Q}$. From the block structure of W_c and w_{c1}, we therefore have

$$\mathbb{E}_k\big(V^*(x_{k+1})\big) \le \|\mathbf{c}_{k+1}\|^2_{W_c} + 2w_{c1}^T \mathbf{c}_k = V^*(x_k) - \big(\|c^*_{0|k}\|^2_S + 2v^T c^*_{0|k}\big),$$

and hence from (7.17)

$$\mathbb{E}_k\big(\|x_{k+1}\|^2_P\big) \le \|x_k\|^2_P - \|x_k\|^2 + \beta^2\Big(V^*(x_k) - \mathbb{E}_k\big(V^*(x_{k+1})\big)\Big) + \gamma^2\mathbb{E}\big(\|w_k\|^2\big).$$

Taking expectations and summing over $k = 0, \ldots, r-1$, we have

$$\frac{1}{r}\sum_{k=0}^{r-1}\mathbb{E}_0\big(\|x_k\|^2\big) \le \gamma^2\mathbb{E}\big(\|w\|^2\big) + \frac{1}{r}\Big(\|x_0\|_P - \mathbb{E}_0\big(\|x_r\|^2_P\big)\Big) + \frac{\beta^2}{r}\Big(V^*(x_0) - \mathbb{E}_k\big(V^*(x_r)\big)\Big).$$

In the limit as $r \to \infty$, this implies the bound (7.16) since the second and third terms on the RHS of this inequality are necessarily bounded from above. □

Theorem 7.2 demonstrates the existence of a finite upper bound on the gain between the mean-square value of the additive disturbance and that of the closed-loop system state. But this result gives no indication of how the gain bound depends on the distribution of multiplicative model uncertainty, and hence it is a weaker result than Theorem 7.1. It is possible, however, to generalize the result of Theorem 7.1 for the case of the cost (7.14) with $\kappa > 1$, since then, using (6.32a–6.32c) and the expression (6.31) for the predicted cost, it can be shown that

$$\mathbb{E}_k\big(J^*(x_{k+1})\big) \le J^*(x_k) - \big(\|x_k\|^2_Q + \|u_k\|^2_R - \kappa^2 l_{ss}\big) \tag{7.18}$$

where $J^*(x_k) = J(x_k, \mathbf{c}^*_k)$ is the value of the cost (7.14) at the solution of the MPC optimization (7.15). By the argument that is used in the proof of Theorem 7.1, it follows from (7.18) that the closed-loop system under Algorithm 7.2 satisfies the quadratic stability condition

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r} \mathbb{E}_0\big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) \le \kappa^2 l_{ss} \tag{7.19}$$

whenever $\kappa > 1$.

7.2.3 Supermartingale Convergence Analysis

The quadratic bounds of Theorems 7.1 and 7.2 provide an indication of the asymptotic behaviour of the mean-square value of the closed-loop system state under Algorithms 7.1 and 7.2. However, except for the special case in which the model (7.1a, 7.1b) contains no additive disturbance, these results do not demonstrate asymptotic convergence of the state to a particular neighbourhood of the origin. Yet on the basis of the bounds on the evolution of the optimal value of the cost in (7.13) and (7.18), it is possible to state a convergence result for the state of the closed-loop system.

In order to do this, we define the ellipsoidal set

$$\Omega_\kappa \doteq \big\{x : x^T Q x \le \kappa^2 l_{ss}\big\}$$

and, given a sequence of states $\{x_0, x_1, \ldots\}$, we define the sequence $\{\hat{x}_0, \hat{x}_1, \ldots\}$ by $\hat{x}_0 = x_0$ and

$$\hat{x}_k = \begin{cases} x_k & \text{if } x_i \notin \Omega_\kappa \text{ for all } i < k \\ \hat{x}_{k-1} & \text{if } x_i \in \Omega_\kappa \text{ for some } i < k \end{cases} \tag{7.20}$$

for all $k > 0$. If x_k satisfies (7.13) for $\kappa = 1$ or (7.18) for $\kappa > 1$, then the sequence $\{J^*(\hat{x}_0), J^*(\hat{x}_1), \ldots\}$ is a supermartingale, namely a sequence of random variables with the property that $\mathbb{E}_k\big(J^*(\hat{x}_{k+1})\big) \le J^*(\hat{x}_k)$ for all $k \ge 0$ [6]. This follows from the fact that (7.13) or (7.18) imply

$$\mathbb{E}_k\big(J^*(\hat{x}_{k+1})\big) \le J^*(\hat{x}_k) - \big(\|\hat{x}_k\|_Q^2 - \kappa^2 l_{ss}\big) \le J^*(\hat{x}_k)$$

if $x_i \notin \Omega_\kappa$ for all $i \le k$ whereas $J^*(\hat{x}_{k+1}) = J^*(\hat{x}_k)$ if $x_i \in \Omega_\kappa$ for any $i \le k$, by (7.20). The stochastic convergence properties of supermartingales are well known, see e.g., [7], and in particular the following result (adapted from [8]) is useful in the current context.

Theorem 7.3 *Under Algorithm 7.1 with* $\kappa = 1$ *or Algorithm 7.2 with* $\kappa > 1$, *the state of the closed-loop system satisfies* $x_k \in \Omega_\kappa$ *for some* k *with probability* 1.

Proof Define the function $l(x)$ by

$$l(x) = \begin{cases} \|x\|_Q^2 - \kappa^2 l_{ss} & \text{if } x \notin \Omega_\kappa \\ 0 & \text{if } x \in \Omega_\kappa \end{cases}$$

and note that $l(x) > 0$ if and only if $x \notin \Omega_\kappa$. Then from (7.13), (7.18) and (7.20) we have, for all $k \geq 0$,

$$\mathbb{E}_k\big(J^*(\hat{x}_{k+1})\big) - J^*(\hat{x}_k) \leq -l(\hat{x}_k), \tag{7.21}$$

and summing over all $k < r$ yields, for any $r > 0$,

$$\sum_{k=0}^{r-1} \mathbb{E}_0\big(l(\hat{x}_k)\big) \leq J^*(x_0) - \mathbb{E}_0\big(J^*(\hat{x}_r)\big).$$

The RHS of this inequality has a finite upper bound because $J^*(x)$ is bounded from below for all x; it follows (by the Borel–Cantelli Lemma—see [6]) that $l(\hat{x}_k) \to 0$ with probability 1 and hence $\hat{x}_k \to \Omega_\kappa$ with probability 1. □

Theorem 7.3 implies that every state trajectory of the closed-loop system converges to the set Ω_κ. Although subsequently the state may not remain in Ω_κ, successive applications of Theorem 7.3 show that it must continually return to Ω_κ. The convergence of $\hat{x}$ to Ω_κ with probability 1 is equivalent to convergence in probability [8] since $\Pr\big(l(\hat{x}_k) \geq \epsilon\big) \to 0$ as $k \to \infty$ for all $\epsilon > 0$.

Analogous stability and convergence results can also be obtained for problems incorporating soft constraints that may be violated as often as required. For example, in applications that involve model uncertainty with unbounded support and for which the satisfaction of probabilistic constraints cannot be guaranteed, feasibility can be maintained by performing the optimization (7.11) (or (7.15) for $\kappa > 1$) whenever this is feasible, and otherwise minimizing the worst-case constraint violation subject to

$$J(x_k, c_k) \leq J\big(x_k, (c^*_{1|k-1}, \ldots, c^*_{N-1|k-1}, 0)\big).$$

Clearly this approach cannot guarantee that the closed-loop system will satisfy constraints of the form (6.7) or (6.8) and (6.9) at the required level of probability. However, it can impose a supermartingale-like condition such as (7.13) (or (7.18)), thus ensuring the quadratic stability condition of (7.12) (or (7.16)) and the convergence property of Theorem 7.3.

7.3 Probabilistically Invariant Ellipsoids

We next consider a generalized form of the probabilistic constraints (6.9). Rather than constraining the probability of the variable $Fx_k + Gu_k$ exceeding some threshold, we consider constraints on n_ψ output variables that are defined as the elements of the vector [9]

$$\psi_k = F_k x_k + G_k u_k + \eta_k. \tag{7.22}$$

The parameters $F_k \in \mathbb{R}^{n_\psi \times n_x}$, $G_k \in \mathbb{R}^{n_\psi \times n_u}$ and the noise process $\eta_k \in \mathbb{R}^{n_\psi}$ are subject to stochastic uncertainty which is described in terms of the random variable $q_k = (q_k^{(1)}, \ldots, q_k^{(m)})$ appearing in the model (7.1a, 7.1b) as $(F_k, G_k, \eta_k) = \big(F(q_k), G(q_k), \eta(q_k)\big)$, where

$$\big(F(q), G(q), \eta(q)\big) = (F^{(0)}, G^{(0)}, 0) + \sum_{j=1}^{m} (F^{(j)}, G^{(j)}, \eta^{(j)}) q^{(j)}. \tag{7.23}$$

Furthermore we consider a form of probabilistic constraint in which the output variable ψ_k is allowed to lie outside a prescribed interval $I_\psi = [\underline{\psi}, \bar{\psi}]$ provided the average probability of this happening over a given horizon N_c does not exceed a given limit:

$$\frac{1}{N_c} \sum_{i=0}^{N_c - 1} \Pr_k(\psi_{i|k} \notin I_\psi) \le \frac{N_{\max}}{N_c}. \tag{7.24}$$

Here, $N_{\max}$ is a limit on the expected number of times the output variable can lie outside the prescribed interval over a horizon of N_c steps.

The form of constraint in (7.24) is applicable to situations in which it is not realistic to invoke a pointwise in time probabilistic constraint of the form of (6.8) at every time instant, but where it is desirable to constrain the average rate of violations. For example in the design of supervisory controllers for large wind turbines with the aim of damping structural vibrations (such as fore-aft tower oscillations [10]) so as to reduce fatigue damage, it may not be possible to constrain the material stresses to a given range with a pre-specified probability at each sampling instant. But to limit potential fatigue damage, it is essential that the expected rate of such violations does not exceed a given threshold.

Propagating stochastic uncertainty over a prediction horizon can present considerable computational challenges, so to provide an efficient method of invoking the constraints (7.24), here we make use of probabilistic bounds on the model parameters. Thus it is assumed that with a probability of at least p, the parameter q lies in a known set $\mathcal{Q}_p$. In the following, we assume $\mathcal{Q}_p$ to be a compact convex polytope defined in terms of its vertices, namely

$$\Pr(q \in \mathcal{Q}_p) \ge p, \qquad \mathcal{Q}_p \doteq \mathrm{Co}\{q_{p,1}, \ldots, q_{p,\nu}\}. \tag{7.25}$$

Here $\mathcal{Q}_p$ defines a confidence region for the uncertain parameters which is non-unique in general, and this can be a source of suboptimality. For example, a probabilistic constraint such as $\Pr\bigl(f(c,q) \le 0\bigr) \ge p$, where c is a decision variable and f a given function, is implied by the condition $f(c,q) \le 0$ for all $q \in \mathcal{Q}_p$. However, replacing the original probabilistic constraint in an optimization problem by the condition $f(c,q) \le 0$ for all $q \in \mathcal{Q}_p$ will, in general, result in a suboptimal solution for c if $\mathcal{Q}_p$ is taken to be a fixed set rather than an optimization variable.

The confidence region $\mathcal{Q}_p$ of (7.25) is computationally convenient in that it allows a straightforward algebraic reformulation of probabilistic constraints and it can be computed offline. Thus, if q were normally distributed with zero mean and identity covariance matrix, then $q^T q$ would have a chi-squared distribution and hence the radius of a sphere centred at the origin and containing q with probability p could be computed using the chi-squared distribution with m degrees of freedom. Consequently $\mathcal{Q}_p$ could be defined as any polytopic set that over-bounds this sphere. Normal distributions do not of course have finite support, but this approach can be used to approximate $\mathcal{Q}_p$ if q has a truncated normal distribution. The implication of (7.25) is that

$$
\begin{aligned}
&\Pr\Bigl(\Phi_k \in \Phi^{(0)} + \mathrm{Co}\bigl\{\Phi(q_{p,1}), \ldots, \Phi(q_{p,\nu})\bigr\}\Bigr) \ge p \\
&\Pr\Bigl(B_k \in B^{(0)} + \mathrm{Co}\bigl\{B(q_{p,1}), \ldots, B(q_{p,\nu})\bigr\}\Bigr) \ge p \\
&\Pr\Bigl(w_k \in \mathrm{Co}\bigl\{w(q_{p,1}), \ldots, w(q_{p,\nu})\bigr\}\Bigr) \ge p
\end{aligned}
\tag{7.26}
$$

where $\Phi_k = A_k + B_k K$ and $\Phi(q) = \Phi^{(0)} + \sum_{j=1}^{m} \Phi^{(j)} q^{(j)}$.

The concept of probabilistic invariance is defined as follows.

Definition 7.1 A set $\mathcal{S} \subset \mathbb{R}^{n_x}$ is invariant with probability p for a system with state x_k if $\Pr_k(x_{1|k} \in \mathcal{S}) \ge p$ for all $x_k \in \mathcal{S}$.

To determine conditions under which an ellipsoidal set is invariant with probability p, we make use of the lifted autonomous state-space formulation of (6.16) in Sect. 6.2:

$$
z_{i+1|k} = \Psi_{k+i} z_{i|k} + \bar{D} w_{k+i}, \quad z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}, \tag{7.27}
$$

where $z_{i|k} \in \mathbb{R}^{n_z}$, $n_z = n_x + N n_u$, and $(\Psi_k, w_k) = \bigl(\Psi(q_k), w(q_k)\bigr)$ with

$$
(\Psi(q), w(q)) = (\Psi^{(0)}, 0) + \sum_{j=1}^{m} (\Psi^{(j)}, w^{(j)}) q^{(j)},
$$

$$
\Psi^{(j)} = \begin{bmatrix} \Phi^{(j)} & B^{(j)} \\ 0 & M \end{bmatrix}, \quad \Phi^{(j)} = A^{(j)} + B^{(j)} K, \quad \bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}.
$$

The implied predicted control law is given by

$$u_{i|k} = Kx_{i|k} + c_{i|k}$$

with $c_{i|k} = 0$ for all $i \geq N$. As before, $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ is a decision variable at time k. We consider ellipsoidal sets $\mathcal{E}_z \subset \mathbb{R}^{n_z}$ defined in terms of $P_z \succ 0$, and their projections onto the x-subspace:

$$\mathcal{E}_z \doteq \{z : z^T P_z z \leq 1\}, \qquad \mathcal{E}_x \doteq \{x : x^T P_x x \leq 1\},$$

where, as discussed in Chap. 2 (Sect. 2.7.2), $P_x = \left(\begin{bmatrix} I_{n_x} & 0 \end{bmatrix} P_z^{-1} \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix}^T \right)^{-1}$.

Theorem 7.4 *The set $\mathcal{E}_z$ is invariant with probability p for the system (7.27) if a scalar λ exists such that, for $j = 1, \ldots, \nu$,*

$$\begin{bmatrix} P_z^{-1} & \Psi(q_{p,j})P_z^{-1} & \bar{D}w(q_{p,j}) \\ \star & \lambda P_z^{-1} & 0 \\ \star & \star & 1-\lambda \end{bmatrix} \succeq 0. \tag{7.28}$$

Proof A sufficient condition for invariance of $\mathcal{E}_z$ with probability p is that $\|z_{1|k}\|^2_{P_z} \leq 1$ whenever $\|z_{0|k}\|^2_{P_z} \leq 1$, for all $q \in \mathcal{Q}_p$. Using (7.27) to express $z_{1|k}$ in terms of $z_{0|k}$ and applying the S-procedure [11] to these two inequalities, we obtain the equivalent condition that $\lambda > 0$ should exist such that

$$1 - \big(\Psi(q)z + \bar{D}w(q)\big)^T P_z \big(\Psi(q)z + \bar{D}w(q)\big) \geq \lambda(1 - z^T P_z z)$$

for all $z \in \mathbb{R}^{n_z}$ and all $q \in \mathcal{Q}_p$. An equivalent condition is that

$$\begin{bmatrix} \lambda P_z & 0 \\ 0 & 1-\lambda \end{bmatrix} - \begin{bmatrix} \Psi^T(q) \\ w^T(q)\bar{D}^T \end{bmatrix} P_z \begin{bmatrix} \Phi(q) & \bar{D}w(q) \end{bmatrix} \succeq 0$$

for all q in $\mathcal{Q}_p$. Using Schur complements, it can be shown that this is equivalent to an LMI in $\Psi(q)$ and $w(q)$, which, when invoked for all q in the polytope $\mathcal{Q}_p$, is by linearity and convexity equivalent to the LMIs (7.28) corresponding to the vertices $q_{p,j}$, $j = 1, \ldots, \nu$ of $\mathcal{Q}_p$. □

The theorem gives conditions under which the state of (7.27) returns, at the next time instant, to $\mathcal{E}_z$ with probability p. A second confidence polytope, $\mathcal{Q}_{\tilde{p}}$, corresponding to a confidence level of $\tilde{p}$, can be used to state conditions such that $\psi_{0|k} \in I_\psi$ with probability $\tilde{p}$ for all $z_{0|k} \in \mathcal{E}_z$. To do this, ψ is first expressed as a function of z and q:

$$\psi_{0|k} = C(q_k)z_{0|k} + \eta(q_k)$$
$$C(q) = \begin{bmatrix} F(q) + G(q)K & G(q)E \end{bmatrix}.$$

Corollary 7.1 *Let $q_{\tilde{p},j}$ for $j = 1, \ldots, \nu$ denote the vertices of $\mathcal{Q}_{\tilde{p}}$. Then* $\Pr(\psi_{0|k} \in I_\psi \mid z_{0|k} \in \mathcal{E}_z) \geq \tilde{p}$ *if*

$$\underline{\psi} \leq \eta(q_{\tilde{p},j}) \leq \bar{\psi} \tag{7.29a}$$

and

$$[C(q_{\tilde{p},j})P_z^{-1}C(q_{\tilde{p},j})^T]_{l,l} \leq [\bar{\psi} - \eta(q_{\tilde{p},j})]_l^2 \tag{7.29b}$$

$$[C(q_{\tilde{p},j})P_z^{-1}C(q_{\tilde{p},j})^T]_{l,l} \leq [\eta(q_{\tilde{p},j}) - \underline{\psi}]_l^2 \tag{7.29c}$$

for $j = 1, \ldots, \nu$ and $l = 1, \ldots, n_\psi$, where $[\cdot]_{l,l}$ and $[\cdot]_l$ denote, respectively, the lth diagonal element of the matrix $[\cdot]$ and the lth element of the vector $[\cdot]$.

Proof For any given q, the maximum absolute value of the lth element of $C(q)z_{0|k}$ over all $z_{0|k} \in \mathcal{E}_z$ is equal to $[C(q)P_z^{-1}C(q)^T]_{l,l}^{1/2}$. It follows that $\Pr(\psi_{0|k} \in I_\psi \mid z_{0|k} \in \mathcal{E}_z) \geq \tilde{p}$ if (7.29a) holds and

$$[C(q)P_z^{-1}C(q)^T]_{l,l}^{1/2} \leq [\bar{\psi} - \eta(q)]_l$$

$$[C(q)P_z^{-1}C(q)^T]_{l,l}^{1/2} \leq [\eta(q) - \underline{\psi}]_l$$

for all $q \in \mathcal{Q}_{\tilde{p}}$ and $l = 1, \ldots, n_\psi$. Given the affine dependence of C and η on q, these conditions are convex in q and it follows that they are equivalent to the conditions (7.29b, 7.29c), which correspond to the vertices $q_{\tilde{p},1}, \ldots, q_{\tilde{p},\nu}$ of $\mathcal{Q}_{\tilde{p}}$. □

Theorem 7.4 and Corollary 7.1 can be used to invoke the constraint (7.24) by deploying a Markov chain model. To illustrate this, we consider a pair of ellipsoidal sets $\mathcal{E}_1, \mathcal{E}_2 \subset \mathbb{R}^{n_x}$ where $\mathcal{E}_1 \subset \mathcal{E}_2$, and $\mathcal{E}_1$ is probabilistically invariant with probability $p_{1,1}$ whereas $\mathcal{E}_2$ is robustly invariant (i.e. invariant with probability 1). Although two sets are considered here, the approach is also applicable to a larger number of nested sets. Define $\mathcal{S}_1 \doteq \mathcal{E}_1$ and $\mathcal{S}_2 \doteq \mathcal{E}_2 - \mathcal{E}_1$, and assume that the predicted state x is steered by (7.27) from $\mathcal{S}_l$ to $\mathcal{S}_j$ in a single time step with probability $p_{j,l}$ for $j = 1, 2$, $l = 1, 2$, i.e.

$$\Pr_{k+i}(x_{i+1|k} \in \mathcal{S}_j \mid x_{i|k} \in \mathcal{S}_l) = p_{j,l}.$$

Also let the probability that $\psi_{i|k} \notin I_\psi$ given that $x_{i|k} \in \mathcal{S}_j$ be p_j, for $j = 1, 2$, i.e.

$$\Pr_{k+i}(\psi_{i|k} \notin I_\psi \mid x_{i|k} \in \mathcal{S}_j) \leq p_j.$$

Then it follows that

$$\Pr_{k+i}\{\psi_{i|k} \notin I_\psi\} \leq \begin{bmatrix} p_1 & p_2 \end{bmatrix} \Pi^i e \tag{7.30}$$

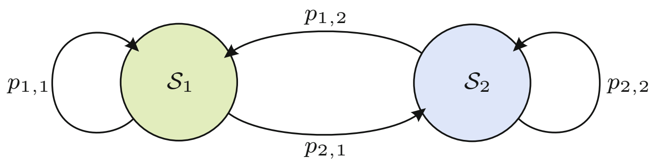

Fig. 7.2 Markov chain model with two discrete states

where

$$\Pi = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix} \quad \text{and} \quad e = \begin{cases} [1\ 0]^T & \text{if } x_k \in \mathcal{S}_1 \\ [0\ 1]^T & \text{if } x_k \in \mathcal{S}_2 \end{cases}$$

By definition Π is the transition matrix of a Markov chain [12] and therefore satisfies $\begin{bmatrix}1 & 1\end{bmatrix}\Pi = \begin{bmatrix}1 & 1\end{bmatrix}$ since the successor state must belong to either $\mathcal{S}_1$ or $\mathcal{S}_2$ (Fig. 7.2). This implies that $\begin{bmatrix}1 & 1\end{bmatrix}$ is a left eigenvector of Π with 1 as the corresponding eigenvalue. Thus the eigenvector decomposition of Π can be written as

$$\Pi = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}, \quad 0 \le \lambda_2 \le 1. \tag{7.31}$$

Together with (7.30), this implies that constraint (7.24) will be satisfied if

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} w_1 v_1^T e_j + \frac{(\lambda_2 - \lambda_2^{N_c})}{N_c(1-\lambda_2)} \begin{bmatrix} p_1 & p_2 \end{bmatrix} w_2 v_2^T e_j \le \frac{N_{\max}}{N_c}, \quad j = 1, 2 \tag{7.32}$$

However, we must have $N_{\max} = \mu N_c$ for some $\mu \in (0, 1)$, and hence this condition can always be satisfied for sufficiently large N_c so long as

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} w_1 v_1^T e_j \le \mu, \quad j = 1, 2. \tag{7.33}$$

In the current setting, we can choose $\mathcal{E}_1$ as $\mathcal{E}_x$, the x-subspace projection of an ellipsoidal set $\mathcal{E}_z = \{z : z^T P_z z \le 1\}$ that is constrained to be invariant with a specified probability, $p_{1,1}^0$, through the conditions Theorem 7.4. Similarly, $\mathcal{E}_2$ can be defined as the x-subspace projection of a robustly invariant ellipsoidal set. Theorem 7.4 then implies that $p_{1,1} \ge p_{1,1}^0$ and $p_{2,1} \le p_{2,1}^0 \doteq 1 - p_{1,1}^0$. However, since the outer ellipsoid $\mathcal{E}_2$ contains all states of the system for which this MPC strategy can be applied, it is reasonable to assume that the probability, p_2, of $\psi \notin I_\psi$ given $x \in \mathcal{E}_2$ will be higher than, p_1, the corresponding probability given $x \in \mathcal{E}_1$. In this case, replacing $p_{1,1}$ and $p_{2,1}$ in Π with their conservative bounds, $p_{1,1}^0$ and $p_{2,1}^0$, has the effect of reinforcing the inequality (7.30) and thus the condition in (7.32) for satisfaction of the constraint (7.24) will continue to hold.

For $x_k \in \mathcal{E}_1$, the MPC cost can be defined, for example, as the expected quadratic cost $J(x_k, \mathbf{c}_k)$ of Sect. 7.2.1. However, whenever $x_k \notin \mathcal{E}_1$, rather than minimize this cost, a sensible strategy is to steer the nominal successor state towards $\mathcal{E}_1$ so as to increase the value of $p_{1,2}$. This can be done by minimizing an objective function such as $\|\Psi^{(0)} z_k\|^2_{P_z}$. The implied MPC algorithm can be stated as follows.

Algorithm 7.3 At times $k = 0, 1, \ldots$:

(i) If $x_k \in \mathcal{E}_1$, compute

$$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \ J(x_k, \mathbf{c}_k) \ \text{ subject to } \ z_k^T P_z z_k \le 1 \tag{7.34}$$

(ii) Otherwise (i.e. if $x_k \notin \mathcal{E}_1$), compute

$$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \ z_k^T \Psi^{(0)^T} P_z \Psi^{(0)} z_k \ \text{ subject to } \ J(x_k, \mathbf{c}_k) \le J(x_k, M\mathbf{c}_{k-1}^*) \tag{7.35}$$

(iii) Apply the control law $u_k = Kx_k + c_{0|k}^*$ where $\mathbf{c}_k^* = (c_{0|k}^*, \ldots, c_{N-1|k}^*)$. ◁

The algorithm requires the minimization of a quadratic objective function subject to a single quadratic constraint in the optimization problems in each of steps (i) and (ii). The online MPC optimization is thus convex and can be solved efficiently. The constraint in (7.35) ensures that the time-average of the expected value of $\|x_k\|^2_Q + \|u_k\|^2_R$ satisfies an asymptotic bound, as shown by the following result.

Theorem 7.5 *For the system (7.1a, 7.1b) under the control law of Algorithm 7.3 with $J(x_k \mathbf{c}_k)$ defined by (7.10), the online optimization (7.34) or (7.35) is recursively feasible and the closed-loop state and control trajectories satisfy the asymptotic mean-square bound*

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}_0\big(\|x_k\|^2_Q + \|u_k\|^2_R\big) \le l_{ss}. \tag{7.36}$$

Proof By construction, $\mathcal{E}_2$ is robustly invariant under Algorithm 7.3 and if $x_k \in \mathcal{E}_2$, then $\mathbf{c}_{k+1} = M\mathbf{c}_k^*$ is necessarily feasible for (7.34) and (7.35) at time $k+1$, thus establishing recursive feasibility. To demonstrate the bound (7.36), let $J^*(x_k) = J(x_k, \mathbf{c}_k^*)$, then

$$\begin{aligned}\mathbb{E}_k\big(J^*(x_{k+1})\big) = {} & \mathbb{E}_k\big(J^*(x_{k+1}) \mid x_{k+1} \in \mathcal{E}_1\big)\Pr_k(x_{k+1} \in \mathcal{E}_1) \\ & + \mathbb{E}_k(J^*(x_{k+1}) \mid x_{k+1} \notin \mathcal{E}_1\big)\Pr_k(x_{k+1} \notin \mathcal{E}_1)\end{aligned}$$

where the feasibility of $\mathbf{c}_{k+1} = M\mathbf{c}_k^*$ implies that $J^*(x_{k+1}) \le J(x_{k+1}, M\mathbf{c}_k^*)$ for all realizations of q_k at time k (this follows from the objective in (7.34) and the constraint in (7.35)). Therefore,

$$\mathbb{E}_k\big(J^*(x_{k+1})\big) \le J^*(x_k) - \big(\|x_k\|^2_Q + \|u_k\|^2_R - l_{ss}\big)$$

and the bound (7.36) follows from the argument of Theorem 7.1. □

Algorithm 7.3 needs to be initialized through the offline design of the parameters $p_{1,1}, p_{1,2}, p_1, p_2$ and P_z. A possible procedure for this is as follows. Specify the initial values, $p^0_{1,1}$ and $p^0_{1,2}$, of $p_{1,1}$ and $p_{1,2}$ (and hence also the values of $p^0_{2,1} = 1 - p^0_{1,1}$ and $p^0_{2,2} = 1 - p^0_{1,2}$). Then, to make the constraints on $\mathcal{E}_1$ and $\mathcal{E}_2$ as unrestrictive as possible, set $p_2 = 1$ and choose p_1 as the maximum allowable value for p_1 according to (7.32). Next, construct the confidence polytopes $\mathcal{Q}_p$ (with $p = p^0_{1,1}$) and $\mathcal{Q}_{\tilde{p}}$ (with $\tilde{p} = p_1$) and maximize the volume of $\mathcal{E}_1$ by solving the optimization problem

$$\underset{P_z^{-1},\lambda}{\text{maximize}} \quad \det(P_x^{-1}) \quad \text{subject to (7.28) and (7.29b, 7.29c)} \tag{7.37}$$

to determine P_z. This problem is convex for fixed values of λ in (7.28) and it can therefore be solved via a univariate search over $\lambda \in (0, 1)$. The optimization (7.37) can also be used to determine a robustly invariant ellipsoid $\mathcal{E}_z$ with maximum volume x-subspace projection, although the vertices of $\mathcal{Q}$ such that $q_k \in \mathcal{Q}$ with probability 1 must be used in place of those of $\mathcal{Q}_p$ in the constraint (7.28), and the constraints (7.29b, 7.29c) are not needed in (7.37) given that $p_2 = 1$ is assumed.

For given $\mathcal{E}_1$ and $\mathcal{E}_2$, the actual value of $p_{1,2}$ can be determined through Monte Carlo simulations (for example, by searching over the boundary of $\mathcal{E}_2$ for the minimum probability of inclusion of the successor state in $\mathcal{E}_1$). In order to ensure satisfaction of (7.32) and hence of the constraint (7.24), we require that $p_{1,2} \geq p^0_{1,2}$. If this condition is not satisfied, the procedure must be repeated with reduced values for the initial guesses $p^0_{1,1}$ and $p^0_{1,2}$.

Problems involving constraints on the rate of accumulation of fatigue damage typically place higher weighting on larger amplitude fluctuations of the constrained output ψ. This suggests using a number of intervals for ψ and modifying the constraint (7.24) in order to define an upper bound on a weighted sum of expected constraint violations over an interval. Further improvements in the accuracy with which the closed-loop system satisfies the probabilistic constraints of the problem can be obtained by using a larger number of sets $\mathcal{S}_j$, since the accuracy of the Markov chain model in predicting constraint violations improves as a finer discretization of the state space is employed. These modifications are considered in Sect. 7.4, which also introduces a mode 1 prediction horizon and tubes in order to obtain more accurate approximations of predicted probability distributions based on Markov chains.

7.4 Markov Chain Models Based on Tubes with Polytopic Cross Sections

This section uses a Markov chain model to approximate the evolution of the probability distribution of the states of the model (7.1a, 7.1b) over a prediction horizon. In this setting, the Markov chain imposes a discretization of the state space based on a sequence of nested tubes. The approach determines offline bounds on the transition probabilities between the sets that form the tube cross sections at successive time steps so as to meet probabilistic constraints. Confidence bounds on the model parameters can be used to apply these probabilistic bounds (together with the hard constraints of the problem) to predicted state and control trajectories which are constructed during the online MPC optimization.

Tubes with polytopic cross sections are considered here. These are taken to be low-complexity polytopes such as the sets considered in Sects. 3.6.2 and 5.4. Thus the shapes of the sets defining the tube cross sections are fixed but their centres and scalings along a set of fixed directions can be adjusted online. At each prediction time step $i = 0, 1, \ldots$, we define μ polytopic cross sections:

$$\mathcal{X}^{(j)}_{i|k} \doteq \{z : \underline{z}^{(j)}_{i|k} \le z \le \bar{z}^{(j)}_{i|k}\}, \quad j = 1, \ldots, \mu, \tag{7.38}$$

where z denotes the transformed state $z = Vx$ for a non-singular matrix $V \in \mathbb{R}^{n_x \times n_x}$. The dynamics of z are given by

$$z_{i+1|k} = \tilde{\Phi}_{k+i} z_{i|k} + \tilde{B}_{k+i} c_{i|k} + \tilde{D} w_{i|k}, \quad i = 0, 1, \ldots$$

where $\tilde{\Phi}_{k+i} = V\Phi_{k+i}V^{-1}$, $\tilde{B}_{k+i} = VB_{k+i}$, $\tilde{D} = VD$.

The methodology of this section is not limited to low-complexity polytopic tubes, and general complexity polytopic tube cross sections such as those considered in Sect. 5.5 could be used in place of the sets defined in (7.38). In this more general case, the set inclusion conditions that are developed in this section could be imposed using an approach based on Farkas Lemma such as that of Lemma 5.6. For simplicity however we present the ideas here using low-complexity tubes.

We assume that V is fixed and designed offline, while $\underline{z}^{(j)}_{i|k}$ and $\bar{z}^{(j)}_{i|k}$, for $i = 0, \ldots, N$, $j = 1, \ldots, \mu$, are online optimization variables that are computed simultaneously with $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ at each time k. The tube cross sections defined in (7.38) are constrained to be nested (Fig. 7.3):

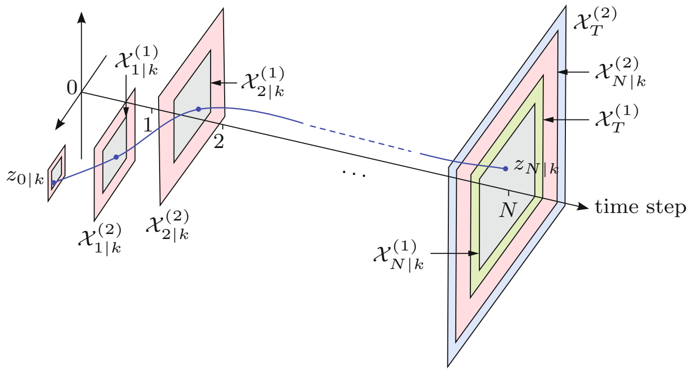

Fig. 7.3 Low-complexity polytopic tubes $\{\mathcal{X}^{(j)}_{0|k}, \ldots, \mathcal{X}^{(j)}_{N|k}\}$ and terminal sets $\mathcal{X}^{(j)}_{T}$ for $j = 1, 2$, for the case of a 2-dimensional state space

$$\mathcal{X}_{i|k}^{(1)} \subseteq \mathcal{X}_{i|k}^{(2)} \subseteq \cdots \subseteq \mathcal{X}_{i|k}^{(\mu)}, \quad i = 0, \ldots, N. \tag{7.39}$$

At the start of the prediction horizon, we require $z_{0|k} = Vx_k$ to lie $\mathcal{X}_{0|k}^{(j)}$ for some $l = 1, \ldots, \mu$, and terminal condition involving a polytopic terminal set is imposed on each tube layer,

$$\mathcal{X}_{N|k}^{(j)} \subseteq \mathcal{X}_{T}^{(j)} = \{z : |z| \leq z_T^{(j)}\}, \quad j = 1, \ldots, \mu, \tag{7.40}$$

where the terminal sets are also nested:

$$\mathcal{X}_{T}^{(1)} \subseteq \mathcal{X}_{T}^{(2)} \subseteq \cdots \subseteq \mathcal{X}_{T}^{(\mu)}. \tag{7.41}$$

The nested conditions of (7.39) and (7.41) can be invoked simply by imposing the constraints

$$\underline{z}_{i|k}^{(1)} \geq \underline{z}_{i|k}^{(2)} \geq \cdots \geq \underline{z}_{i|k}^{(\mu)}, \tag{7.42a}$$
$$\bar{z}_{i|k}^{(1)} \leq \bar{z}_{i|k}^{(2)} \leq \cdots \leq \bar{z}_{i|k}^{(\mu)}, \tag{7.42b}$$
$$z_{T}^{(1)} \leq z_{T}^{(2)} \leq \cdots \leq z_{T}^{(\mu)}. \tag{7.42c}$$

The constraints of the problem are taken to be the same as those considered in Sect. 7.3 (i.e. in (7.24)), but here we separate the constraints into probabilistic and hard constraints through the definition of two output vectors, ψ^p and ψ^h. These are given in terms of $z_{i|k}$ and $c_{i|k}$ as

$$\psi_{i|k}^{p} = F_p z_{i|k} + G_p c_{i|k} \tag{7.43a}$$
$$\psi_{i|k}^{h} = F_h z_{i|k} + G_h c_{i|k} \tag{7.43b}$$

For simplicity, the parameters F_p and G_p are assumed to be deterministic, although constraints such as (7.22)–(7.23) can be handled by a simple extension of the same approach. We consider the constraints:

$$\frac{1}{N_c}\sum_{i=1}^{N_c} \Pr_k(\psi_{i|k}^{p} > \mathbf{1}) \leq \frac{N_{\max}}{N_c} \tag{7.44a}$$
$$\psi_{i|k}^{h} \leq \mathbf{1}. \tag{7.44b}$$

It is straightforward to show that with an appropriate choice of the parameters F_p, G_p, (7.44a) has the same form as (7.24).

The strategy for handling the probabilistic constraints (7.44a) resembles that of Sect. 7.3 in that conditions are imposed on the probability with which the predicted state makes a transition between different tube layers and on the probability of the

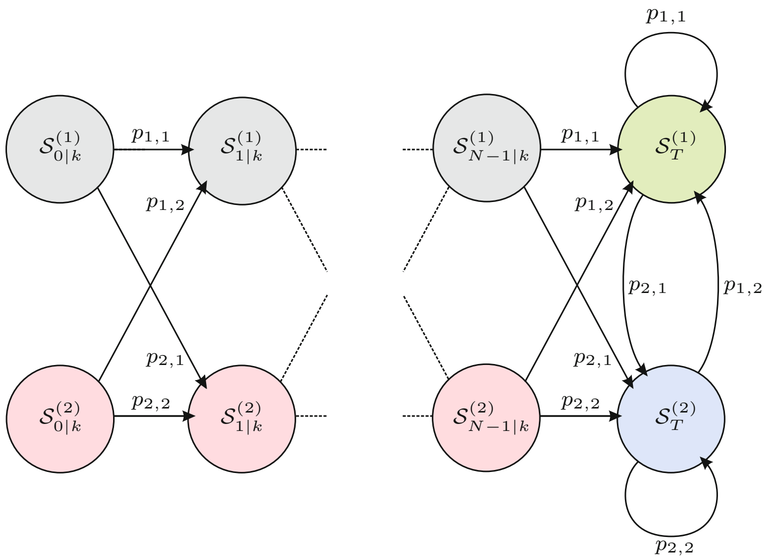

Fig. 7.4 Markov chain model for the case of $\mu = 2$

constrained output variables in (7.43a, 7.43b) exceeding threshold values within each tube layer. Define sets $\mathcal{S}_{i|k}^{(j)}$ for $i = 0, \ldots, N$ by

$$\mathcal{S}_{i|k}^{(j)} = \begin{cases} \mathcal{X}_{i|k}^{(1)} & j = 1 \\ \mathcal{X}_{i|k}^{(j)} \setminus \mathcal{X}_{i|k}^{(j-1)} & j = 2, \ldots, \mu \end{cases}$$

and let $p_{j,m}$ be the probability that $z_{i+1|k}$ lies in $\mathcal{S}_{i+1|k}^{(j)}$ given that $z_{i|k}$ lies in $\mathcal{S}_{i|k}^{(m)}$ (Fig. 7.4). Then

$$\begin{bmatrix} \Pr_k\big(z_{i+1|k} \in \mathcal{S}_{i+1|k}^{(1)}\big) \\ \vdots \\ \Pr_k\big(z_{i+1|k} \in \mathcal{S}_{i+1|k}^{(\mu)}\big) \end{bmatrix} = \Pi \begin{bmatrix} \Pr_k\big(z_{i|k} \in \mathcal{S}_{i|k}^{(1)}\big) \\ \vdots \\ \Pr_k\big(z_{i|k} \in \mathcal{S}_{i|k}^{(\mu)}\big) \end{bmatrix}, \quad \Pi \doteq \begin{bmatrix} p_{1,1} & \cdots & p_{1,\mu} \\ \vdots & \ddots & \vdots \\ p_{\mu,1} & \cdots & p_{\mu,\mu} \end{bmatrix} \tag{7.45}$$

and, for $z_{0|k} \in \mathcal{S}_{0|k}^{(j)}$, we have

$$\begin{bmatrix} \Pr_k\big(z_{i|k} \in \mathcal{S}_{i|k}^{(1)}\big) \\ \vdots \\ \Pr_k\big(z_{i|k} \in \mathcal{S}_{i|k}^{(\mu)}\big) \end{bmatrix} = \Pi^i e_j \tag{7.46}$$

where e_j is the jth column of the identity matrix. If, in addition, the probability that $\psi^p_{i+1|k} > 1$ given $z_{i|k} \in \mathcal{S}^{(m)}_{i|k}$ is no greater than p_m for each $m = 1, \ldots, \mu$, then the bound

$$\Pr_k(\psi^p_{i+1|k} > \mathbf{1}) \le \begin{bmatrix} p_1 \cdots p_\mu \end{bmatrix} \Pi^i e_j$$

holds for any given i whenever $z_{0|k} \in \mathcal{S}^{(j)}_{0|k}$, and this in turn can be used to ensure that (7.44a) is satisfied.

In order to obtain a computationally convenient set of constraints for the online MPC optimization, we formulate the constraints in terms of the probabilities $\tilde{p}_{j,m}$ of transition from $\mathcal{X}^{(m)}_{i|k}$ to $\mathcal{X}^{(j)}_{i+1|k}$ rather than transition probabilities between $\mathcal{S}^{(m)}_{i|k}$ and $\mathcal{S}^{(j)}_{i+1|k}$. Furthermore these conditions are imposed through inequalities rather than equality constraints. The required set of constraints is as follows.

(i) Transition probability constraints, for $j, m = 1, \ldots, \mu$

$$\Pr_{k+i}\bigl(z_{i+1|k} \in \mathcal{X}^{(j)}_{i+1|k} \mid z_{i|k} \in \mathcal{X}^{(m)}_{i|k}\bigr) \ge \tilde{p}_{j,m}, \quad i = 0, \ldots, N-1 \tag{7.47a}$$

$$\Pr_{k+i}\bigl(z_{i+1|k} \in \mathcal{X}^{(j)}_{T} \mid z_{i|k} \in \mathcal{X}^{(m)}_{T}\bigr) \ge \tilde{p}_{j,m}, \quad i \ge N. \tag{7.47b}$$

(ii) Probabilistic output constraints, for $j = 1, \ldots, \mu$

$$\Pr_{k+i}\bigl(\psi^p_{i+1|k} > \mathbf{1} \mid z_{i|k} \in \mathcal{X}^{(j)}_{i|k}\bigr) \le p_j, \quad i = 0, \ldots, N-1 \tag{7.48a}$$

$$\Pr_{k+i}\bigl(\psi^p_{i+1|k} > \mathbf{1} \mid z_{i|k} \in \mathcal{X}^{(j)}_{T}\bigr) \le p_j, \quad i \ge N. \tag{7.48b}$$

(iii) Robust output constraints

$$\psi^h_{i+1|k} \le \mathbf{1} \;\; \forall z_{i|k} \in \mathcal{X}^{(\mu)}_{i|k}, \quad i = 0, \ldots, N-1 \tag{7.49a}$$

$$\psi^h_{i+1|k} \le \mathbf{1} \;\; \forall z_{i|k} \in \mathcal{X}^{(\mu)}_{T}, \quad i \ge N. \tag{7.49b}$$

(iv) Initial and terminal constraints, for $j = 1, \ldots, \mu$

$$V x_k \in \mathcal{X}^{(\mu)}_{0|k} \tag{7.50a}$$

$$\mathcal{X}^{(j)}_{N|k} \subseteq \mathcal{X}^{(j)}_{T}. \tag{7.50b}$$

The transition probability constraints are invoked through inequalities in (7.47a, 7.47b), which implies that the transition probabilities of the Markov chain model (7.45) will not, in general, hold with equality. However it is still possible to ensure satisfaction of the constraints (7.44a, 7.44b) if we make the following assumption on the probabilities $\tilde{p}_{j,m}$, p_j chosen by the designer.

Assumption 7.1 The probabilities $\tilde{p}_{j,m}$ and p_j satisfy:

$$\tilde{p}_{j,m} \geq \tilde{p}_{j,m+1}, \quad j, m = 1, \ldots, \mu - 1 \tag{7.51a}$$
$$\tilde{p}_{\mu,m} = 1, \quad m = 1, \ldots, \mu \tag{7.51b}$$
$$p_{j+1} \geq p_j, \quad j = 1, \ldots, \mu. \tag{7.51c}$$

These conditions correspond to a unimodality assumption on the distributions of probabilistically constrained outputs. In particular, the conditions of (7.51c) require that the probability with which the one-step ahead output satisfies $\psi^p \leq \mathbf{1}$ should be smaller at points further from the centre of the tube. Likewise, (7.51a) requires that the probability of transition to any given layer should decrease away from the tube centre, while (7.51b) ensures that the outer tube layer bounds robustly the predicted state trajectories for all possible uncertainty realizations.

We next use (7.47)–(7.50) to derive an upper bound on the probability $\Pr_k(\psi^p_{i+1|k} > 1)$ for any $i \geq 0$.

Lemma 7.2 *Under conditions (7.47)–(7.50) and Assumption 7.1, the probability that $\psi^p_{i+1|k} > \mathbf{1}$ given that $z_{0|k} \in \mathcal{S}^{(j)}_{0|k}$ is bounded by*

$$\Pr_k\big(\psi^p_{i+1|k} > \mathbf{1}\big) \leq \big[p_1 \, \ldots \, p_\mu\big]\, (T\tilde{\Pi})^i e_j \tag{7.52}$$

for any $i \geq 0$, where $\tilde{\Pi}$ and T are defined by

$$\tilde{\Pi} = \begin{bmatrix} \tilde{p}_{1,1} & \cdots & \tilde{p}_{1,\mu} \\ \vdots & \ddots & \vdots \\ \tilde{p}_{\mu,1} & \cdots & \tilde{p}_{\mu,\mu} \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}. \tag{7.53}$$

Proof First, let $\tilde{v}_{i|k}$ and $v_{i|k}$ denote the vectors

$$\tilde{v}_{i|k} = \Big[\Pr_k\big(z_{i|k} \in \mathcal{X}^{(1)}_{i|k}\big) \, \cdots \, \Pr_k\big(z_{i|k} \in \mathcal{X}^{(\mu)}_{i|k}\big)\Big]^T$$
$$v_{i|k} = \Big[\Pr_k\big(z_{i|k} \in \mathcal{S}^{(1)}_{i|k}\big) \, \cdots \, \Pr_k\big(z_{i|k} \in \mathcal{S}^{(\mu)}_{i|k}\big)\Big]^T$$

and note that the nested property (7.41) and the definition of $\mathcal{S}^{(j)}_{i|k}$ as the set difference $\mathcal{X}^{(j)}_{i|k} \setminus \mathcal{X}^{(l-1)}_{i|k}$ implies that

$$\Pr_k\big(z_{i|k} \in \mathcal{X}^{(j)}_{i|k}\big) = \Pr_k\big(z_{i|k} \in \mathcal{X}^{(l-1)}_{i|k}\big) + \Pr_k\big(z_{i|k} \in \mathcal{S}^{(j)}_{i|k}\big), \quad l = 2, \ldots, \mu$$

and hence $v_{i|k} = T\tilde{v}_{i|k}$. Next we show by induction that if $z_{0|k} \in \mathcal{X}^{(j)}_{0|k}$, then

$$\tilde{v}_{i|k} \geq (\tilde{\Pi}T)^{i-1}\tilde{\Pi}e_j. \tag{7.54}$$

Specifically, conditions (7.47a, 7.47b) imply $\Pr_{k+i}\bigl(z_{i+1|k} \in \mathcal{X}^{(j)}_{i+1|k} \mid z_{i|k} \in \mathcal{S}^{(m)}_{i|k}\bigr) \geq \tilde{p}_{j,m}$, and since $\Pr\bigl(z_{0|k} \in \mathcal{S}^{(j)}_{0|k}\bigr) = 1$, we therefore have $\tilde{v}_{1|k} \geq \tilde{\Pi}e_j \geq 0$. Furthermore condition (7.51a) of Assumption 7.1 ensures that the elements of $\tilde{\Pi}T$ are non-negative and it follows that

$$\tilde{v}_{i+1|k} \geq \tilde{\Pi}v_{i|k} = \tilde{\Pi}T\tilde{v}_{i|k}$$

whenever $\tilde{v}_{i|k} \geq 0$. Therefore (7.54) holds for all $i > 0$.

Finally we obtain a bound on the probability that $\psi^p_{i+1|k} > \mathbf{1}$ using (7.54) and

$$\Pr_k\bigl(\psi^p_{i+1|k} > \mathbf{1}\bigr) \geq \bigl[p_1 \ \dots \ p_\mu\bigr]\, v_{i|k} = \bigl[p_1 \ \dots \ p_\mu\bigr]\, T\tilde{v}_{i|k}. \tag{7.55}$$

Here the first $\mu - 1$ elements of the row vector $\bigl[p_1 \ \dots \ p_\mu\bigr]\, T$ are non-positive because of (7.51c), whereas every element of $\tilde{v}_{i|k}$ except the last (which by (7.51b) is equal to 1) is over-estimated by (7.54). Hence replacing $\tilde{v}_{i|k}$ in (7.55) by the RHS of (7.54) yields an upper bound on $\Pr_k\bigl(\psi^p_{i+1|k} > \mathbf{1}\bigr)$ and the bound in (7.52) then follows since $T(\tilde{\Pi}T)^{i-1}\tilde{\Pi} = (T\tilde{\Pi})^i$ for $i > 0$. □

The bounds of Lemma 7.2 provide sufficient conditions for the satisfaction of the probabilistic constraint (7.44a). Summing (7.52) over $i = 0, \dots, N_c - 1$ yields directly the result that (7.44a) necessarily holds if the probabilities $\tilde{p}_{j,m}$ and p_j, for $j, m = 1, \dots, \mu$ are chosen so as to satisfy

$$\frac{1}{N_c}\sum_{i=0}^{N_c-1}\bigl[p_1 \ \dots \ p_\mu\bigr]\,(T\tilde{\Pi})^i e_j \leq \frac{N_{\max}}{N_c} \tag{7.56}$$

for all $j = 1, \dots, \mu$.

Before stating the stochastic MPC algorithm, we first show that the constraints (7.47)–(7.50) can be expressed as linear inequalities in the optimization variables $\mathbf{c}_k$, $\underline{z}^{(j)}_{i|k}$ and $\bar{z}^{(j)}_{i|k}$, $i = 0, \dots, N$, $j = 1, \dots, \mu$. There is little advantage in including the parameters $z^{(j)}_T$ of the terminal sets in the list of degrees of freedom; instead, it is suggested that these parameters are chosen offline in order to maximize the volume of each terminal set, which, given that $\mathcal{X}^{(j)}_T$, $j = 1, \dots, \mu$ are orthotopes, can be taken to be the product of the elements of $z^{(j)}_T$. Clearly, such a maximization has to be subject to conditions (7.47b)–(7.49b) or the equivalent inequalities to be presented below. The implied optimization can be solved using the techniques described in Sect. 5.4.1.

Each of the constraints (7.47)–(7.50) is conditioned on either $z_{i|k} \in \mathcal{X}_{i|k}^{(j)}$ or $z_{i|k} \in \mathcal{X}_T^{(j)}$. Moreover, if these constraints are imposed using convex confidence polytopes for the uncertain model parameters, then as we show below, they each depend linearly on $z_{i|k}$. Therefore, (7.47)–(7.50) are equivalent to the corresponding constraints invoked only at the vertices of $\mathcal{X}_{i|k}^{(j)}$ and $\mathcal{X}_T^{(j)}$. Furthermore, these conditions can be invoked using the recursive bounding technique of Sect. 5.4.2 without the need to evaluate the vertices of the tube cross sections.

Conditions (7.47)–(7.48) are probabilistic and can be converted into linear inequalities by invoking them at the vertices of confidence polytopes for the uncertain model parameters. However these polytopes must be defined separately for each of the probabilities $\tilde{p}_{j,m}$ involved in (7.47)–(7.48). Therefore we define $\mathcal{Q}(\tilde{p}_{j,m})$ as a confidence polytope such that $\Pr\bigl(q \in \mathcal{Q}(\tilde{p}_{j,m})\bigr) \geq \tilde{p}_{j,m}$, and denote its vertices by $q^{(s)}(\tilde{p}_{j,m})$, $s = 1, \ldots, \nu(\tilde{p}_{j,m})$. With these definitions, the constraints of (7.47)–(7.50) can be expressed as the following linear inequalities.

(i) Transition probability constraints, for $j, m = 1, \ldots, \mu$:

$$\tilde{\Phi}^+\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr)\bar{z}_{i|k}^{(m)} - \tilde{\Phi}^-\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr)\underline{z}_{i|k}^{(m)} + \tilde{B}\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr)c_{i|k} + \tilde{D}w\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr) \leq \bar{z}_{i+1|k}^{(j)}, \quad i = 0, \ldots, N-1 \tag{7.57a}$$

$$\tilde{\Phi}^+\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr)\underline{z}_{i|k}^{(m)} - \tilde{\Phi}^-\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr)\bar{z}_{i|k}^{(m)} + \tilde{B}\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr)c_{i|k} + \tilde{D}w\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr) \geq \underline{z}_{i+1|k}^{(j)}, \quad i = 0, \ldots, N-1 \tag{7.57b}$$

$$\bigl|\tilde{\Phi}\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr)\bigr|\bar{z}_T^{(m)} + \tilde{D}w\bigl(q^{(s)}(\tilde{p}_{j,m})\bigr) \leq \bar{z}_T^{(j)} \tag{7.57c}$$

(ii) Probabilistic output constraints, for $j = 1, \ldots, \mu$:

$$\Bigl(F_p\tilde{\Phi}\bigl(q^{(s)}(1-p_j)\bigr)\Bigr)^+\bar{z}_{i|k}^{(j)} - \Bigl(F_p\tilde{\Phi}\bigl(q^{(s)}(1-p_j)\bigr)\Bigr)^-\underline{z}_{i|k}^{(j)} + F_p\tilde{B}\bigl(q^{(s)}(1-p_j)\bigr)c_{i|k} + F_p\tilde{D}w\bigl(q^{(s)}(1-p_j)\bigr) + G_p c_{i|k} \leq \mathbf{1}, \quad i = 0, \ldots, N-1 \tag{7.58a}$$

$$\bigl|F_p\tilde{\Phi}\bigl(q^{(s)}(1-p_j)\bigr)\bigr|\bar{z}_T^{(j)} + F_p\tilde{D}w\bigl(q^{(s)}(1-p_j)\bigr) \leq \mathbf{1} \tag{7.58b}$$

(iii) Robust output constraints

$$F_h^+\bar{z}_{i|k}^{(\mu)} - F_h^-\underline{z}_{i|k}^{(\mu)} + G_h c_{i|k} \leq \mathbf{1}, \quad i = 0, \ldots, N-1 \tag{7.59a}$$

$$|F_h|\bar{z}_T^{(\mu)} \leq \mathbf{1} \tag{7.59b}$$

(iv) Initial and terminal constraints, for $j = 1, \ldots, \mu$

$$\underline{z}_{0|k}^{(\mu)} \le V x_k \le \bar{z}_{0|k}^{(\mu)} \tag{7.60a}$$

$$|\bar{z}_{N|k}^{(j)}| \le z_T^{(j)} \tag{7.60b}$$

$$|\underline{z}_{N|k}^{(j)}| \le z_T^{(j)} \tag{7.60c}$$

Here $A^+ \doteq \max\{A, 0\}$ and $A^- \doteq \max\{-A, 0\}$ denote the absolute values of the positive and negative elements of a matrix A.

Having defined the constraints of the online optimization problem, we now formulate a stochastic MPC algorithm with the objective of minimizing the expected quadratic cost $J(x_k, \mathbf{c}_k)$ of Sect. 7.2.1. This requires the online solution of a quadratic program.

Algorithm 7.4 At each time instant $k = 0, 1, \ldots$:

1. Solve the optimization

$$\underset{\substack{\mathbf{c}_k,\\ (\bar{z}_{0|k}^{(1)},\ldots,\bar{z}_{0|k}^{(\mu)})\cdots(\bar{z}_{N|k}^{(1)},\ldots,\bar{z}_{N|k}^{(\mu)})\\ (\underline{z}_{0|k}^{(1)},\ldots,\underline{z}_{0|k}^{(\mu)})\cdots(\underline{z}_{N|k}^{(1)},\ldots,\underline{z}_{N|k}^{(\mu)})}}{\text{minimize}} \quad J(x_k, \mathbf{c}_k) \tag{7.61}$$

subject to (7.42a, 7.42b), (7.57a, 7.57b), (7.58a), (7.59a) and (7.60a–7.60c)

2. Implement the control law $u_k = K x_k + c_{0|k}^*$ where $\mathbf{c}_k^* = (c_{0|k}^*, \ldots, c_{N-1|k}^*)$. $\lhd$

By construction, this algorithm is recursively feasible and hence Theorem 7.1 demonstrates that the closed-loop system satisfies a quadratic stability condition. These properties can be summarized as follows.

Corollary 7.2 *For the system (7.1a, 7.1b) with the control law of Algorithm 7.4, if $\tilde{p}_{j,m}$ and p_j, $j, m = 1, \ldots, \nu$ satisfy (7.56), then the optimization (7.61) is recursively feasible and the closed-loop system satisfies the constraints (7.44) and the asymptotic mean-square bound*

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}_0\big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) \le l_{ss}. \tag{7.62}$$

We conclude this section by noting that, in the interests of optimality, K should ideally be chosen as the unconstrained optimal feedback gain discussed in Sect. 6.2. However it may be necessary to detune this feedback gain in order to use the methods of Sect. 5.4.1 in the design of V and the terminal set parameters $\bar{z}_T^{(j)}$. It should also be noted that the number of optimization variables can be reduced by using tube cross sections that are parameterized by scalar variables, for example by redefining $\mathcal{X}_{i|k}^{(j)}$ as the set $\{z : |z - z_{i|k}^{(0)}| \le \alpha_{i|k}^{(j)} \bar{z}_T^{(j)}\}$, where the scalar $\alpha_{i|k}^{(j)}$ and the vector $z_{i|k}^{(0)}$ are decision variables.

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Chapter 8
Explicit Use of Probability Distributions in SMPC

The previous chapter introduced the use of tubes with ellipsoidal or polytopic cross sections in stochastic MPC. However the probabilistic constraints on predicted states and control inputs were handled using confidence regions for stochastic model parameters, namely sets determined offline that contain the uncertain parameters of the model with a specified probability. This provides a computationally convenient but indirect way to exploit the available information on the probability distribution of uncertainty.

In this chapter the probability distributions of model parameters are directly employed in the formulation of constraints in the online MPC optimization. We do this first in the context of stochastic additive model disturbances. Then, towards the end of this chapter, we consider stochastic multiplicative model uncertainty.

Sections 8.1 and 8.2 consider pointwise-in-time probabilistic constraints that apply to individual scalar random variables for problems with additive disturbances. This makes it possible to transfer the majority of the computational burden of handling probabilistic constraints to offline calculations, and thus allows highly efficient online implementations. Section 8.3 describes a method of adapting constraints according to the number of past constraint violations with the aim of maintaining a specified average rate of violations in closed-loop operation. Still considering the case of additive model uncertainty, Sect. 8.4 deals with joint probabilistic constraints that apply simultaneously to several random variables by constructing stochastic tubes containing the uncertain component of the predicted model state. Section 8.5 is likewise concerned with the design of stochastic tube MPC strategies, but here polytopic tubes are constructed online for the case in which both multiplicative and additive model uncertainty are present.

In Sects. 8.1 and 8.2, we avoid discussing the details of the computational methods that are needed to determine probabilistic bounds on random variables. This is possible because these sections deal with simple cases of scalar random variables, which can be handled for example by numerical integration or random sampling

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performed offline. Numerical methods of computing probabilistic bounds are discussed in more detail in Sects. 8.4 and 8.5. In particular, the approach of Sect. 8.4 lends itself to methods of computing probabilistic bounds that are based on numerical integration, whereas that of Sect. 8.5 is naturally suited to an online optimization based on random sampling.

8.1 Polytopic Tubes for Additive Disturbances

We begin by considering linear systems that are subject only to additive model uncertainty. As in Chaps. 3 and 4, the system model is given by

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \tag{8.1}$$

with state $x_k \in \mathbb{R}^{n_x}$, which is assumed to be known to the controller at time k, control input $u_k \in \mathbb{R}^{n_u}$, and disturbance input $w_k \in \mathbb{R}^{n_w}$ which is unknown to the controller at time k. The disturbance w_k is taken to be the realization at time k of a bounded i.i.d. random variable. Unlike the model employed in Chaps. 3 and 4, the probability distribution of w_k is assumed to be known. We further assume that the distribution of w_k is finitely supported with $w_k \in \mathcal{W}$ for all k, where $\mathcal{W}$ is a compact convex polytopic set that contains the origin.

The aim of the stochastic MPC strategy is to minimize a predicted cost, which is designed to ensure that the closed-loop system is stable in a suitable sense, subject to a pointwise-in-time probabilistic constraint of the form (6.8). However, in this section and in Sects. 8.2 and 8.3, we treat (6.8) as a collection of n_C individual probabilistic constraints, each of which is defined by a row of F and G, namely

$$\Pr_k(F_j x_{1|k} + G_j u_{1|k} \leq 1) \geq p, \quad j = 1, \ldots, n_C \tag{8.2}$$

where F_j, G_j denote the jth rows of F, G, and p is a specified probability. This differs from the pointwise-in-time probabilistic constraints considered in Chaps. 6 and 7, as well as those to be considered in Sects. 8.4 and 8.5, which are treated as constraints on the probability that all elements of a random vector should not exceed a given threshold.

Hard constraints can be included in the problem formulation by setting the probability p in (8.2) equal to 1. Also the treatment of intersections of constraint sets corresponding to different probabilities presents no particular challenges. As discussed in Sect. 7.1, the assumption of a compact bounding set $\mathcal{W}$ for w_k is needed in order to establish a guarantee of recursive feasibility.

The control strategy considered in this section minimizes the infinite horizon expected quadratic cost discussed in Sect. 6.2, subject to the constraint (8.2), with the aim of asymptotically steering the state to a neighbourhood of the origin. This asymptotic target set can be taken to be the minimal robust invariant set of Definition 3.4 under the control law $u = Kx$, with K defined as the unconstrained optimal state feedback gain discussed in Sect. 6.2. We thus formulate the stochastic MPC objective as the minimization of the predicted cost of (6.15):

$$\sum_{i=0}^{\infty} \mathbb{E}_k\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}\big) \tag{8.3}$$

where $l_{ss} = \mathrm{tr}\big(\Theta(Q + K^T RK)\big)$ and Θ is the solution of the Lyapunov equation (6.10). Note that, for the case considered here, the expectation operator on the LHS of (6.10) is not needed because the model (8.1) is not subject to multiplicative uncertainty. For the dual prediction mode strategy, namely

$$u_{i|k} = Kx_{i|k} + c_{i|k} \tag{8.4}$$

with $c_{i|k} = 0$ for all $i \geq N$, Theorem 6.1 shows that the cost (8.3) can be expressed as a quadratic function, $J(x_k, \mathbf{c}_k)$, of the vector of degrees of freedom $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$. Furthermore, given that K is the unconstrained optimal feedback gain, Corollary 6.1 gives

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \mathbb{E}_k\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}\big) = x_k^T W_x x_k + \mathbf{c}_k^T W_c \mathbf{c}_k + w_1$$

where $w_1 = -\mathrm{tr}(W_x \Theta)$ and W_x, W_c are given by (6.23)–(6.24), which reduce to the certainty equivalent conditions of Theorem 2.10 since no multiplicative uncertainty is included in the model (8.1). The minimization of the predicted cost is to be performed subject to the constraint (8.2), which is to be invoked in a manner that allows a guarantee of recursive feasibility of the online MPC optimization problem. This results in a set of constraints that apply at each time step of the initial N-step prediction horizon (Mode 1) and a terminal constraint requiring $x_{N|k}$ to lie in a set that is robustly invariant under the terminal control law $u = Kx$.

When considering constraints, it is convenient to use the decomposition of prediction dynamics introduced in Sect. 3.2:

$$s_{i+1|k} = \Phi s_{i|k} + Bc_{i|k} \tag{8.5a}$$

$$e_{i+1|k} = \Phi e_{i|k} + Dw_{i|k} \tag{8.5b}$$

where $\Phi = A + BK$ and $x_{i|k} = s_{i|k} + e_{i|k}$, with $s_{0|k} = x_k$ and $e_{0|k} = 0$. The convenience of this decomposition is due to the fact that the nominal system in (8.5a) is deterministic and the effects of uncertainty are treated by (8.5b). Furthermore, the uncertain component, $e_{i|k}$, of the predicted state is independent of x_k since the initial condition for (8.5b) is taken to be $e_{0|k} = 0$ and $\{w_0, w_1, \ldots\}$ is a stationary process. Hence the effects of model uncertainty on the constraints of the problem can be computed offline. As explained in this section, which is based on the approach of

[1, 2], the explicit use of such information makes it possible to construct constraints that are as tight as possible for the open-loop dual-mode prediction strategy of (8.4).

In order to formulate the constraints of the stochastic MPC algorithm, we first consider the conditions on the vector $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ of decision variables under which the constraints (8.2) are satisfied at each future time step of an infinite prediction horizon.

Lemma 8.1 *The predictions generated by (8.4) and (8.5a, 8.5b) satisfy the constraints* $\Pr_k(F_j x_{i|k} + G_j u_{i|k} \leq \mathbf{1}) \geq p$ *for* $j = 1, \ldots, n_C$ *and* $i = 1, 2, \ldots$ *if and only if*

$$\tilde{F} s_{i|k} + G c_{i|k} \leq \mathbf{1} - \gamma_i, \quad i = 1, 2, \ldots \tag{8.6}$$

where $\tilde{F} = F + GK$ *and where the elements of* $\gamma_i = (\gamma_{i,1}, \ldots, \gamma_{i,n_C})$ *are defined by*

$$\begin{aligned} \gamma_{i,j} \doteq \min_{\gamma_{i,j}} \quad & \gamma_{i,j} \\ \text{subject to} \quad & \Pr\Big(\tilde{F}_j(\Phi^{i-1} D w_i + \cdots + D w_1) \leq \gamma_{i,j}\Big) \geq p \end{aligned} \tag{8.7}$$

for $j = 1, \ldots, n_C$ *and* $i = 1, 2, \ldots$.

Proof This is a consequence of the predicted control sequence (8.4) and the decomposition $x_{i|k} = s_{i|k} + e_{i|k}$, according to which $F x_{i|k} + G u_{i|k} = \tilde{F} s_{i|k} + G c_{i|k} + \tilde{F} e_{i|k}$, whereas (8.5b) with $e_{0|k} = 0$ implies

$$e_{i|k} = \Phi^{i-1} D w_k + \cdots + D w_{k+i-1}. \tag{8.8}$$

Hence the definition (8.7) implies that the jth element of γ_i has the minimum value such that $\tilde{F}_j e_{i|k} \leq \gamma_{i,j}$ with probability p. It follows that the conditions $\Pr_k(F_j x_{i|k} + G_j u_{i|k} \leq \mathbf{1}) \geq p$ are satisfied if and only if (8.6) holds. □

The constraints in (8.6) are analogous to constraints that were derived in Sect. 3.2 for systems with additive model uncertainty, and which were based on tubes bounding the uncertain components of predicted state and control trajectories. The formulation here likewise uses polytopic uncertainty tubes and hence results in linear constraints on nominal predictions.

A key observation concerns the computation of $\gamma_{i,j}$ in (8.7), which requires knowledge of the probability distribution of $\tilde{F}_j e_{i|k}$. However $e_{i|k}$ does not depend on the system state at time k but is instead a function of additive disturbance realizations whose distributions are known a priori (this explains the absence of the time index k in (8.7)), and hence $\gamma_{i,j}$ can be computed for each given i and j offline. In practice the computation of $\gamma_{i,j}$ in (8.7) has to be performed approximately, for example by numerically approximating the associated convolution integrals or by random sampling methods (see e.g. [3, 4]).

Although the conditions of Lemma 8.1 impose the probabilistic constraints in (8.2) on the predicted state and control trajectories over an infinite prediction horizon, these conditions do not ensure existence of a feasible decision variable $\mathbf{c}_{k+1}$ at time $k+1$ given feasibility at time k. However it is possible to guarantee recursive feasibility by imposing the constraint at time k that $\mathbf{c}_{k+1} = M\mathbf{c}_k$ will be feasible at time $k+1$ for all realizations of w_k, where M is the shift matrix defined in (2.26b). As discussed in Sect. 7.1, this approach requires that the disturbances $w_k, \ldots, w_{k+i-1}$ are handled robustly (by considering their worst-case values) in the probabilistic constraints imposed at time k on the predicted state and control input i steps ahead, whereas w_{k+i} is treated probabilistically in these i steps ahead constraints. The resulting constraint set is defined as follows.

Theorem 8.1 *If* $\mathbf{c}_k$ *satisfies the constraints defined at time* k *by*

$$\tilde{F}s_{i|k} + Gc_{i|k} \leq \mathbf{1} - \beta_i, \quad i = 1, 2, \ldots \tag{8.9}$$

where β_i *and* α_i *are defined for all* $i \geq 1$ *by*

$$\beta_i \doteq \gamma_1 + \sum_{j=1}^{i-1} a_j \tag{8.10a}$$

$$a_i \doteq \max_{w \in \mathcal{W}} \tilde{F}\Phi^i D w, \tag{8.10b}$$

then the constraints (8.2) hold, and, at time $k+1$, $\mathbf{c}_{k+1} = M\mathbf{c}_k$ *will necessarily satisfy* $\tilde{F}s_{i|k+1} + Gc_{i|k+1} \leq 1 - \beta_i$ *for* $i = 1, 2, \ldots$.

Proof First note that, in order that the constraints (8.2) hold for $\mathbf{c}_k = M^k\mathbf{c}_0$ at all times $k \geq 0$ it is sufficient (and also necessary) that: (i) the i steps ahead constraints $\Pr_0(F_j x_{i|0} + G_j u_{i|0} \leq \mathbf{1}) \geq p, j = 1, \ldots, n_C$ hold for all $i \geq 1$ at time $k = 0$, and (ii) for each $i \geq 1$ these constraints are feasible with $\mathbf{c}_k = M^k\mathbf{c}_0$ at times $k = 1, \ldots, i-1$. To prove the theorem, we show that both (i) and (ii) are ensured by the constraints of (8.9) time $k = 0$.

Consider the constraints $\Pr_0(F_j x_{i|0} + G_j u_{i|0} \leq \mathbf{1}) \geq p, j = 1, \ldots, n_C$, which apply to the i steps ahead state and control input predicted at time $k = 0$ for some particular i. By Lemma 8.1, these constraints are satisfied if (8.6) holds at time $k = 0$.

To ensure that at time $k = 1$ the predictions generated by $\mathbf{c}_1 = M\mathbf{c}_0$ will satisfy the $i-1$ steps ahead constraints, we also require that

$$\Pr_1\Big(\max_{w_0 \in \mathcal{W}} (F_j x_{i|0} + G_j u_{i|0}) \leq \mathbf{1}\Big) \geq p, \quad j = 1, \ldots, n_C$$

holds at time $k = 0$. Using $Fx_{i|k} + Gu_{i|k} = \tilde{F}s_{i|k} + Gc_{i|k} + \tilde{F}e_{i|k}$ and (8.8), this condition can be expressed as the constraint

$$\tilde{F}s_{i|0} + Gc_{i|0} \le \mathbf{1} - \gamma_{i-1} - \max_{w_0 \in \mathcal{W}} \tilde{F}\Phi^{i-1}Dw_0$$
$$= \mathbf{1} - \gamma_{i-1} - a_{i-1}.$$

To ensure that the predictions generated at time $k = 2$ by $\mathbf{c}_2 = M^2\mathbf{c}_0$ will satisfy the $i-2$ steps ahead constraints, we require that

$$\Pr_2\Big(\max_{\{w_0,w_1\} \in \mathcal{W}\times\mathcal{W}} (F_j x_{i|0} + G_j u_{i|0}) \le \mathbf{1}\Big) \ge p, \quad j = 1, \ldots, n_C$$

holds at time $k = 0$. From (8.8) this is equivalent to

$$\tilde{F}s_{i|0} + Gc_{i|0} \le \mathbf{1} - \gamma_{i-2} + \max_{w_0 \in \mathcal{W}} \tilde{F}\Phi^{i-1}Dw_0 - \max_{w_1 \in \mathcal{W}} \tilde{F}\Phi^{i-2}Dw_1$$
$$= \mathbf{1} - \gamma_{i-2} - a_{i-2} - a_{i-1}.$$

Repeating this argument for the predictions generated by $\mathbf{c}_k = M^k\mathbf{c}_0$ at times $k = 3, \ldots, i-1$, we obtain the conditions:

$$\tilde{F}s_{i|0} + Gc_{i|0} \le \mathbf{1} - \max\big\{\gamma_i,\ (\gamma_{i-1} + a_{i-1}),\ \ldots,\ (\gamma_1 + a_1 + \cdots + a_{i-1})\big\}. \tag{8.11}$$

But $\gamma_i \le \gamma_{i-1} + a_{i-1}$ for all $i > 1$ since (8.7) and (8.10b) imply that

$$\Pr\Big(\tilde{F}_j(\Phi^{i-1}Dw_i + \cdots + Dw_1) \le \gamma_{i-1,j} + a_{i-1,j}\Big) \ge p, \quad j = 1, \ldots, n_C,$$

and it follows that (8.11) is equivalent to the constraint $\tilde{F}s_{i|0} + Gc_{i|0} \le 1 - \beta_i$. These conditions, when invoked for all $i \ge 1$ and any given $k \ge 0$, are equivalent to (8.9). □

The constraints of (8.9) consist of an infinite number of inequalities which correspond to constraints applied to the predicted trajectories of the model (8.1) over an infinite prediction horizon. However these conditions can be expressed equivalently in terms of a finite number of inequalities using the approach of Sect. 3.2.1. Of course, these conditions are only meaningful if the constraints (8.9) are feasible for some x_k and $\mathbf{c}_k$, and, since the prediction model is by assumption stable, we therefore require that β_i is strictly less than 1 for all $i \ge 1$. The following lemma shows that the limit $\bar{\beta} \doteq \lim_{i\to\infty} \beta_i$ exists and provides bounds on its value.

Lemma 8.2 *The sequence* $\{\beta_1, \beta_2, \ldots\}$ *is monotonically non-decreasing and converges to a limit* $\bar{\beta} \doteq \lim_{i\to\infty} \beta_i$*, where the lth element of* $\bar{\beta}$ *is bounded by*

$$\bar{\beta}_l \le \gamma_{1,l} + \sum_{j=1}^{\rho-1} a_{j,l} + \frac{\lambda^\rho}{1-\lambda}\|\tilde{F}_l^T\|_S, \quad l = 1, \ldots, n_C \tag{8.12}$$

for any non-negative integer ρ, and for $S \succ 0$ and λ satisfying the conditions

$$\max_{w \in \mathcal{W}} \|Dw\|_{S^{-1}} \le 1 \tag{8.13a}$$

$$\Phi S \Phi^T \preceq \lambda^2 S, \quad \lambda \in [0, 1). \tag{8.13b}$$

Proof The non-decreasing property of the sequence $\beta_1, \beta_2, \ldots$ follows from the definition (8.10a) and from the fact that $a_i \ge 0$ for all i, which follows from (8.10b). Also (8.10a) implies

$$\bar{\beta} = \lim_{i \to \infty} \beta_i = \gamma_1 + \sum_{i=1}^{\infty} a_i \tag{8.14}$$

and, by condition (8.13a), we have

$$a_{i,l} = \max_{w \in \mathcal{W}} \tilde{F}_l \Phi^i D w \le \max_{\|v\|_{S^{-1}} \le 1} \tilde{F}_l \Phi^i v \le \left\| {\Phi^i}^T \tilde{F}_l^T \right\|_S \tag{8.15}$$

for $l = 1, \ldots, n_C$. However, (8.13b) implies that

$$\left\| {\Phi^i}^T \tilde{F}_l^T \right\|_S \le \lambda \left\| {\Phi^{i-1}}^T \tilde{F}_l^T \right\|_S,$$

which, combined with (8.15), gives $a_{i,l} \le \lambda^i \|\tilde{F}_l^T\|_S$ for $l = 1, \ldots, n_C$. Replacing $a_{i,l}$ in (8.14) with this bound for all $i \ge \rho$ gives the bound in (8.12). □

By assumption Φ is strictly stable and hence (8.13b) will necessarily have solutions for S and $\lambda \in [0, 1)$. These can be scaled so that (8.13a) will be met. Furthermore, it follows from (8.14) and the non-negative property of a_i that the maximum error in the bound in (8.12) on the lth element of $\bar{\beta}$ can be no greater than $\lambda^\rho \|\tilde{F}_l^T\|_S / (1 - \lambda)$, which can be made as small as desired by using a sufficiently large value of ρ. To ensure the existence of feasible x_k and $\mathbf{c}_k$ satisfying the conditions of Theorem 8.1, we therefore assume

$$\bar{\beta} < \mathbf{1}. \tag{8.16}$$

The constraints of (8.9) are more convenient to handle (both in terms of computation and notation) using the lifted prediction dynamics introduced in Sect. 2.7 and extended to the case of additive model uncertainty in Sect. 3.2:

$$\bar{F} \Psi^i z_{0|k} \le \mathbf{1} - \beta_i, \quad i = 1, 2, \ldots$$

where

$$z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Phi & BE \\ 0 & M \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} \tilde{F} & GE \end{bmatrix}$$

with E and M as defined in (2.26b). An equivalent formulation of (8.9) in terms of a finite number of inequalities is given by the following result.

Theorem 8.2 *If (8.16) holds, then for any given z, $\bar{F}\Psi^i z \le \mathbf{1} - \beta_i$ is satisfied for all $i \ge 1$ if and only if*

$$\bar{F}\Psi^i z \le \mathbf{1} - \beta_i, \quad i = 1, 2, \ldots, \nu \tag{8.17}$$

where ν is the smallest integer such that $\bar{F}\Psi^{\nu+1} z \le \mathbf{1} - \beta_{\nu+1}$ holds for all z satisfying (8.17).

Proof The necessity of (8.17) is obvious and we therefore prove sufficiency. For ν satisfying the conditions of the theorem, define $\mathcal{Z}^{(\nu)}$ as the set

$$\mathcal{Z}^{(\nu)} \doteq \{z : \bar{F}\Psi^i z \le 1 - \beta_i, \ i = 1, 2, \ldots, \nu\}$$

then for all $z \in \mathcal{Z}^{(\nu)}$we have:

$$\bar{F}\Psi z \le \mathbf{1} - \beta_1 \tag{8.18a}$$

$$\bar{F}\Psi^{i+1} z \le \mathbf{1} - \beta_{i+1}, \quad i = 1, \ldots, \nu \tag{8.18b}$$

and since $\beta_{i+1} = \beta_i + a_i \ge \beta_i + \tilde{F}\Phi^i D w$ for all $w \in \mathcal{W}$, (8.18b) implies, for all $w \in \mathcal{W}$

$$\bar{F}\Psi^i (\Psi z + \bar{D} w) \le \mathbf{1} - \beta_i, \quad i = 1, \ldots, \nu$$

where $\bar{D} = [D^T \ 0]^T$. Therefore $\mathcal{Z}^{(\nu)}$ is a robustly invariant set for the system with dynamics $z_{k+1} = \Psi z_k + \bar{D} w_k$, $w_k \in \mathcal{W}$ and constraints $\bar{F}\Psi z_k \le \mathbf{1} - \beta_1$. This property and the equivalence of (8.18a) for $z = (x_k, \mathbf{c}_k)$ with the constraints of (8.2) imply that $\bar{F}\Psi^i z \le \mathbf{1} - \beta_i$ holds for all $i \ge 1$ whenever $z \in \mathcal{Z}^{(\nu)}$. □

There necessarily exists a finite integer ν satisfying the conditions of Theorem 8.2 whenever $(\Psi, \bar{F})$ is observable. This can be shown using an argument identical to the one used in the proof of Theorem 3.1. It also follows from the argument of Theorem 3.1 that $\mathcal{Z}^{(\nu)}$ is the maximal RPI set for the dynamics $z_{k+1} = \Psi z_k + \bar{D} w_k$, $w_k \in \mathcal{W}$ and constraints $\bar{F}\Psi z_k \le \mathbf{1} - \beta_1$, and from this it can be concluded that every feasible pair $(x_k, \mathbf{c}_k)$ for the conditions of (8.9) must lie in the set $\mathcal{Z}^{(\nu)}$. The value of ν can be determined by solving a sequence of linear programs to determine for $i = 0, 1, \ldots$ the maximum of $\bar{F}\Psi^{i+1} z$ over $z \in \mathcal{Z}^{(i)}$ in order to check whether $\bar{F}\Psi^{i+1} z \le \mathbf{1} - \beta_{i+1}$ for all $z \in \mathcal{Z}^{(i)}$.

A stochastic MPC algorithm based on the constraint formulation of this section and the quadratic cost $J(x_k, \mathbf{c}_k)$ of (8.3) can be stated as follows. The online optimization in step (i) requires the solution of a QP at each sampling instant.

Algorithm 8.1 At each time instant $k = 0, 1, \ldots$

(i) Perform the optimization:

$$\begin{aligned} &\underset{\mathbf{c}_k}{\text{minimize}} \quad J(x_k, \mathbf{c}_k) \\ &\text{subject to} \quad \bar{F}\Psi^i \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \leq \mathbf{1} - \beta_i, \quad i = 1, \ldots, \nu \end{aligned} \tag{8.19}$$

(ii) Implement the control law $u_k = Kx_k + c^*_{0|k}$ where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N|k})$ is the minimizing argument of (8.19). ◁

The recursive feasibility of the optimization (8.19), is guaranteed by Theorem 8.1. Also the definition of MPC cost $J(x_k, \mathbf{c}_k)$ in 8.3 implies, by Theorem 7.1, that the closed-loop system satisfies a quadratic stability condition. For completeness, these properties are summarized in the following corollary.

Corollary 8.1 *Algorithm 8.1 applied to the system (8.1) is feasible at all times $k = 1, 2, \ldots$ if it is feasible at $k = 0$. The constraints of (8.2) hold for all $k \geq 0$ and the mean-square bound*

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}_0\big(\|x_k\|^2_Q + \|u_k\|^2_R\big) \leq l_{ss} \tag{8.20}$$

where $l_{ss} = \lim_{k\to\infty}(\|x_k\|^2_Q + \|u_k\|^2_R)$, is satisfied along trajectories of the closed-loop system. If $u = Kx$ is the optimal feedback law for the cost (8.3) in the absence of constraints, then x_k converges with probability 1 to the minimal RPI set for the dynamics (8.1) under this feedback law as $k \to \infty$.

8.2 Striped Prediction Structure with Disturbance Compensation in Mode 2

In this section, we consider the system description of (8.1) but the matrix D defining the disturbance input map is, for simplicity, taken to be equal to the identity matrix I. The disturbance input is again stochastic with a known distribution. For convenience, we express the constraints in terms of an output vector $\psi_k \in \mathbb{R}^{nc}$:

$$\psi_k = \tilde{G}x_{k+1} + \tilde{F}u_k \tag{8.21}$$

and hence the constraints take the form

$$\Pr_k(\psi_k \leq h) \geq p. \tag{8.22}$$

As in the case of the constraints assumed in Sect. 8.4, the constrained output variable here depends on the unknown additive disturbance at time k and represents a mixture of state and input constraints. The formulation of constraints in terms of (8.22) rather than (8.2) simplifies the presentation of this section but is no less general than (8.2).

The approach employed in Sect. 8.1 does not make use of disturbance feedback. However, feedback can be used to attenuate the effects of the future disturbances w_{k+j} for $j = 0, \dots, i-1$ on the i steps ahead predicted state and control input $x_{i|k}$ and $u_{i|k}$ at time k. These predicted disturbance values are not known a priori at time k but will be available to the controller at time $k+i$. For this reason, when considering the i steps ahead state and control input we refer in this section to the predictions, $w_{j|k}$, of w_{k+j}, for $j = 0, \dots, i-1$ at time k as known future disturbances (KFD), whereas $w_{j|k}$ for all $j \geq i$ are referred to as unknown future disturbances (UFD). This section describes the approach of [5], which introduces disturbance feedback with a striped structure into the predicted control trajectories of a stochastic MPC strategy.

To cater for KFD, the predicted control inputs and the resulting predicted dynamics are decomposed as

$$u_{i|k} = Kx_{i|k} + c_{i|k} + v_{i|k}, \quad i = 0, 1, \dots \tag{8.23}$$

where $c_{i|k} = 0$ for all $i \geq N$, $\mathbf{c}_k = (c_{0|k}, \dots, c_{N-1|k})$ is an optimization variable at time k and $v_{i|k}$ depends on the KFD linearly, as explained later in this section. We again use the decomposition of prediction dynamics introduced in Sect. 3.2 into nominal ($s_{i|k}$) and uncertain ($e_{i|k}$) components

$$\begin{aligned} s_{i+1|k} &= \Phi s_{i|k} + Bc_{i|k} \\ e_{i+1|k} &= \Phi e_{i|k} + Bv_{i|k} + w_{i|k} \end{aligned}$$

where

$$x_{i|k} = s_{i|k} + e_{i|k} \tag{8.24}$$

with $s_{0|k} = x_k$ and $e_{0|k} = 0$.

Like the striped parameterized tube MPC approach of Sect. 4.2.3 (and unlike the disturbance affine of Sect. 4.2.1), the component v of the predicted control input is applied throughout the infinite prediction horizon, and thereby enables disturbance compensation to extend to the predicted control law of Mode 2. This allows constraints to be relaxed and therefore leads to larger sets of feasible initial conditions. The scheme of (8.24) generates predictions for the constrained output according to

$$\psi_{i|k} = (Gs_{i|k} + Fc_{i|k}) + (Ge_{i|k} + Fv_{i|k}) + \tilde{G}w_{i|k}$$

where

$$G = \tilde{G}\Phi + \tilde{F}K, \quad F = \tilde{G}B + \tilde{F}$$

This leads to the overall vector of constraint output predictions

$$\boldsymbol{\psi}_k = \boldsymbol{\zeta}_k + \boldsymbol{\epsilon}_k + \boldsymbol{\eta}_k \tag{8.25}$$

where

$$\begin{aligned} \boldsymbol{\zeta}_k &= C_3 s_k + C_{1,N}\mathbf{c}_k \\ \boldsymbol{\epsilon}_k &= C_1 \mathbf{v}_k + C_2 \mathbf{w}_k \\ \boldsymbol{\eta}_k &= \operatorname{diag}\{\tilde{G}, \tilde{G}, \tilde{G}, \ldots\}\mathbf{w}_k \end{aligned} \tag{8.26}$$

with bold symbols being used to indicate the entire sequence of predictions over an infinite future horizon; of these only $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N|k})$ is finite-dimensional. Here $C_{1,N}$ denotes the matrix consisting of the first N block-columns of matrix C_1 and the matrices C_1, C_2, C_3 are defined as

$$C_1 = \begin{bmatrix} F & 0 & 0 & \cdots \\ GB & F & 0 & \cdots \\ G\Phi B & GB & F & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ G & 0 & 0 & \cdots \\ G\Phi & G & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad C_3 = \begin{bmatrix} G \\ G\Phi \\ G\Phi^2 \\ \vdots \end{bmatrix}.$$

Equation (8.25) separates the sequence of predicted outputs into three components:

(i) $\boldsymbol{\zeta}_k$, generated by the nominal (disturbance free) dynamics;
(ii) $\boldsymbol{\epsilon}_k$, associated with the KFD;
(iii) $\boldsymbol{\eta}_k$, associated with the UFD.

The first of these is to be controlled by the variables $c_{i|k}, i = 0, \ldots, N-1$, the second can be compensated for through the use of $v_{i|k}, i = 1, 2, \ldots$, while the third is beyond control since it depends on unknown future disturbances.

In the special case of the number of constraints being equal to the number of inputs and where the dynamics defined by the state space model (Φ, B, G, F) are minimum-phase, the effects of the KFD can be completely eliminated by setting $\boldsymbol{\epsilon}_k = 0$, which gives

$$\mathbf{v}_k = -C_1^{-1}C_2\mathbf{w}_k = L\mathbf{w}_k, \tag{8.27}$$

where L is a lower block triangular matrix whose first block row is zero. In this case, the predictions of the component v take the form:

$$\begin{aligned} v_{0|k} &= 0 \\ v_{i|k} &= L_{i,1}w_{0|k} + L_{i,2}w_{1|k} + \cdots + L_{i,j}w_{i-1|k}. \end{aligned} \tag{8.28}$$

In general, however, complete cancellation of the KFD is not be possible and instead the parameters $L_{i,j}$ for $j = 1, \ldots, i$, $j = 1, \ldots, N-1$ could be computed online. This is the approach that is employed by the disturbance affine MPC algorithms proposed by [6, 7] for the robust case, and by [8] for the stochastic case. Clearly the number of degrees of freedom in such algorithms grows quadratically with the prediction horizon, and this could result in unacceptably high online computational loads for long horizons. It was seen in Sect. 4.2.3 that in the case of robust MPC there may be no loss of performance in using a striped prediction structure, particularly if disturbance compensation is allowed to extend into the Mode 2 prediction horizon, in which case SPTMPC can outperform PTMPC.

To develop the idea, consider first the case with no disturbance compensation. As was seen in the previous section, for this case recursive feasibility is achieved through the satisfaction of the constraint

$$C_3 x_k + C_{1,N} \mathbf{c}_k \leq 1 - \beta \tag{8.29}$$

where the ith block elements of β are given by the sum of: (i) γ_1 and (ii) $\sum_{j=0}^{i-1} \max_{w \in \mathcal{W}} G\Phi^j w$. Thus γ_1, the jth element of which is defined here as the minimum value of $\gamma_{1,j}$ satisfying the condition $\Pr(\tilde{G}_j w < \gamma_{1,j}) = p$, for $j = 1, \ldots, n_C$, accounts for the term $\boldsymbol{\eta}_k$ in (8.25) which is associated with the UFD and is treated probabilistically. On the other hand, (ii) accounts for of the term $\boldsymbol{\epsilon}_k$ in (8.25) which is associated with the KFD and therefore has to be treated robustly. It is noted that although (8.29) involves an infinite number of inequalities, as explained in Sect. 8.1 it is only necessary to consider the inequalities implied by the ith block for $i = 1, \ldots, \nu$, for some finite integer ν which is independent of x_k and can be determined offline. The implied inequality is here denoted by

$$\hat{C}_3 x_k + \hat{C}_{1,N} \mathbf{c}_k \leq 1 - \hat{\beta}. \tag{8.30}$$

Allowing now for disturbance compensation, a non-zero vector $\mathbf{v}_k$ can be used to reduce the amount of constraint tightening, β, in (8.29) needed to account for $\boldsymbol{\epsilon}_k$. Adopting a striped structure, $\mathbf{v}_k$ is written as

$$\mathbf{v}_k = L \mathbf{w}_k \tag{8.31}$$

where

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots \\ L_1 & 0 & \cdots & 0 & 0 & \cdots \\ L_2 & L_1 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ L_{N-1} & L_{N-2} & \cdots & L_1 & 0 & \cdots \\ 0 & L_{N-1} & \cdots & L_2 & L_1 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{bmatrix} \tag{8.32}$$

The aim is to choose L so as to reduce the constraint tightening parameter $\hat{\beta}$. In particular, by substituting (8.31) into (8.26), $\tilde{\beta}$ must satisfy

$$(\tilde{C}_1 L + \tilde{C}_2)\mathbf{w}_k \leq \tilde{\beta} - \hat{\gamma} \leq \hat{\beta} - \hat{\gamma} \tag{8.33}$$

where $\hat{\gamma}$ is a vector of commensurate dimensions whose block elements are equal to the corresponding blocks of γ.

In order to keep the computational load low, and in fact to make it comparable to classical MPC, we begin by selecting L offline. This is done in a sequential manner, designing L_1, then L_2, and so on. Not only is such a design computationally convenient but it favours the design of the L parameters that apply at the beginning of the prediction horizon at the expense of those applying at later prediction times, which is consistent with the expectation that constraints will be more stringent during the initial transients and are less likely to be active as the predicted state and control trajectories tend towards the steady state.

Before stating an algorithm for the design of the elements of L, note that $\hat{C}_1 L + \hat{C}_2$ has ν row blocks and that all the column blocks of this matrix beyond the νth are zero. Thus the infinite-dimensional $\mathbf{w}_k$ appearing in (8.33) can be replaced by the vector, denoted $\hat{\mathbf{w}}_k$, that contains only the first ν blocks of $\mathbf{w}_k$. For the same reason, L is replaced by $\hat{L}$, which comprises only the first ν row blocks and column blocks of L. Then let $\hat{\mathcal{W}}$ denote the set $\{\hat{\mathbf{w}}_k : w_{i|k} \in \mathcal{W},\ i = 0, \ldots, \nu\}$ and let E_i be the matrix which is such that $E_i M$ gives the ith block of M.

Algorithm 8.2

(i) Solve the minimization

$$\left(L_1^{(1)}, L_2^{(1)}, \ldots, L_{N-1}^{(1)}, r^{(1)}\right) = \arg \min_{L_1,\ldots,L_{N-1},r} r$$

subject to

$$\begin{aligned} E_1(\hat{C}_1\hat{L} + \hat{C}_2)\hat{\mathbf{w}} &\leq r\mathbf{1}, & \forall\, \hat{\mathbf{w}} \in \hat{\mathcal{W}} \\ (\hat{C}_1\hat{L} + \hat{C}_2)\hat{\mathbf{w}} &\leq r\hat{\beta} - \hat{\gamma}, & \forall\, \hat{\mathbf{w}} \in \hat{\mathcal{W}} \end{aligned}$$

(ii) Then, for each $i = 2, \ldots, N-1$, set $(L_1, \ldots, L_{i-1}) = \left(L_1^{(1)}, \ldots, L_{i-1}^{(i-1)}\right)$ and solve the minimization

$$\left(L_i^{(i)}, L_{i+1}^{(i)}, \ldots, L_{N-1}^{(i)}, r^{(i)}\right) = \arg \min_{L_1,\ldots,L_{N-1},r} r$$

subject to

$$\begin{aligned} E_i(\hat{C}_1\hat{L} + \hat{C}_2)\hat{\mathbf{w}} &\leq r\mathbf{1}, & \forall\, \hat{\mathbf{w}} \in \hat{\mathcal{W}} \\ (\hat{C}_1\hat{L} + \hat{C}_2)\hat{\mathbf{w}} &\leq r\hat{\beta} - \hat{\gamma}, & \forall\, \hat{\mathbf{w}} \in \hat{\mathcal{W}} \end{aligned}$$

◁

Since the constraints of the algorithm depend linearly on $\hat{\mathbf{w}}$, they need only be invoked at the vertices of $\hat{\mathcal{W}}$, which therefore implies that the minimizations in steps (i) and (ii) of the algorithm are linear programming problems. It is also noted that Algorithm 8.2 is necessarily feasible under the assumption that $\lim_{i\to\infty} \beta_i < \mathbf{1}$, since then $L_i = 0$, $i = 1, \ldots, N-1$ gives at least one feasible solution.

The reduced tightening parameters derived by Algorithm 8.2 are given as

$$\tilde{\beta}_i = E_i|C_1 L + C_2|\boldsymbol{\alpha} + \gamma, \quad i = 1, \ldots, N+m \tag{8.34}$$

where $\boldsymbol{\alpha}$ is an infinite-dimensional vector whose blocks are α. By definition these tightening parameters satisfy the condition $\tilde{\beta}_i \le \hat{\beta}_i$, and they can be shown by simple algebra to share the monotonically increasing property of $\hat{\beta}_i$ and to tend to a limit $\tilde{\beta}_\infty \le \beta_\infty$.

Using the approach described in this section, disturbance compensation can be introduced into the stochastic MPC strategy of Sect. 8.1 simply by replacing the constraint tightening parameters $\hat{\beta}_i = \beta_i$ with $\tilde{\beta}_i$. However two further issues must also be addressed. The first concerns the definition of a finite constraint set, which may lead to a different value of ν, say $\tilde{\nu}$. The definition of this is exactly the same as for ν, namely through the conditions (8.17) of Theorem 8.2, but with β_i replaced by $\tilde{\beta}_i$ (and ν replaced by $\tilde{\nu}$). The second issue concerns the extension of disturbance compensation into Mode 2, which causes the steady-state value of the expectation of the predicted stage cost, say $\tilde{l}_{ss}$, to differ from the value $l_{ss} = \mathrm{tr}\big(\Theta(Q + K^T RK)\big)$ that is subtracted from the stage cost of (6.15). This steady state expected stage cost can be computed, given the feedback gain matrix L, through a straightforward modification of the method of computing l_{ss} in Sect. 6.2.

8.3 SMPC with Bounds on Average Numbers of Constraint Violations

The probabilistic constraints considered thus far allow constraint violations to occur but require that the expected frequency of violations remains at all times below a threshold of $1 - p$. Maintaining the frequency of constraint violations below a given threshold is essential in some applications. For example, the specification of a wind turbine control problem might require that the power captured from the wind is maximized while the frequency with which the material stresses in the turbine blades and tower violate certain thresholds is kept below a required rate defined by fatigue damage considerations. In such a scenario, formulating constraints in the probabilistic manner presented in Sects. 8.1 and 8.2 can be conservative because it does not account for the fact that, during certain periods of the past, the average number of constraint violations may fortuitously have been low (for example because of periods of low turbulence). In such circumstances, the controller is able to be more aggressive, causing a higher number of violations while still maintaining the overall average of constraint violations within acceptable limits.

To take advantage of such situations a stochastic MPC strategy is proposed in [9] that controls the average violation of a given constraint

$$g^T x_{k+1} \le h$$

where for convenience we consider here only a single constraint, i.e. $g \in \mathbb{R}^{n_x}$. This average is defined as $v_k = L_k/s_k$ where L_k is the accumulated loss (weighted by a forgetting factor $\gamma \in [0, 1]$):

$$L_k = \sum_{i=0}^{k} \gamma^{k-i} l(g^T x_i - h)$$

and s_k is the normalizing factor $s_k = \sum_{i=0}^{k} \gamma^{k-i}$. The function $l(\cdot)$ is a non-decreasing (and lower semi-continuous) loss function; it could for example be the indicator function that is equal to 1 when $g^T x > h$ and 0 otherwise. The strategy adopted is to try and keep v_k below a given threshold ξ. More precisely, if the current average is below ξ then the constraint

$$\mathbb{E}_k\big(L_{k+1}/s_{k+1}\big) \le \xi \tag{8.35}$$

is employed. Otherwise, namely if $v_k > \xi$, then the aim is to return v_k, with probability 1, to a value below ξ.

Given that $L_{k+1} = \gamma L_k + l(g^T x_{k+1} - h)$ it follows that (8.35) can be written as

$$\mathbb{E}_k\Big(l(g^T x_{k+1} - h)\Big) \le \gamma(\xi s_k - L_k) + \xi$$

and to allow for a relaxation of this constraint at times when L_k is small, this condition is replaced by

$$\begin{aligned} \mathbb{E}_k\Big(l(g^T x_{k+1} - h)\Big) &\le \beta_k \\ \beta_k &= \max\big\{\gamma(\xi s_k - L_k) + \xi, \alpha\big\}, \quad \alpha \le \xi \end{aligned}$$

Thus at times of low average violation (i.e. when $L_k/s_k \le \xi$), we have $\beta_k > \alpha$ thereby resulting in a relaxed constraint which still meets the requirement of (8.35). At all other times, the expected 1-step-ahead loss $\mathbb{E}_k\big(l(g^T x_{k+1} - h)\big)$ is forced to be less than or equal to α thereby ensuring that the average loss will, at some point in the future, be no greater than ξ.

To ensure recursive feasibility, the successor state is constrained to lie at each time instant in a robust controlled invariant set, $\mathcal{S}$, which satisfies the condition:

$$\forall x \in \mathcal{S},\ \exists u \in \mathcal{U} \ \text{ such that } \ Ax + Bu + Dw \in \mathcal{S}, \quad \forall w \in \mathcal{W}, \tag{8.36a}$$

$$\mathbb{E}_k\Big(l\big(g^T(Ax + Bu + Dw) - h\big)\Big) \le \alpha \tag{8.36b}$$

Here $\mathcal{U}$ denotes a set of allowable inputs and $\mathcal{W}$ defines a set of admissible disturbances, with $\mathcal{U}$ and $\mathcal{W}$ assumed to be convex and polytopic. At times of low average constraint violation, restricting the successor state to lie in $\mathcal{S}$ is unnecessarily conservative because, as argued above, it is possible to relax the constraint of the expected loss. Instead it is possible to define a set of nested reachable sets according to the recursion

$$\mathcal{S}_{j+1} = \{x : \exists u \in \mathcal{U} \ \text{ such that } \ Ax + Bu + Dw \in \mathcal{S}_j \ \forall w \in \mathcal{W}\}, \quad \mathcal{S}_1 = \mathcal{S},$$

so that the successor state can be allowed to lie in a set, $\mathcal{S}_{j(k)}$, larger than $\mathcal{S}$ without affecting recursive feasibility. The choice of the largest index $j(k)$ which retains recursive feasibility depends on the current value of accumulated constraint violations, as discussed in [9]. The overall constraint set then becomes

$$u \in \mathcal{U} \tag{8.37a}$$

$$Ax + Bu + Dw \in \mathcal{S}_{j(k)} \quad \forall w \in \mathcal{W} \tag{8.37b}$$

$$\mathbb{E}_k\Big(l\big(g^T(Ax + Bu + Dw) - h\big)\Big) \le \beta_k \tag{8.37c}$$

and this can be grafted into a stochastic MPC algorithm with guaranteed recursive feasibility.

In Sects. 8.1 and 8.2 probabilistic constraints involving affine functions of future disturbances were converted in to affine constraints on the degrees of freedom through numerical integration techniques. Similarly here, given the assumptions on the loss function $l(\cdot)$, it is possible to use numerical integration to convert (8.36b) into an affine inequality in u. In particular, (8.36b) can be written as

$$g^T(Ax + Bu) \le h + q(\alpha)$$

where

$$q(\alpha) = \sup\left\{\mu : \int_{-\infty}^{\infty} l(\mu + y) f_{g^T Dw}(y)\, dy \le \alpha\right\}$$

with $f_{g^T Dw}(y)$ denoting the probability density function of $g^T Dw$. The same reasoning applies to the constraint (8.37c), with $q(\alpha)$ replaced by $q(\beta_k)$, however in this case β_k is not known a priori. To avoid computing $q(\beta_k)$ online, it is possible to compute $q(\hat{\beta}_j)$ for a large number of predefined points $\hat{\beta}_j$ in the range of possible values for β_k and select the value of $\hat{\beta}_{j(\beta_k)}$ that is nearest to β_k and such that $\hat{\beta}_{j(\beta_k)} \le \beta_k$. Note that, since (8.36b) can be converted into an inequality which is affine in x and u, it follows that $\mathcal{S}$ and hence $\mathcal{S}_j$ are polyhedra. These can be determined as the maximal such sets or as inner approximations of the maximal sets; the latter offering a compromise between the demands of computation of these sets and the relaxation of (8.37b).

We conclude the section by noting that a stochastic MPC algorithm which minimizes an appropriate predicted cost subject to the constraints

$$\begin{aligned} u &\in \mathcal{U} \\ Ax + Bu + Dw &\in \mathcal{S}_{j(k)} \quad \forall w \in \mathcal{W} \\ g^T(Ax + Bu + Dw) &\leq h + q\big(\hat{\beta}_{j(\beta_k)}\big) \end{aligned}$$

will preserve feasibility at all future times given feasibility at initial time, and in closed-loop operation will meet the average violation constraints, either instantaneously or with probability 1 at some time in the future.

8.4 Stochastic Quadratic Bounds for Additive Disturbances

Each of the probabilistic constraints considered in Sects. 8.1–8.3 is specified in terms of a bound on the probability of a scalar linear function exceeding a given threshold. This leads to computationally efficient algorithms in which the bulk of the computation that is needed to convert probabilistic constraints on random variables into deterministic conditions on optimization variables is performed offline. We now return to the case of multiple linear constraints that are required to jointly hold with a specified probability. Using the approach of [10], this can be achieved by constructing tubes with ellipsoidal cross sections that are defined on the basis of information on the distribution of a quadratic function of the unknown model disturbance input, w_k. The approach is computationally convenient since it characterizes the effects of model uncertainty on the predicted state and control trajectories in terms of a scalar stochastic variable that defines the scaling of the tube cross sections.

We consider again the model of (8.1), which is subject to an i.i.d. additive disturbance sequence $\{w_0, w_1, \ldots\}$, where the probability distribution of w_k is known and assumed to be finitely supported. The system is assumed to be subject to the pointwise-in-time probabilistic constraints of (6.8), defined in terms of given matrices F and G and a specified probability $p \in (0, 1]$ by

$$\Pr_k(Fx_{1|k} + Gu_{1|k} \leq \mathbf{1}) \geq p, \tag{8.38}$$

where, as before, $x_{i|k}$ and $u_{i|k}$ denote the i steps ahead predictions at time k of the system state x_{k+i} and control input u_{k+i}. Hard constraints can be handled in this framework by setting the probability p equal to 1.

In the following development, we consider ellipsoidal sets defined for given $V_w \succ 0$ by

$$\mathcal{E}_w(\alpha) = \{w : \ w^T V_w w \leq \alpha\},$$

where α is a random variable whose realization at time k is defined by $\alpha_k \doteq w_k^T V_w w_k$. We assume that it is possible to determine the distribution of α from knowledge of the distribution of w, and in particular we assume that the cumulative distribution function

$$\mathcal{F}_\alpha(a) = \Pr(\alpha_k \leq a)$$

can be computed, for example by numerical integration. The assumption that w_k is bounded implies that $w_k \in \mathcal{W}$ with probability 1 for some bounded set $\mathcal{W}$. On account of this bound it is possible to determine an upper bound $\bar{\alpha}$ for α_k; this is formally stated in the following assumption.

Assumption 8.1 $\mathcal{F}_\alpha(\bar{\alpha}) = 1$, where $\bar{\alpha} = \max_{w \in \mathcal{W}} w^T V_w w$.

Using the dual-mode predicted control strategy of (8.4), the i steps ahead predicted state, $x_{i|k}$, at time k is decomposed into nominal ($s_{i|k}$) and uncertain ($e_{i|k}$) components according to (8.5a, 8.5b). We begin by considering the constraints of the Mode 1 prediction horizon. For each $i > 0$ let $\mathcal{E}_e(\beta_i)$ denote an ellipsoidal set,

$$\mathcal{E}_e(\beta_i) \doteq \{e : e^T V_e e \leq \beta_i\},$$

that contains $e_{i|k}$. Then, since $e_{i|k}$ is a stochastic variable, the minimum β_i such that $e_{i|k} \in \mathcal{E}_e(\beta_i)$ is likewise stochastic. Given the distribution $\mathcal{F}_\alpha$ of the parameter α_k that determines the scaling of the set $\mathcal{E}_w(\alpha_k)$, it is possible to compute a probability distribution for β_i. Moreover a value b_i such that $e_{i|k} \in \mathcal{E}_e(b_i)$ with a probability of at least p can be computed on the basis of this distribution. Then the constraint $\Pr_k(F x_{i|k} + G u_{i|k} \leq \mathbf{1}) \geq p$ can be transformed, using (8.4) and (8.5a, 8.5b), into the linear deterministic constraint

$$\tilde{F} s_{i|k} + G c_{i|k} \leq \mathbf{1} - b_i^{1/2} h,$$

where $\tilde{F} = F + GK$ and the vector $h = (h_1, \ldots, h_{n_C}) \in \mathbb{R}^{n_C}$ is defined by

$$h_j = (\tilde{F}_j V_e^{-1} \tilde{F}_j^T)^{1/2}, \quad j = 1, \ldots, n_C,$$

with $\tilde{F}_j$ denoting the jth row of $\tilde{F}$. The justification for this transformation is that $b_i^{1/2} h_j$ is the attainable upper bound for $\tilde{F}_j e_{i|k}$ over all $e_{i|k} \in \mathcal{E}_e(b_i)$. To simplify notation we use β_i and b_i in preference to $\beta_{i|k}$ and $b_{i|k}$; these variables do not depend on k because the probability distribution of $e_{i|k}$ is independent of k. For the same reason the notation α_i is used here instead of $\alpha_{i|k}$.

To obtain the probability distributions of β_i, $i = 1, 2, \ldots$ we need to determine how $\mathcal{E}_e(\beta_i)$ evolves over the prediction horizon. A recurrence relation governing β_i can be deduced from the requirement that the sequence $\beta_0, \beta_1, \ldots$ must satisfy the 1-step-ahead inclusion condition

$$\max_{e \in \mathcal{E}_e(\beta_i),\, w \in \mathcal{E}_w(\alpha_i)} (\Phi e + D w)^T V_e (\Phi e + D w) \leq \beta_{i+1} \tag{8.39}$$

in order to ensure that $e_{i+1|k} \in \mathcal{E}_e(\beta_{i+1})$ whenever $e_{i|k} \in \mathcal{E}_e(\beta_i)$. However, the problem of determining the minimum β_{i+1} satisfying (8.39) is nonconvex (in fact it is NP-complete [11]), and instead we make use of the following sufficient condition.

Theorem 8.3 *The 1-step-ahead inclusion condition, that $e_{i+1|k} \in \mathcal{E}_e(\beta_{i+1})$ whenever $e_{i|k} \in \mathcal{E}_e(\beta_i)$, holds if*

$$\beta_{i+1} = \lambda\beta_i + \alpha_i \tag{8.40a}$$

$$V_e^{-1} - \frac{1}{\lambda}\Phi V_e^{-1}\Phi^T \succeq D V_w^{-1} D^T \tag{8.40b}$$

for some $V_e \succ 0$ and $\lambda > 0$. Furthermore there exist $V_e \succ 0$ and $\lambda \in (0, 1)$ satisfying (8.40b) if Φ is strictly stable.

Proof Using the S-procedure [11], sufficient conditions for (8.39) are given by

$$\beta_{i+1} \geq \lambda\beta_i + \mu\alpha_i \tag{8.41a}$$

$$\begin{bmatrix} \Phi^T \\ D^T \end{bmatrix} V_e \begin{bmatrix} \Phi & D \end{bmatrix} \preceq \lambda \begin{bmatrix} I \\ 0 \end{bmatrix} V_e \begin{bmatrix} I & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ I \end{bmatrix} V_w \begin{bmatrix} 0 & I \end{bmatrix} \tag{8.41b}$$

for some scalars $\lambda, \mu > 0$. Scaling β_i, β_{i+1} and V_e by μ^{-1} removes μ from these conditions. The equivalence of the scaled version of (8.41b) with (8.40b) can be shown using Schur complements, whereas (8.41a) implies (8.40a) since we are interested in the minimum value of β_{i+1} satisfying (8.39). If all eigenvalues of Φ are no greater than ρ in absolute value and $\rho < 1$, then (8.41a) has a solution $V_e \succ 0$ whenever $\lambda \in (\rho^2, 1)$ since, for any $S = S^T \succ 0$ and λ in this interval, the Lyapunov matrix equation $V - \frac{1}{\lambda}\Phi V \Phi^T = S$ has a solution $V \succ 0$. □

Theorem 8.3 makes it possible to propagate the distribution of β_i over the prediction horizon given the distribution of β_0. Before doing this we state the following corollary to Theorem 8.1.

Corollary 8.2 *If $\lambda \in (0, 1)$, then β_i lies in the interval $\beta_i \in [0, \bar{\beta}_i]$ for all i, where $\bar{\beta}_0 = 0$ and*

$$\bar{\beta}_{i+1} = \lambda\bar{\beta}_i + \bar{\alpha}$$

Furthermore $\bar{\beta}_i \leq \bar{\beta}$ for all i, where $\bar{\beta} \doteq \bar{\alpha}/(1-\lambda)$.

Proof If $\beta_i \in [0, \bar{\beta}_i]$ for some i, then from (8.40a) and Assumption 8.1 we obtain $\beta_{i+1} \in [0, \lambda\bar{\beta}_i + \bar{\alpha}]$, and, since $e_{0|k} = 0$ implies $\beta_0 = \bar{\beta}_0 = 0$, it follows that $\beta_i \in [0, \bar{\beta}_i]$ for all i. For any given $\lambda \in (0, 1)$ the asymptotic bound $\lim_{i\to\infty} \bar{\beta}_i = \bar{\beta} = \bar{\alpha}/(1-\lambda)$ therefore holds, and from $\bar{\beta} - \bar{\beta}_{i+1} = \lambda(\bar{\beta} - \bar{\beta}_i)$ we have $\bar{\beta}_i \leq \bar{\beta}_{i+1} \leq \bar{\beta}$ for all i. □

The recursion in (8.40a) expresses β_{i+1} as the sum of two random variables, namely $\lambda\beta_i$ and α_i. Therefore the distribution function for β_{i+1} is given by a convolution integral (see e.g. [12]),

$$\mathcal{F}_{\beta_{i+1}}(\tau) = \lambda \int_0^{\bar{\beta}} \mathcal{F}_{\beta_i}(\theta) f_\alpha(\tau - \lambda\theta) d\theta, \tag{8.42}$$

where $f_\alpha(\cdot)$ is the probability density function of α and $\mathcal{F}_{\beta_i}(\cdot)$ is the cumulative distribution function of β_i. In general it is not possible to perform this integration analytically but it can be approximated using a Markov chain similar to those introduced in Sects. 7.3 and 7.4.

Consider, for example, subdividing the interval $[0, \bar{\beta}]$ into r intervals $[\tau_i, \tau_{i+1})$, $i = 0, \ldots, r-1$, where

$$0 = \tau_0 < \tau_1 < \cdots < \tau_r = \bar{\beta},$$

and approximating the distribution function $\mathcal{F}_{\beta_i}(\tau)$ by a piecewise constant function $\hat{\mathcal{F}}_{\beta_i}$, defined by

$$\hat{\mathcal{F}}_{\beta_i}(\tau) = \begin{cases} \pi_{i,j} & \tau \in [\tau_j, \tau_{j+1}) \\ \pi_{i,r} = 1 & \tau \geq \tau_r. \end{cases}$$

Under mild assumptions on the continuity of f_α (see e.g. [10]), it can be shown that a generic numerical quadrature approximation of (8.42) provides uniform convergence of the approximation error, $\mathcal{F}_{\beta_i} - \hat{\mathcal{F}}_{\beta_i} \to 0$ for given i, as $\max_j(\tau_{j+1} - \tau_j) \to 0$. Let π_i denote the vector $\pi_i = (\pi_{i,0}, \ldots, \pi_{i,r})$. Then numerical integration applied to the convolution integral (8.42) results in a linear relationship defining π_{i+1} in terms of π_i:

$$\pi_{i+1} = P\pi_i. \tag{8.43}$$

where $\pi_0 = \mathbf{1}$ since $\beta_0 = 0$ implies $\mathcal{F}_{\beta_0}(\tau) = 1$ for all $\tau \geq 0$.

The transition matrix P in (8.43) has as elements the probabilities $p_{l,m}$ that β_{i+1} lies in the interval $[0, \tau_l)$ given that β_i lies in the interval $[0, \tau_m)$. Thus P in (8.43) differs from the transition matrix Π of (7.45) in Sect. 7.4 in that it relates cumulative probabilities, but it can be converted into analogous form by pre-multiplying (8.43) by the matrix T of (7.53) and writing

$$T\pi_{i+1} = \Pi(T\pi_i), \quad \Pi = TPT^{-1}. \tag{8.44}$$

Therefore it may be concluded (as was done in Sects. 7.3 and 7.4) that TPT^{-1}, and hence also P, has one eigenvalue equal to 1, while all other eigenvalues of P are less than 1 in absolute value. The implication of this is that π_{ss}, the eigenvector of P that corresponds to the eigenvalue at 1 describes the steady-state behaviour of the approximation π_i of $\mathcal{F}_{\beta_i}$ as $i \to \infty$.

Now, by construction P is such that the elements $\pi_{i,j}$ of π_i satisfy the inequality $\pi_{i,j} \le \pi_{i,j+1}$ and $\pi_{i,r} = 1$ so that for any given probability p it is possible to determine the smallest j such that $\beta_i \le \tau_j$ with probability at least p. Formally this can be achieved through the use of the function $b(\pi_i, p)$ defined by

$$\begin{aligned} \operatorname{ind}(\pi_i, p) &= \min\{j : \pi_{i,j} \ge p\}, \\ b(\pi_i, b) &= \tau_j, \quad j = \operatorname{ind}(\pi_i, p). \end{aligned}$$

With this definition, we can state that the i steps ahead prediction $e_{i|k}$ lies with a probability of at least p in the ellipsoid $\mathcal{E}_e\big(b(\pi_i, p)\big)$, i.e.

$$\Pr_k\Big(e_{i|k} \in \mathcal{E}_e\big(b(\pi_i, p)\big)\Big) \ge p. \tag{8.45}$$

The probabilistic inclusion condition in (8.45) defines a stochastic tube with ellipsoidal cross sections $\mathcal{E}_e\big(b(\pi_i, p)\big)$ containing the uncertain component $e_{i|k}$ of the predicted state with probability at least p. Equivalently, it defines a stochastic tube with cross sections $\{s_{i|k}\} \oplus \mathcal{E}_e\big(b(\pi_i, p)\big)$ that will contain the predicted state $x_{i|k}$ with the same probability. It is important to note that these tubes can be computed offline and hence the dimension r of π_i can be taken to be as large as desired. This allows the error in the approximation of the integral in (8.43) to be made insignificant without increasing the online computational load. We next use these tubes to derive linear inequalities that ensure the state predictions satisfy constraints (8.38).

Lemma 8.3 *The constraint* $\Pr_k(Fx_{i|k} + Gu_{i|k} \le \mathbf{1}) \ge p$ *is satisfied by the predictions of the model (8.5) for given i if*

$$\tilde{F}s_{i|k} + Gc_{i|k} \le \mathbf{1} - \big(b(\pi_i, p)\big)^{1/2} h \tag{8.46}$$

where $h_j = (\tilde{F}_j V_e^{-1} \tilde{F}_j^T)^{1/2}$ *for* $j = 1, \dots, n_C$, *and* $\tilde{F} = [\,\tilde{F}_1^T \ \cdots \ \tilde{F}_{n_C}^T\,]^T$.

Proof From the state decomposition (8.5a, 8.5b), we have that $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$ whenever $\tilde{F}e_{i|k} \le \mathbf{1} - (\tilde{F}s_{i|k} + Gc_{i|k})$. But $e_{i|k}$ lies in $\mathcal{E}_e(b)$ with probability p, where $b = b(\pi_i, p)$, and hence $Fx_{i|k} + Gu_{i|k} \le \mathbf{1}$ with probability p if the maximum of each element of $\tilde{F}e$ over all e in the ellipsoid $\mathcal{E}_e(b)$ is no greater than the corresponding element of $\mathbf{1} - (\tilde{F}s_{i|k} + Gc_{i|k})$. This condition is ensured by the inequality of (8.46) since $\max_{e \in \mathcal{E}_e(b)} \tilde{F}_j e = b^{1/2}(\tilde{F}_j V_e^{-1} \tilde{F}_j^T)^{1/2}$. □

Lemma 8.3 allows the probabilistic constraint $\Pr_k(Fx_{i|k} + Gu_{i|k} \le \mathbf{1}) \ge p$ on the i steps ahead predicted state and control input to be imposed for $i = 1, 2, \dots$ through linear constraints on the online optimization variable $\mathbf{c}_k$. However these constraints are not necessarily recursively feasible and hence they cannot ensure the future feasibility of the probabilistic constraint in (8.38). This can be explained using the reasoning of Sect. 7.1 as follows. Suppose $\mathbf{c}_k$ is such that (8.46) is satisfied at time k, thus ensuring that the i steps ahead constraint $\Pr_k(Fx_{i|k} + Gu_{i|k} \le \mathbf{1}) \ge p$ holds for given i. Then, to ensure feasibility of the corresponding $i - 1$ steps ahead probabilistic constraint at time $k+1$, we require that $\mathbf{c}_{k+1} = (c_{1|k}, \ldots, c_{N-1|k}, 0)$ satisfies (8.46), with k and i replaced by $k+1$ and $i-1$, respectively. But at time $k+1$, the disturbance input w_{k+1} has already been realized and therefore it cannot be treated as a stochastic variable; hence the stochastic tube that was used to formulate the probabilistic constraint in (8.46) at time k is no loger valid.

In order to ensure that $\mathbf{c}_{k+1}$ is feasible for the $i-1$ steps ahead probabilistic constraint at time $k+1$, it is necessary at time k to take into account the effect of the worst-case value of w_{k+1} on the i-step-ahead probabilistic constraint. This can be done by considering the stochastic tube $\{\mathcal{E}_e(\bar{\beta}_1), \mathcal{E}_e(\beta_2), \ldots, \mathcal{E}_e(\beta_i)\}$. For this tube the distribution functions $\mathcal{F}_{\beta_j}$ for $j \geq 2$ are again governed by (8.42), but with the initial condition

$$\mathcal{F}_{\beta_1}(\tau) = \begin{cases} 0 & \tau < \bar{\beta}_1 \\ 1 & \tau \geq \bar{\beta}_1 \end{cases}$$

which corresponds to $\beta_1 = \bar{\beta}_1$, and the approximation of $\mathcal{F}_{\beta_j}$ therefore evolves according to (8.43), but with an initial condition π_1 corresponding to $\beta_1 = \bar{\beta}_1$. The constraint that $\Pr_{k+1}(Fx_{i-1|k+1} + Gu_{i-1|k+1} \leq \mathbf{1}) \geq p$ should hold for all realizations of w_{k+1} then has the same form as (8.46), but with the RHS of the inequality adjusted to account for the new value of π_1. The feasibility of this constraint would then be ensured at $k+1$, but we need also to guarantee that it remains feasible at times $k+2, \ldots, k+i-1$, which requires that the worst-case bounds on β_i for $i = 2, \ldots, i-1$ must similarly be taken into account in the constraints imposed at time k.

To simplify the analysis, we introduce the notation $\pi_{i|j}$ to denote the approximation of the distribution of β_i when β_j, for some given $j \leq i$, assumes its maximum value of $\bar{\beta}_j$. We therefore define

$$\pi_{i|j} = P^{i-j}\pi_{j|j}, \qquad \pi_{j|j} = \mathbf{u}(\bar{\beta}_j), \tag{8.47}$$

where the bound $\bar{\beta}_j$ is given by Corollary 8.2 as

$$\bar{\beta}_j = \frac{1-\lambda^j}{1-\lambda}\bar{\alpha} \tag{8.48}$$

and where $\mathbf{u}(\bar{\beta}_j)$ is the vector of 0s and 1s, the lth element of which is equal to 1 if $\tau_{l+1} < \bar{\beta}_j$ and is equal to 0 otherwise (i.e. if $\tau_{l+1} \geq \bar{\beta}_j$). Then ensuring that constraint (8.46) is feasible j steps ahead implies that it must also hold with $b(\pi_{i|0}, p)$ replaced by $b(\pi_{i|j}, p)$. Invoking this argument for all $j = 0, 1, \ldots, i-1$ results in the constraint

$$\tilde{F}s_{i|k} + Gc_{i|k} \leq \mathbf{1} - \left(\max\{b(\pi_{i|0}, p), \ldots, b(\pi_{i|i-1}, p)\}\right)^{1/2} h.$$

This constraint can be simplified using the following result, which is consistent with the intuition that the maximum of the bounds $b(\cdot, p)$ appearing on the RHS corresponds to the case in which β_{i-1} assumes its worst-case value and α_{i-1} is treated as a stochastic variable.

Lemma 8.4 *For all $i \geq 1$ and $0 \leq j < i$ we have $b(\pi_{i|j}, p) \leq b(\pi_{i|i-1}, p)$.*

Proof Given that $\bar{\beta}_{i-1}$ defines an upper bound on β_{i-1} for all uncertainty realizations, we must have $\bar{\beta}_{i-1} \geq b(\pi_{i-1|j}, 1)$ for all $i \geq 1$ and $0 \leq j < i$, and hence $\pi_{i-1|i-1} = \mathbf{u}(\bar{\beta}_{i-1}) \leq \pi_{i-1|j}$. It follows that $\pi_{i|i-1} \leq \pi_{i|j}$ since the elements of P are non-negative, and this implies that $b(\pi_{i|j}, p) \leq b(\pi_{i|i-1}, p)$ for any given $p \in (0, 1]$. □

Applying Lemma 8.4 to the constraints of this section results in conditions that are equivalent to the constraints of (7.7a, 7.7b), formulated for the general case in Sect. 7.1. These constraints and their recursive feasibility property can be summarized as follows.

Theorem 8.4 *The constraints defined at time k by*

$$\tilde{F} s_{i|k} + G c_{i|k} \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h, \quad i = 1, 2, \ldots \tag{8.49}$$

ensure that (8.38) holds. Furthermore if $\mathbf{c}_k$ satisfies (8.49) at time k, then $\mathbf{c}_{k+1} = M\mathbf{c}_k$ will be feasible at time $k+1$ for (8.49) with k replaced by $k+1$, where M is the shift matrix defined in (2.26b).

The conditions of (8.49) impose an infinite number of constraints on the predicted state and control trajectories over an infinite prediction horizon. Using the approach of Sect. 3.2.1 however, it is possible to impose these constraints through a finite number of inequalities. To demonstrate this, we reintroduce the lifted autonomous prediction dynamics of Sect. 3.2 and thus write

$$z_{i+1|k} = \Psi z_{i|k}, \quad z_{0|k} = \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Phi & BE \\ 0 & M \end{bmatrix}$$

where E and M are defined in (2.26b). This allows the predicted trajectories of the state and control input to be generated as $x_{i|k} = [I\ 0]z_{i|k} + e_{i|k}$ and $u_{i|k} = [K\ E]z_{i|k} + Ke_{i|k}$, and hence the constraints (8.49) can be expressed equivalently (and more conveniently) as

$$\bar{F}\Psi^i z_{0|k} \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h, \quad i = 1, 2, \ldots \tag{8.50}$$

where $\bar{F} = [F + GK\ \ GE]$.

Given that the prediction system is stable, the constraints in (8.50) are necessarily satisfied for some x_k and $\mathbf{c}_k$ if and only if the RHS of the inequality is non-negative for all $i \geq 1$. This condition is satisfied if

$$\bar{\beta}^{1/2} h < \mathbf{1} \tag{8.51}$$

since the definition of $\bar{\beta}_i$ as a bound on β_i implies that $b(\pi_{i|i-1}, p) \leq \bar{\beta}_i$, and by Corollary 8.2 the sequence $\{\bar{\beta}_0, \bar{\beta}_1, \ldots\}$ is monotonically non-decreasing with $\lim_{i\to\infty} \bar{\beta}_i = \bar{\beta}$. Under the assumption that (8.51) holds, the following result, which is closely related to Theorem 3.1, shows that the infinite set of inequalities in (8.50) is equivalent to a finite number of inequality constraints.

Theorem 8.5 *If (8.51) holds, then* $\bar{F}\Psi^i z \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h$ *is satisfied for all* $i \geq 1$ *if and only if*

$$\bar{F}\Psi^i z \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h, \quad i = 1, 2, \ldots, \nu \tag{8.52}$$

where ν *is the smallest integer such that* $\bar{F}\Psi^{\nu+1} z \leq \mathbf{1} - \big(b(\pi_{\nu+1|\nu}, p)\big)^{1/2} h$ *holds for all* z *satisfying (8.52).*

Proof Clearly (8.52) must hold in order that $\bar{F}\Psi^i z \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h$ holds for all $i \geq 1$. To show that (8.52) is also sufficient, let $\mathcal{Z}^{(\nu)}$ denote the set

$$\mathcal{Z}^{(\nu)} = \Big\{z : \bar{F}\Psi^i z \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h,\ i = 1, 2, \ldots, \nu\Big\}.$$

Then, under the conditions of the theorem, $z \in \mathcal{Z}^{(\nu)}$ implies

$$\bar{F}\Psi z \leq \mathbf{1} - \big(b(\pi_{1|0}, p)\big)^{1/2} h \tag{8.53a}$$

$$\bar{F}\Psi^i z \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h, \quad i = 2, \ldots, \nu + 1 \tag{8.53b}$$

and, by Theorem 8.4, the constraints in (8.53b) imply that the conditions

$$\bar{F}\Psi^i(\Psi z + \bar{D}w) \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h, \quad i = 1, \ldots, \nu$$

hold for all $w \in \mathcal{W}$, where $\bar{D} = [D^T\ 0]^T$. Thus $\mathcal{Z}^{(\nu)}$ is robustly invariant for the system

$$z_{k+1} = \Psi z_k + \bar{D} w_k, \quad w_k \in \mathcal{W}, \tag{8.54}$$

while (8.53a) implies that $\Pr_k(F x_{1|k} + G u_{1|k} \leq \mathbf{1}) \geq p$ is satisfied for all $z_k \in \mathcal{Z}^{(\nu)}$ and it follows that $\bar{F}\Psi^i z \leq \mathbf{1} - (b(\pi_{i|i-1}, p))^{1/2} h$ holds for all $i \geq 1$ whenever $z \in \mathcal{Z}^{(\nu)}$. □

A straightforward extension of the arguments of Theorem 3.1 can be used to show that there exists a finite ν satisfying the conditions of Theorem 8.5 if $(\Psi, \bar{F})$ is observable, and also that $\mathcal{Z}^{(\nu)}$ is the maximal RPI set for the system (8.54) and the constraints of (8.53a).

We are now in a position to state the stochastic MPC algorithm. Since the constraints (8.52) are linear in the optimization variable $\mathbf{c}_k$, the online MPC optimization problem requires the solution of a QP at each sampling instant.

Algorithm 8.3 At each time instant $k = 0, 1, \ldots$

(i) Perform the optimization:

$$\begin{aligned} &\underset{\mathbf{c}_k}{\text{minimize}} \ J(x_k, \mathbf{c}_k) \\ &\text{subject to } \bar{F}\Psi^i \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \leq \mathbf{1} - \big(b(\pi_{i|i-1}, p)\big)^{1/2} h, \quad i = 1, \ldots, \nu \end{aligned} \tag{8.55}$$

(ii) Implement the control law $u_k = Kx_k + c^*_{0|k}$ where $\mathbf{c}^*_k = (c^*_{0|k}, \ldots, c^*_{N|k})$ is the minimizing argument of (8.55). $\lhd$

The statement of the algorithm presupposes that the parameters V_e and λ have been designed offline. These parameters can be determined for example by solving the optimization problem

$$(V_e^{-1}, \lambda) = \arg \min_{V_e^{-1},\, \lambda \in (0,1)} \frac{\bar{\alpha}}{1-\lambda} \max_j (\tilde{F}_j V_e^{-1} \tilde{F}_j^T) \ \text{ subject to (8.40b)}$$

The rationale behind this optimization is that it minimizes the steady-state effect of the uncertainty on constraints of (8.55) by minimizing the maximum element of the LHS of (8.51). The optimization can be performed by combining a univariate search over $\lambda \in (0, 1)$ with semidefinite programming to compute the optimal V_e for fixed λ.

Algorithm 8.3 has the guarantee of recursive feasibility provided by Theorem 8.4, and, by Theorem 7.1, the use of the cost $J(x_k, \mathbf{c}_k)$ therefore ensures that the closed-loop system satisfies a quadratic stability condition. These properties are summarized as follows.

Corollary 8.3 *Algorithm 8.3 applied to the system (8.1) is feasible at all times $k = 1, 2, \ldots$ if it is feasible at $k = 0$. The closed-loop system satisfies the constraints of (8.38) for all $k \geq 0$ as well as the mean-square bound*

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r-1} \mathbb{E}_0\big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) \leq l_{ss} \tag{8.56}$$

where $l_{ss} = \lim_{k\to\infty}(\|x_k\|_Q^2 + \|u_k\|_R^2)$. If $u = Kx$ is optimal in the absence of constraints for the cost (8.3), then x_k converges with probability 1 to the minimal RPI set for the dynamics (8.1) under this feedback law as $k \to \infty$.

8.5 Polytopic Tubes for Additive and Multiplicative Uncertainty

The system dynamics considered in Sects. 8.1–8.4 of this chapter are subject only to additive disturbances. We now turn to the case of linear models that are subject to stochastic multiplicative uncertainty as well as additive uncertainty. We consider systems described by the model of (6.2)–(6.3):

$$x_{k+1} = A_k x_k + B_k u_k + D w_k \tag{8.57}$$

where the additive disturbance $w_k \in \mathbb{R}^{n_w}$ and the matrices A_k, B_k that contain the model parameters depend linearly on a set of zero-mean random variables $q_k^{(j)}, j = 1, \ldots, m$ so that $(A_k, B_k, w_k) = \big(A(q_k), B(q_k), w(q_k)\big)$ at time k, with

$$\big(A(q), B(q), w(q)\big) = (A^{(0)}, B^{(0)}, 0) + \sum_{j=1}^{m} (A^{(j)}, B^{(j)}, w^{(j)}) q^{(j)}, \tag{8.58}$$

and $(A^{(j)}, B^{(j)}, w^{(j)})$, $j = 0, 1, \ldots, m$, are known, constant parameters. We assume that the probability distribution of $q_k = (q_k^{(1)}, \ldots, q_k^{(m)})$ is known and time invariant, and that q_k and q_j are statistically independent for all $k \neq j$.

The MPC strategy discussed in this section carries a guarantee of recursive feasibility and is designed for problems with mixed hard and probabilistic constraints. As discussed in Sect. 7.1, this requires knowledge of a bounding set, $\mathcal{Q}$, such that $q_k \in \mathcal{Q}$ with probability 1 for all k. For convenience, we assume that $\mathcal{Q}$ is a compact, convex polytope, with known vertices $q_v^{(1)}, \ldots, q_v^{(\nu)}$. Corresponding to each vertex $q_v^{(l)}$ of $\mathcal{Q}$ is a vertex of the uncertainty set for the model parameters (A, B, w), which we denote as $(A_v^{(l)}, B_v^{(l)}, w_v^{(l)})$, so that

$$\big(A_v^{(l)}, B_v^{(l)}, w_v^{(l)}\big) = \big(A(q_v^{(l)}), B(q_v^{(l)}), w(q_v^{(l)})\big)$$

for $l = 1, \ldots, \nu$.

The system of (8.57) is considered to be subject at all times $k = 0, 1, \ldots$ to a mixture of hard constraints:

$$F_h x_k + G_h u_k \leq \mathbf{1}, \tag{8.59}$$

and probabilistic constraints:

$$\Pr_k(F_p x_{1|k} + G_p u_{1|k} \leq \mathbf{1}) \geq p, \tag{8.60}$$

for a given set of matrices F_h, G_h, F_p, G_p and a given probability $p \in (0, 1]$. Here (8.60) requires, similarly to the pointwise probabilistic constraint (6.8), that the joint probability of all elements of the vector $F_p x_{1|k} + G_p u_{1|k}$ being less than or equal to unity should be no less than p, where $F_p x_{1|k} + G_p u_{1|k}$ is the 1 step ahead prediction of $F_p x + G_p u$ at time k. As in Sect. 8.1 it is straightforward to extend the approach of this section to intersections of probabilistic constraints (8.60) with different probabilities p.

Using an open-loop prediction strategy, the predicted control sequence is parameterized in terms of a perturbed linear feedback law:

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \quad i = 0, 1, \ldots$$

with $c_{i|k} = 0, i = N, N+1, \ldots$. The dynamics governing predicted state trajectories therefore assume the form

$$x_{i+1|k} = \Phi_{k+i} x_{i|k} + B_{k+i} c_{i|k} + D w_{k+i}$$

where $\Phi_k = A_k + B_k K$. The feedback gain K is assumed to be stabilizing in the sense that $x_{k+1} = \Phi_k x_k$ is mean-square stable in the absence of constraints. Hence the quadratic predicted cost, which is taked to be (6.15), can be evaluated using the approach of Sect. 6.2.

It was mentioned in Chap. 7 that propagating the effects of multiplicative uncertainty over a prediction horizon can cause computational difficulties because both Φ_{k+i} and $x_{i|k}$ are uncertain. Instead, similarly to the robust MPC strategies considered in Chap. 5, we construct tubes with polytopic cross sections [13, 14], defined by

$$\mathcal{X}_{i|k} = \{x_{i|k} : Vx_{i|k} \leq \alpha_{i|k}\}, \tag{8.61}$$

that contain the predicted state trajectories. The vectors $\alpha_{i|k}$ for $i = 0, \ldots, N$ are treated as variables in the online MPC optimization whereas the matrix $V \in \mathbb{R}^{n_V \times n_x}$ is determined offline and remains fixed online. The choice of V is based on the considerations detailed in Sects. 5.5 and 5.6, summarized by the following assumption.

Assumption 8.2 V is chosen so that the set $\mathcal{X} = \{x : Vx \leq \mathbf{1}\}$ is λ-contractive for the dynamics $x_{k+1} = \Phi_k x_k + D w_k$, for some $\lambda < 1$.

On account of the requirement for a recursively feasible MPC strategy (and also satisfaction of hard constraints, when these are present), the constraint that the predicted state $x_{i|k}$ should lie in the tube cross section $\mathcal{X}_{i|k}$ must be handled robustly. Through the application of Farkas' Lemma discussed in Chap. 5, Lemma 5.6 shows that this is achieved by the conditions, for $H^{(l)} \geq 0$ and $i = 0, 1, \ldots$,

$$\alpha_{i+1|k} \geq H^{(l)} \alpha_{i|k} + V B_v^{(l)} c_{i|k} + V D w_v^{(l)} \tag{8.62a}$$

$$H^{(l)} V = V \Phi_v^{(l)} \tag{8.62b}$$

for $l = 1, \ldots, \nu$, where $\Phi_v^{(l)} \doteq A_v^{(l)} + B_v^{(l)} K$. Recursive feasibility with respect to satisfaction of the hard constraints is then guaranteed by the existence of a matrix $H_h \geq 0$ such that, for $i = 0, 1, \ldots$,

$$H_h\alpha_{i|k} + G_h c_{i|k} \le \mathbf{1} \tag{8.63a}$$

$$H_h V = \tilde{F}_h. \tag{8.63b}$$

where $\tilde{F}_h \doteq F_h + G_h K$.

In order to handle the probabilistic constraints however, we require a probabilistic extension of Farkas' Lemma. This is provided by the following result.

Theorem 8.6 ([14]) *Let*

$$\mathcal{X}_1 = \{x \,:\, V_1 x \le b_1\}, \quad \mathcal{X}_2 = \{x \,:\, \Pr(V_2 x \le b_2) \ge p\}$$

where V_2 and b_2 are random variables. Then $\mathcal{X}_1 \subseteq \mathcal{X}_2$ (i.e. $\Pr(V_2 x \le b_2) \ge p$ for all x such that $V_1 x \le b_1$) if and only if there exists a random variable $H \ge 0$ satisfying

$$HV_1 = V_2 \tag{8.64a}$$

$$\Pr(Hb_1 \le b_2) \ge p. \tag{8.64b}$$

Proof Sufficiency follows from the fact that, by (8.64a), for $x \in \mathcal{X}_1$ we can write

$$V_2 x = HV_1 x \le Hb_1$$

which, from (8.64b), implies that

$$\Pr(V_2 x \le b_2) \ge p.$$

Thus every $x \in \mathcal{X}_1$ also belongs to $\mathcal{X}_2$, thereby implying that $\mathcal{X}_1 \subseteq \mathcal{X}_2$.

To prove necessity, assume that $\mathcal{X}_1 \subseteq \mathcal{X}_2$ holds. Then

$$\Pr(\mu_i \le b_{2,i}) \ge p$$

with

$$\mu_i = \max_x \{V_{2,i} x \,:\, V_1 x \le b_1\} \tag{8.65}$$

where $V_{2,i}$ and $b_{2,i}$ denote the ith row and ith element of V_2 and b_2, respectively. By strong duality, the dual of the linear program (8.65) for a given realization of V_2 gives

$$\mu_i = \min_h \{h^T b_1 \,:\, h^T V_1 = V_{2,i},\ h \ge 0\}. \tag{8.66}$$

Let h_i^* be the minimizing argument of this dual LP and define H as the matrix with ith row equal to h_i^*. Note that h_i^* is a continuous piecewise affine function of the random variable $V_{2,i}$ since (8.66) has the form of a (right-hand side) parametric linear program in the parameter $V_{2,i}$. Hence h_i^* is itself a random variable [15]. The constraints of (8.66) imply that $H \ge 0$ and that H satisfies (8.64a). From the objective of (8.66) it follows that H also satisfies (8.64b). □

Theorem 8.6 can be used to derive conditions which ensure that the predicted states $x_{i|k}$ satisfy the probabilistic constraints of (8.60) for all $x_{i|k} \in \mathcal{X}_{i|k}$, $i = 0, 1, \ldots$, namely that

$$\begin{aligned}
&\Pr_{k+i}(F_p x_{i+1|k} + G_p u_{i+1|k}) \\
&\qquad = \Pr_{k+i}\big(\tilde{F}_p \Phi_{k+i} x_{i|k} + \tilde{F}_p (B_{k+i} c_{i|k} + D w_{k+i}) + G_p c_{i+1|k} \le \mathbf{1}\big) \ge p
\end{aligned}$$

where $\tilde{F}_p = F_p + G_p K$. From (8.64a, 8.64b), these conditions are equivalent to the requirement that there exists $H_p \ge 0$ satisfying, for $i = 0, 1, \ldots$

$$\Pr\big(H_p \alpha_{i|k} + \tilde{F}_p (B c_{i|k} + D w) + G_p c_{i+1|k} \le \mathbf{1}\big) \ge p \tag{8.67a}$$

$$H_p V = \tilde{F}_p \Phi. \tag{8.67b}$$

Since Φ is a function of the random variable q, in general H_p satisfying (8.67b) will also be a random variable. This means the method of handling (8.67a, 8.67b) differs from that of (8.62a, 8.62b) and (8.63a, 8.63b); however given knowledge of the distribution of q it is possible to construct a computationally tractable online optimization, as discussed below.

In summary therefore, a recursively feasible set of conditions that impose the constraints of (8.59–8.60) are:

(i) Tube inclusion constraints—(8.62a, 8.62b);
(ii) Hard constraints—(8.63a, 8.63b);
(iii) Probabilistic constraints—(8.67a, 8.67b).

The degree of conservativeness of the conditions of (8.62), (8.63) and (8.67) would be minimized if V, $H^{(l)}$, H_h and H_p were computed online for each prediction instant, $i = 0, 1, \ldots$. However this strategy is unlikely to be implementable as it would require the solution of a large nonconvex optimization problem online. Instead we discuss how to design these matrices offline. Thus V is to be chosen, as described in Assumption 8.2, so as to define a λ-contractive set for the dynamics $x_{k+1} = \Phi_k x_k$, whereas each of the matrices $H^{(l)}$, H_h and H_p is designed to have minimum row sum with the aim of relaxing the associated constraints. In particular the ith rows of these matrices are selected according to

$$H_i^{(l)\,T} = \arg\min_h \; \mathbf{1}^T h \text{ subject to } h^T V = V_i \Phi_v^{(l)} \text{ and } h \ge 0, \quad l = 1, \ldots, \nu \tag{8.68a}$$

$$H_{h,i}^T = \arg\min_h \; \mathbf{1}^T h \text{ subject to } \; h^T V = \tilde{F}_{h,i} \text{ and } h \ge 0 \tag{8.68b}$$

$$H_{p,i}^T = \arg\min_h \; \mathbf{1}^T h \text{ subject to } \; h^T V = \tilde{F}_{p,i} \Phi \text{ and } h \ge 0. \tag{8.68c}$$

The values of $H^{(l)}$ and H_h in (8.68a) and (8.68b) are fixed for given V, $\Phi_v^{(l)}$ and $\tilde{F}_h$ as the deterministic solutions of a set of linear programs. However (8.68c) specifies H_p as a random variable, the probability distribution of which is defined by the parametric solutions of a set of linear programs. In particular, from (8.58) the linear program (8.68c) is equivalent to

$$
\begin{aligned}
h^*(q) = \ \arg\min_{h} \quad & \mathbf{1}^T h \\
\text{subject to} \quad & h^T V = \tilde{F}_{p,i}(A^{(0)} + B^{(0)}K) + \sum_{j=1}^{m} \tilde{F}_{p,i}(A^{(j)} + B^{(j)}K)q^{(j)} \\
& h \geq 0.
\end{aligned}
\tag{8.69}
$$

Since the constraints of this problem depend linearly on both h and the random variable $q = (q^{(1)}, \ldots, q^{(m)})$, it can be shown that the solution $h^*(q)$ is a continuous, piecewise affine function of q [16]. Thus H_p is given by (8.68c) as a continuous and piecewise affine function of q. For most problems of practical interest it is unlikely to be computationally feasible to determine the probability distribution of H_p by using multiparametric linear programming to solve (8.68c) for H_p as an explicit function of q. Instead (8.68c) can be used as a means of generating random samples of the distribution of H_p given samples of q; the approach is discussed further at the end of this section.

The matrices $H^{(l)}$, H_h and H_p are necessarily sparse in the sense that each of their rows can have at most n_x non-zero elements. This follows from the fact that the problems posed in (8.68a–8.68c) for given q, have $n_V - n_x$ active constraints so that $n_V - n_x$ of the elements of the optimizing h must in each case be zero. This affords computational advantages.

Using the argument of Sect. 7.1, conditions (8.62a), (8.63a) and (8.67a) would ensure recursive feasibility if they were applied over an infinite prediction horizon, but this would of course require an infinite number of constraints and is clearly not implementable. This difficulty can be avoided through the use of terminal conditions, as described in Sect. 5.5. For example, let the sequence of parameters $\{\alpha_{0|k}, \ldots, \alpha_{N|k}\}$ satisfy the constraints of (8.62a), (8.63a) and (8.67a) for $i = 0, \ldots, N-1$ and impose terminal constraints on $\alpha_{N|k}$:

$$H^{(l)}\alpha_{N|k} + VDw_v^{(l)} \leq \alpha_{N|k}, \quad l = 1, \ldots, \nu \tag{8.70a}$$

$$H_h\alpha_{N|k} \leq \mathbf{1} \tag{8.70b}$$

$$\Pr(H_p\alpha_{N|k} \leq \mathbf{1}) \geq p. \tag{8.70c}$$

The following lemma shows that, under Assumption 8.2, the matrices $H^{(l)}$ satisfy the condition $\|H^{(l)}\|_\infty \leq 1$, which is necessary for feasibility of (8.70a). With these constraints we are able to state the following result.

Lemma 8.5 *Under Assumption 8.2, the definition of $H^{(l)}$ in (8.68a) implies that $H^{(l)}\mathbf{1} + VDw_v^{(l)} \leq \lambda\mathbf{1}$ for all $l = 1, \ldots, \nu$.*

Proof Since the set $\{x : Vx \leq \mathbf{1}\}$ is λ-contractive for $x_{k+1} = \Phi_k x_k + Dw_k$, we have $V\Phi_v^{(l)} x + VDw_v^{(l)} \leq \lambda \mathbf{1}$ for all $l = 1, \ldots, \nu$ and x such that $Vx \leq \mathbf{1}$, so the bound $H^{(l)}\mathbf{1} + VDw_v^{(l)} \leq \lambda \mathbf{1}$ follows from the constraints of (8.68a). □

Theorem 8.7 *If V is chosen according to Assumption 8.2, then the constraints of (8.62a), (8.63a) and (8.67a), invoked for $i = 0, \ldots, N-1$, and the terminal and initial constraints, (8.70a–8.70c) and $Vx_k \leq \alpha_{0|k}$, are jointly recursively feasible for the system (8.57–8.58) under the control law $u_k = Kx_k + c_{0|k}$.*

Proof Suppose that $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ and $\{\alpha_{0|k}, \ldots, \alpha_{N|k}\}$ satisfy the constraints of the theorem at time k. Then a feasible set of parameters at time $k+1$ is given by

$$\begin{aligned}
\mathbf{c}_{k+1} &= (c_{1|k}, \ldots, c_{N-1|k}, 0) \\
\alpha_{i|k+1} &= \alpha_{i+1|k}, \quad i = 0, \ldots, N-1 \\
\alpha_{N|k+1} &= \alpha_{N|k}
\end{aligned}$$

since these parameters give $\mathcal{X}_{i|k+1} = \mathcal{X}_{i+1|k}$ for $i = 0, 1, \ldots, N-1$ and $\mathcal{X}_{N|k+1} = \mathcal{X}_{N|k}$. It follows that (8.62a), (8.63a) and (8.67a), with k replaced by $k+1$, hold for $i = 0, \ldots, N-1$. Also the conditions (8.70a–8.70c) are trivially satisfied when k is replaced by $k+1$ if $\alpha_{N|k+1} = \alpha_{N|k}$. Furthermore, we have $Vx_{k+1} \leq \alpha_{0|k+1}$ since $x_{k+1} \in \mathcal{X}_{1|k}$ for all realizations of model uncertainty at time k. □

We can now formulate the stochastic MPC algorithm. For simplicity the objective function to be minimized online is chosen here as the quadratic predicted cost of (6.15) in Sect. 6.2:

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \mathbb{E}\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - l_{ss}\big). \tag{8.71}$$

By Theorem 6.1, this cost is a quadratic function of the vector, $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$, of free variables in the predicted control sequence.

Algorithm 8.4 At each time instant $k = 0, 1, \ldots$

(i) Perform the optimization:

$$\begin{aligned}
&\underset{\substack{\mathbf{c}_k \\ \alpha_{0|k}, \ldots, \alpha_{N|k}}}{\text{minimize}} \quad J(x_k, \mathbf{c}_k) \\
&\text{subject to } \ (8.62\text{a}), (8.63\text{a}), (8.67\text{a}) \text{ for } i = 0, \ldots, N-1, \\
&\qquad\qquad\quad (8.70\text{a-c}) \text{ and } Vx_k \leq \alpha_{0|k}.
\end{aligned} \tag{8.72}$$

(ii) Implement the control law $u_k = Kx_k + c_{0|k}^*$ where $\mathbf{c}_k^* = (c_{0|k}^*, \ldots, c_{N|k}^*)$ is the minimizing argument of (8.72). ◁

The constraints of Algorithm 8.4 are recursively feasible by Theorem 8.7. Furthermore, Theorem 7.1 of Chap. 7.2.1 demonstrates that the closed-loop system satisfies the constraints (8.59–8.60) for all $k = 0, 1, \ldots$ and the optimal value of the cost, $J^*(x_k)$ satisfies

$$\mathbb{E}_k\big(J^*(x_{k+1})\big) \le J^*(x_k) - \big(\|x_k\|_Q^2 + \|u_k\|_R^2 - l_{ss}\big).$$

Therefore the quadratic stability condition holds for the closed-loop system:

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r} \mathbb{E}_0\big(\|x_k\|_Q^2 + \|u_k\|_R^2\big) \le l_{ss}.$$

The online MPC optimization in step (i) of Algorithm 8.4 is not stated in form that can be implemented directly. This is because the constraints (8.67a) and (8.70c) involve products of optimization variables $\alpha_{i|k}$ and $c_{i|k}$ with the random variables H_p and B [19], and because the probability distribution of H_p is implicitly defined by (8.68c). A way to circumvent these difficulties is to use methods for imposing probabilistic constraints based on random sampling [3, 4, 20].

Let $q^{[j]}$, $j = 1, \ldots, n_s$ denote a set of n_s independent samples drawn from the known probability distribution for q. Given these samples, the corresponding samples of B and w:

$$B^{[j]} = B(q^{[j]}), \quad w^{[j]} = w(q^{[j]}), \quad j = 1, \ldots, n_s$$

are generated by (8.58). Likewise samples of H_p are obtained by defining the ith row of $H_p^{[j]}$ using (8.69) as

$$H_{p,i}^{[j]} = \big(h^*(q^{[j]})\big)^T, \quad j = 1, \ldots, n_s.$$

Using this set of samples, the probabilistic constraints (8.67a) and (8.70c) in the online MPC optimization can be approximated using sampled constraints defined by

$$H_p^{[j]}\alpha_{i|k} + \tilde{F}_p(B^{[j]}c_{i|k} + Dw^{[j]}) + G_p c_{i+1|k} + s_{i|k}^{[j]} = \mathbf{1}, \quad i = 0, \ldots, N-1 \tag{8.73a}$$

$$H_p^{[j]}\alpha_{N|k} + s_{N|k}^{[j]} = \mathbf{1} \tag{8.73b}$$

$$s_{i|k}^{[j]} \ge 0, \quad \forall j \in \mathcal{I}_k \subseteq \{1, \ldots, n_s\}, \quad |\mathcal{I}_k| \ge r n_s. \tag{8.73c}$$

Here $|\mathcal{I}_k|$ denotes the number of elements in the set $\mathcal{I}_k$. Thus (8.73c) ensures that the conditions

$$H_p^{[j]}\alpha_{i|k} + \tilde{F}_p(B^{[j]}c_{i|k} + Dw^{[j]}) + G_p c_{i+1|k} \le \mathbf{1}, \quad i = 0, \ldots, N-1$$
$$H_p^{[j]}\alpha_{N|k} \le \mathbf{1}$$

are imposed for all $j \in \mathcal{I}_k$, where $\mathcal{I}_k$ in an index set containing no fewer than $\lceil rn_s \rceil$ of the samples $j \in \{1, \ldots, n_s\}$. The remaining $\lfloor (1-r)n_s \rfloor$ samples are discarded since the corresponding slack variables $s_{i|k}^{[j]}$ are not constrained to be non-negative in (8.73a, 8.73b). Since samples are selected randomly, the constraints (8.73a–8.73c) are not equivalent to (8.67a) and (8.70c) for finite n_s. However it is possible to derive bounds on the probability, which is dependent on n_s and r, such that a solution to the MPC optimization (8.72) with (8.67a) and (8.70c) replaced by (8.73a–8.73c) satisfies the probabilistic constraints of (8.67a) and (8.70c). For details we refer the reader to [4, 17].

This approach therefore approximates the probabilistic constraints by using samples to empirically approximate the distributions of the stochastic variables appearing in (8.72). Since the index set $\mathcal{I}_k$ is an optimization variable, the resulting optimization has the form of a mixed integer quadratic program (MIQP). The effects of varying the number of samples on the confidence of constraint satisfaction are discussed in the following example.

Example 8.1 This example provides a simple illustration of the use of sampling to approximate a probabilistically constrained optimization problem. Consider the minimization

$$\begin{aligned} &\underset{x}{\text{minimize}} \quad f(x) \\ &\text{subject to} \quad g_i(x) \le 0, \quad \forall i \in \mathcal{I} \subseteq \{1, \ldots, n_s\}, \quad |\mathcal{I}| \ge rn_s \end{aligned} \tag{8.74}$$

where the functions $f(x)$ and $g_i(x) = g(x, q^{[i]})$ are convex in the optimization variable x, and where $q^{[i]}$, $i = 1, \ldots, n_s$ are independent samples of a random variable q. For suitable choices of n_s and r, the constraints of (8.74) provide an approximation of the probabilistic constraint

$$\Pr\big(g(x, q) \le 0\big) \ge p. \tag{8.75}$$

Let $F_{n,m}(p)$ denote the binomial distribution function giving the probability of m or fewer successes in n independent trials, each of which has a probability p of success:

$$F_{n,m}(p) \doteq \sum_{i=0}^{m} \binom{n}{i} p^i (1-p)^{n-i}.$$

Then a lower bound on the probability that the solution of the convex program (8.74) satisfies the probabilistic constraint (8.75) is given in [4, 17] as $1 - \epsilon$, where the parameter ϵ satisfies

$$\epsilon \le \binom{\lfloor n_s(1-r) \rfloor + \rho - 1}{\lfloor n_s(1-r) \rfloor} F_{n_s, \lceil rn_s \rceil - \rho}(1-p)$$

Here ρ is the number of support constraints of the problem (8.74), which is essentially the number of constraints of the form $g_i(x) \le 0$ that can be active at the solution of (8.74) (where removal of an active constraint causes a reduction in the value of the objective). For a feasible problem we must clearly have $\rho \le n_x$ [17]. Similarly, the probability that a solution of (8.74) is feasible for (8.75) has an upper bound β, where

$$\beta \le F_{n_s, \lceil rn_s \rceil - 1}(p).$$

The variation with r of these confidence bounds is shown for $p = 0.9$ and various sample sizes n_s, with $\rho = 1$ and $\rho = 2$ in Figs. 8.1 and 8.2. ♢

Example 8.2 This example illustrates the use of random sampling in approximating the probabilistically constrained online MPC optimization of Algorithm 8.4. We consider a system model of the form (8.57) containing multiple independent sources of stochastic uncertainty. The expected values of the system matrices are given by

$$A^{(0)} = \begin{bmatrix} -1.9 & -1.4 \\ 0.7 & 0.5 \end{bmatrix}, \quad B^{(0)} = \begin{bmatrix} 1 \\ -0.25 \end{bmatrix}$$

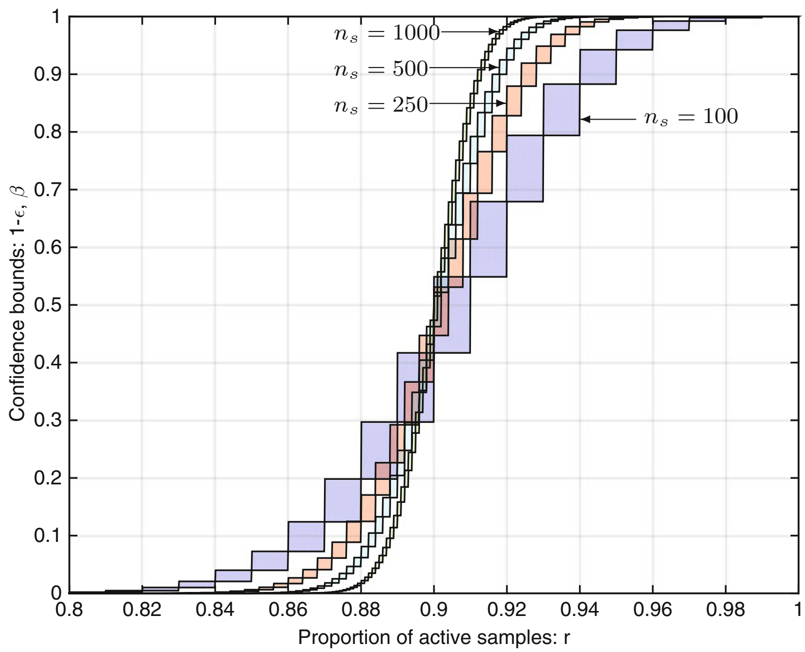

Fig. 8.1 *Upper* and *lower* bounds on the probability that a solution of the sampled program (8.74) satisfies the probabilistic constraint (8.75) for the case of $\rho = 1$

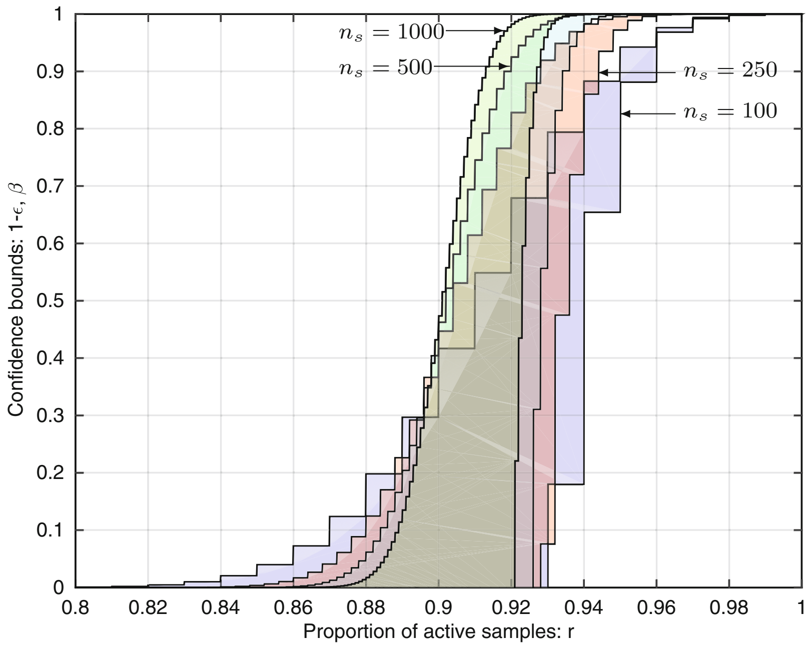

Fig. 8.2 *Upper* and *lower* bounds on the probability that a solution of the sampled program (8.74) satisfies the probabilistic constraint (8.75) with $\rho = 2$

and $D = I$. The realizations of A_k, B_k and w_k are given by

$$\begin{aligned} A_k &= A^{(0)} + A^{(1)} q_k^{(1)} + A^{(2)} q_k^{(2)} + A^{(3)} q_k^{(3)} \\ B_k &= B^{(0)} + B^{(4)} q_k^{(4)} + B^{(5)} q_k^{(5)} \\ w_k &= w^{(6)} q_k^{(6)} + w^{(7)} q_k^{(7)} \end{aligned}$$

where $q^{(j)}$ is a scalar random variable, uniformly distributed on the interval $[-0.5, 0.5]$ for $j = 1, \ldots, 7$, and $q_k = (q_k^{(1)}, \ldots, q_k^{(7)})$ satisfies $\mathbb{E}(q_k q_k^T) = \frac{1}{24} I$ and $\mathbb{E}(q_k q_i^T) = 0$ for all $i \neq k$. The remaining model parameters are

$$A^{(1)} = \begin{bmatrix} 0.03 & 0.15 \\ -0.15 & -0.03 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} -0.03 & -0.15 \\ 0 & -0.03 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} 0 & 0 \\ 0.15 & 0.06 \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} 0.036 \\ -0.024 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} -0.036 \\ 0.024 \end{bmatrix}, \quad w^{(1)} = \begin{bmatrix} 0.04 \\ -0.04 \end{bmatrix}, \quad w^{(2)} = \begin{bmatrix} -0.04 \\ 0.04 \end{bmatrix}.$$

The system is subject to the constraint

$$\Pr_k(Fx_{1|k} \leq 1) \geq 0.9, \quad F = \begin{bmatrix} -5 & 10 \end{bmatrix},$$

and the weighting matrices in the cost (8.71) are defined by $Q = I$ and $R = 1$.

In order to apply the stochastic MPC law of Algorithm 8.4, the probabilistic constraints (8.67a) and (8.70c) in the online optimization (8.72) are replaced by (8.73a–8.73c). A mode 1 horizon of $N = 4$ steps and 250 samples are employed at each prediction time step. For this problem the number of support constraints at each time step is at most $\rho = 1$, and the confidence bounds of Fig. 8.1 with $n_s = 250$ can therefore be used to determine the fraction r of the samples that should be activated in order to achieve a given confidence of feasibility with respect to the probabilistic constraints [18]. For a confidence level of 90 % we need $r = 0.93$, and hence $\lceil rn_s \rceil = 233$ samples activated at each prediction time step.

The MPC optimization incorporating (8.73a–8.73c) is solved approximately using a greedy algorithm. This attempts to identify the optimal samples to be discarded at each prediction time step by successively solving the QP problem that corresponds to a fixed set of discarded samples, and then discarding the samples that correspond to the constraints (8.73a, 8.73b) that have the largest associated multipliers. Note that the implementation of (8.73a–8.73c) in terms of slack variables has the advantage that not all constraints in the online optimization need to be recomputed at each iteration.

We next compare the performance of the stochastic MPC law with that of its robust counterpart, which is obtained by invoking the robust constraints $Fx_{i|k} \leq \mathbf{1}$, $i = 0, 1, \ldots,$ for all realizations of model uncertainty. For problems involving multiple independent sources of uncertainty, the robust MPC approach is likely to be very conservative. In particular, although each uncertain component of the model is uniformly distributed, the model uncertainty combines to give a one step-ahead probability density function for the model state that is heavily centre-weighted and quickly drops to a negligible value a short distance from its centroid. A probabilistically constrained stochastic MPC algorithm can explicitly account for this effect whereas robust MPC must take into account the worst-case value of each source of uncertainty.

The high degree of conservativeness of robust MPC can be seen in Fig. 8.3, which shows how the optimal predicted cost for robust MPC compares with that of stochastic MPC for the initial condition $x_0 = (0.4, 0.4)$ as p varies: clearly a small reduction in p causes a relatively large reduction in predicted cost. The state trajectories of the closed-loop system under SMPC with $p = 0.9$ and RMPC are shown in Figs. 8.4 and 8.5 for 100 model uncertainty sequences. Again it is clear that using robust MPC in this example results in conservative closed-loop responses.

Table 8.1 compares the closed-loop performance of the robust MPC and stochastic MPC algorithms using 500 realizations of model uncertainty. Here the mean cost computed along closed-loop system trajectories is 16 % lower in the stochastic case than the robust, and this is achieved with approximately double the computation

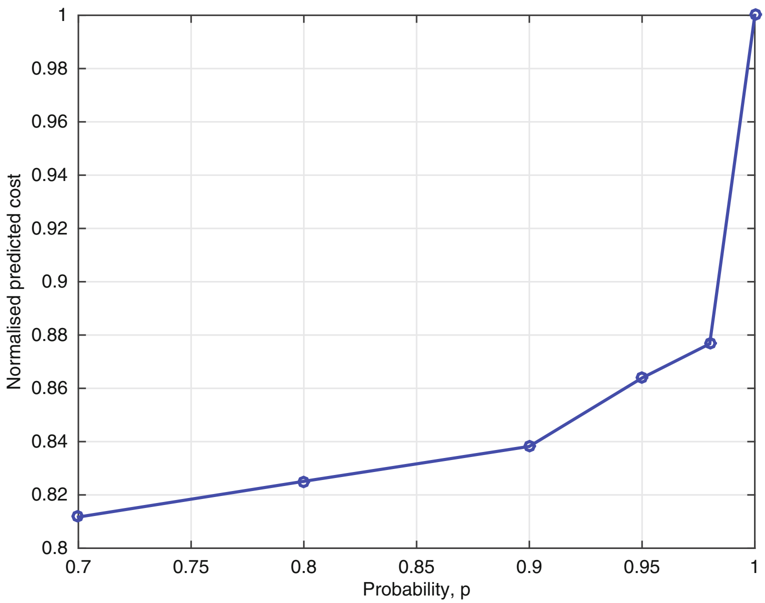

Fig. 8.3 Variation with p of the optimal predicted cost of stochastic MPC for a fixed initial condition, relative to the optimal predicted cost for $p = 1$

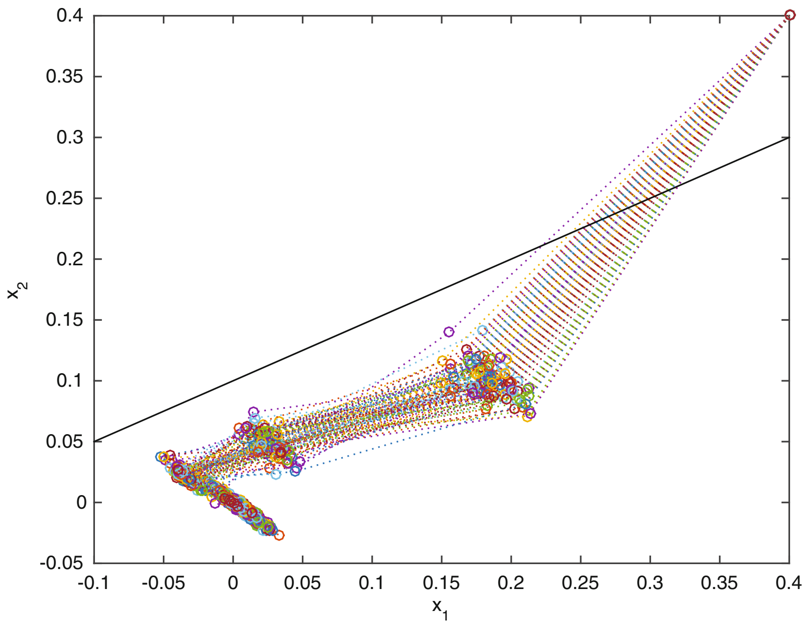

Fig. 8.4 Robust MPC: closed-loop state trajectories for 100 uncertainty sequences

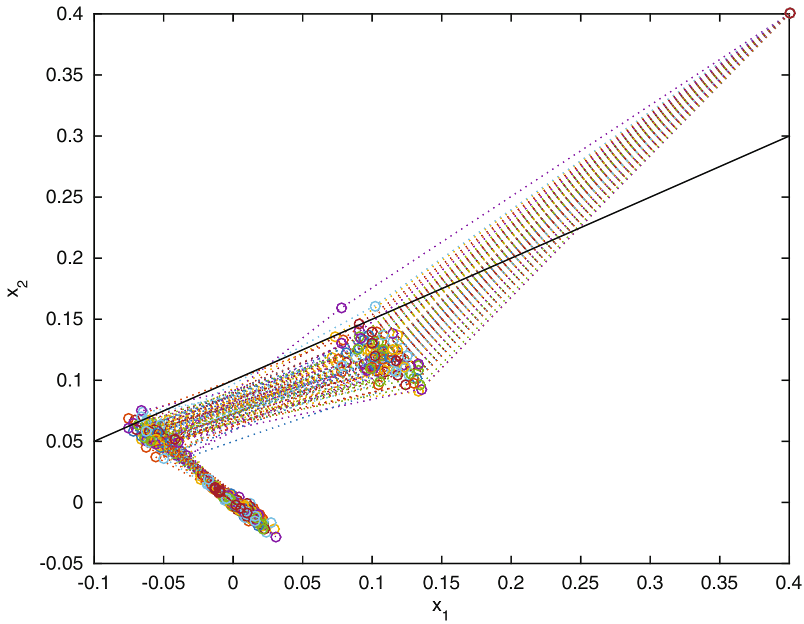

Fig. 8.5 Stochastic MPC: closed-loop state trajectories, 100 uncertainty sequences

Table 8.1 Closed-loop comparison of robust and stochastic MPC: average costs, constraints and computation times for 500 sequences of model uncertainty

	Robust	Stochastic
Mean closed-loop cost	25.7	21.5
Proportion of realizations satisfying constraints	100	93.8
Mean computation time (ms)	26	60

time required for the stochastic algorithm. The proportion of trajectories satisfying constraints (93.8 %) implies a degree of conservativism in the stochastic algorithm. However this is expected from the confidence parameter of 90 % and the relatively small sample size, $n_s = 250$. ♢

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Chapter 9
Conclusions

The aim of this final chapter is to give a short discursive summary of some of the key results presented in this book. We also speculate on extensions that could, in our opinion, be pursued in future.

It is perhaps difficult to pinpoint precisely the origins of predictive control but it appears that the early development of the subject ignored the presence of constraints. The perception of the subject has changed considerably over the last few decades and now the justification and success of predictive control is almost exclusively attributed to its ability to provide near optimal solutions that account for constraints. This feature alone makes MPC a particularly useful tool for the solution of real life problems where typically limits in actuation and safety considerations imply the presence of constraints.

This development brought with it two difficulties, one of which is theoretical and the other practical, namely the guarantee of the stability of the control system and the implementation within the inter-sample interval. The practicability of implementation implied the need to turn what, in essence, was an infinite-dimensional optimization problem into a finite-dimensional problem and this was made possible through the split of the prediction horizon into the near horizon (mode 1), where the control moves are considered to be degrees of freedom, and the far horizon (mode 2), where the control moves are dictated by a prescribed control law. To guarantee feasibility within mode 2, use was made of the concept of invariance and the implied terminal constraints. Thereafter, closed-loop stability could be established by applying a Lyapunov-like analysis to the closed-loop behaviour of the optimal predicted cost. Clearly, feasibility of the MPC optimization was required at each time step, and in general this was guaranteed recursively by ensuring that a specific predicted trajectory satisfied the constraints of the problem at successive time instants.

There has been a proliferation of MPC strategies proposed in the literature over the last few decades and their relative success has been judged mostly on the basis of their ease of computation on the one hand and comparing the size of their respective

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regions of attraction on the other. Often one may also wish to see a comparison of performance, but in general this is example- and initial condition-dependent, thereby rendering such an exercise meaningless. The overall aim has nevertheless been to strive for the best possible balance between ease of computation and optimality. Several attempts, some quite effective, have been proposed and are described in this book. Thus one of the ways of achieving efficiency in computation is to reduce the number of degrees of freedom over which the online repetitive optimization of MPC is to be performed. This can be done for example by input blocking, or interpolating between given trajectories or using the homotopy-based active constraint approach. Alternatively, for low-dimensional systems, one may use a multi-parametric approach which identifies regions in which the optimal predicted control is known and given by affine relationships to the state. The lifted autonomous formulation of the predicted dynamics provides yet another way in which the online computational load can be reduced significantly through the replacement of the optimization by a well-behaved Newton–Raphson procedure. Arguably, there is no unqualified best amongst all these, and an array of other approaches exist, which have not been mentioned in this book. The designer has to choose the approach that best meets the demands (in terms of degree of optimality, efficiency of computation and size of region of attraction) of the particular problem to be addressed. It is to be hoped that more original ideas will come about in future years and that some of the existing approaches will be developed further.

As mentioned earlier, over and above computation, one needs to consider the size of the region of attraction of a particular MPC algorithm. To improve on this, one can use as large a terminal set as possible and for a given terminal control law that leads naturally to the employment of the maximal invariant set. Further improvements are possible through the use of longer prediction horizons but this carries the penalty of increased computational load. An alternative to longer horizons is the introduction of controller dynamics whose action extends across an infinite prediction horizon. Such dynamics can be optimized to give the largest ellipsoidal region of attraction that can be attained over all terminal linear feedback laws. Yet this benefit is attained regardless of the choice of the terminal control law which can be chosen to be the unconstrained optimal.

The body of ideas of classical MPC carry over to robust MPC, but catering for uncertainty clearly requires more intensive online optimization. Low complexity tubes provide a convenient (albeit potentially conservative) way to define sets that contain predictions for all possible realizations of uncertainty. On the basis of these, constraints can be invoked robustly, and, coupled with a monotonically non-increasing property of a cost based on the nominal predictions or on perturbations to an unconstrained optimal control law, this leads to algorithms with guaranteed closed-loop stability. It is also possible to use general (rather than low complexity) tubes and in particular for additive uncertainty only one can employ rigid or homothetic tubes with the attendant inclusion conditions, whereas in the case of multiplicative (and also additive) uncertainty one can construct general tubes through inclusion conditions that are based on the use Farkas' Lemma.

For the case of additive disturbances only, improved results can be achieved through a re-parameterization of predictions that is affine in future disturbances and has a lower triangular structure. The drawback of this approach however is that, through the lower triangular structure, it introduces a greater number of degrees of freedom, which is in fact of the order of N^2, where N is the mode 1 prediction horizon. For the same order of magnitude of degrees of freedom, it is possible to use a more general lower-triangular tube parameterization which is piecewise affine rather than affine in the future disturbances. Both the disturbance affine MPC and PTMPC assume that the uncertainty set is polytopic but the former works with the facets of the set whereas the latter assumes a set description in terms of its vertices. The numbers of facets and vertices could differ significantly and this in turn implies a significant difference in the number of inequalities in the online optimization of the two approaches. This difference could be removed if an extension were found that enabled the methodology of PTMPC to deploy uncertainty facets rather than vertices.

An alternative that reduces computational complexity considerably replaces the lower triangular structure of PTMPC by a striped lower triangular structure, thus leading to a striped PTMPC (or SPTMPC) algorithm. Despite the reduction of the number of degrees of freedom (which are of order N for SPTMPC), this modification allows disturbance compensation to extend to mode 2. On account of this it can potentially outperform the parameterized tube MPC in terms of the size of its region of attraction.

In all of these endeavours, the goal is to get as close to the dynamic programming (DP) solution for the optimal feedback law without restrictive assumptions on controller parameterization, but to do so with a computational load that is tractable. PTMPC has narrowed the gap between available algorithms and the DP solution and indeed produces optimal results for several special cases. However, for fast sampling applications, the computational requirement of PTMPC could be excessive, while the degree of sub-optimality in SPTMPC could be more than desired. The field is wide open for researchers to come up with ideas that sit somewhere between PTMPC and SPTMPC in respect of the balance between optimality and computability. The field is also wide open in respect of re-parameterizations of tube MPC for the case of multiplicative uncertainty.

Robust MPC is clearly not the answer to controlling systems that are subject to random uncertainty with known probability distributions and that are subject to constraints, some of which could be probabilistic. The answer to this problem is provided by stochastic MPC, which has received considerable attention over the last decade. There were significant developments in this field, especially on the control theoretic front, leading to recursive feasibility (through a combined robust and probabilistic treatment of constraints) and stability guarantees. In general this is only possible for model uncertainty with finitely supported distributions. Such distributions are perhaps not as convenient as the Gaussian but accord well with the physical world where variations in the uncertain parameters are almost never unbounded.

These developments considered first the case of additive uncertainty and only more recently have been extended to the case of more general multiplicative uncertainty models. The particular difficulty here is that the multiplication of predicted states, which are random variables, by parameters which themselves are stochastic, makes it difficult to determine the distributions of predictions. The combined use of Farkas' Lemma with sampling circumvents this difficulty in terms of practical implementation and it is anticipated that further advances in this area will carry on being proposed in the near future. It is perhaps to be expected that some re-parameterization of stochastic MPC (e.g. along the lines of PTMPC) might be available for the case of additive model uncertainty, and possibly also the multiplicative uncertainty case.

Another area that may attract attention in future concerns the definition of predicted performance costs that preserve as much of the probabilistic nature of the cost as viable computation allows. Costs expressed in terms of nominal, expected values or worst-case values tend to conceal much of the stochastic nature of the control problem. An attempt at overcoming this difficulty was proposed in the solution to the sustainable problem discussed in Chap. 6 through the definition of a cost on the basis of probabilistic bands, but certainly there will be alternatives which are yet to be worked on. A topic related to the stochastic nature of the problem is the possibility of relaxation of future constraints on the basis of past realizations of uncertainty. Preliminary results in this area have been reported in the last chapter of the book but this area deserves further development.

In conclusion, classical MPC is now mature enough to suggest that further future developments, though still possible, will be few. The same, is not true of MPC applied to uncertain systems, especially for cases in which uncertainty is stochastic.

Solutions to Exercises

Solutions to Exercises for Chap. 2

1 (a) The predicted state and control sequences at time k with $N = 2$ are

$$\mathbf{x} = \begin{bmatrix} x_{0|k} \\ x_{1|k} \\ x_{2|k} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_{0|k} \\ u_{1|k} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1.5 \\ 2.25 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1.5 & 1 \end{bmatrix} \mathbf{u}.$$

Hence the predicted cost for $q = 1$ is $J(x, \mathbf{u}) = \mathbf{x}^T \mathbf{x} + 10\mathbf{u}^T \mathbf{u}$

$$\begin{aligned} J(x, \mathbf{u}) = \mathbf{u}^T & \left(\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1.5 & 1 \end{bmatrix} \right) \mathbf{u} \\ & + 2x \begin{bmatrix} 1 & 1.5 & 2.25 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1.5 & 1 \end{bmatrix} \mathbf{u} + x^2 \begin{bmatrix} 1 & 1.5 & 2.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \\ 2.25 \end{bmatrix} \\ = {} & \mathbf{u}^T H \mathbf{u} + 2x F^T \mathbf{u} + x^2 G, \end{aligned}$$

Since $\mathbf{u} = -H^{-1} F x$, we get the unconstrained MPC law

$$u_k = -\left(\frac{11(1.5 + 1.5^3) - 1.5^3}{11(11 + 1.5^2) - 1.5^2} \right) x_k = -0.350 x_k.$$

(b) Let $u_{i|k} = K x_{i|k}$ for all $i \geq 2$, with $K = -0.88$ (the LQ optimal feedback gain). If q satisfies $q - (A + BK)^2 q = 1 + 10K^2$, i.e. if

$$q = \frac{1 + 10K^2}{1 - (A + BK)^2} = \frac{1 + 10(0.88)^2}{1 - (1.5 - 0.88)^2} = 14.20,$$

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then the predicted cost satisfies

$$J = \sum_{i=0}^{N-1}(x_{i|k}^2 + 10u_{i|k}^2) + qx_{2|k}^2 = \sum_{i=0}^{\infty}(x_{i|k}^2 + 10u_{i|k}^2)$$

so the unconstrained MPC law is identical to LQ optimal control.

(c) Constraints: $-0.5 \le u_{i|k} \le 1$ for $i = 0, 1, \ldots$ imply constraints on the predicted input sequence:

$$-0.5 \le u_{i|k} \le 1, \quad i = 0, 1, \ldots, N + \nu$$

where $N = 2$ and ν must be large enough so that

$$-0.88(1.5 - 0.88)^{\nu+1}x \in [-0.5, 1] \text{ for all } x \text{ such that}$$
$$-0.88(1.5 - 0.88)^{i}x \in [-0.5, 1], \ i = 0, \ldots, \nu$$

Here $1.5 - 0.88 = 0.62$, so $\nu = 0$ is sufficient.

2 (a) The dynamics are stable if and only if $|\alpha| < 1$, which is therefore a requirement for $|y_k| \le 1$ for all $k \ge 0$. Also

$$\begin{aligned}
y_0 &= \begin{bmatrix}1 & 0\end{bmatrix} x_0, && \text{so } |y_0| \le 1 \iff -1 \le \begin{bmatrix}1 & 0\end{bmatrix} x_0 \le 1 \\
y_1 &= \begin{bmatrix}0 & 1\end{bmatrix} x_0, && \text{so } |y_1| \le 1 \iff -1 \le \begin{bmatrix}0 & 1\end{bmatrix} x_0 \le 1 \\
y_2 &= \alpha \begin{bmatrix}0 & 1\end{bmatrix} x_0, && \text{so } |y_2| \le 1 \Longleftarrow \begin{bmatrix}-1 \\ -1\end{bmatrix} \le x_0 \le \begin{bmatrix}1 \\ 1\end{bmatrix}
\end{aligned}$$

so we can conclude that $|y_k| \le 1$ for all $k \ge 0$ if and only if each element of x_0 is less than or equal to 1 in absolute value.
The same result can also be deduced from $y_i = \alpha^{i-1} \begin{bmatrix}1 & 0\end{bmatrix} x_0$ for $i \ge 1$.

(b) If $u_{i|k} = \begin{bmatrix}-\beta & 0\end{bmatrix} x_{i|k}$ for all $i \ge N$, then

$$\sum_{i=N}^{\infty}(y_{i|k}^2 + u_{i|k}^2) = x_{N|k}^T \begin{bmatrix}p_1 & p_{12} \\ p_{12} & p_2\end{bmatrix} x_{N|k}$$

where

$$\begin{bmatrix}p_1 & p_{12} \\ p_{12} & p_2\end{bmatrix} - \begin{bmatrix}0 & 1 \\ 0 & \alpha\end{bmatrix}^T \begin{bmatrix}p_1 & p_{12} \\ p_{12} & p_2\end{bmatrix} \begin{bmatrix}0 & 1 \\ 0 & \alpha\end{bmatrix} = \begin{bmatrix}1 + \beta^2 & 0 \\ 0 & 0\end{bmatrix} \implies \begin{cases} p_1 = 1 + \beta^2 \\ p_{12} = 0 \\ p_2 = \dfrac{p_1}{1 - \alpha^2} \end{cases}$$

which proves (i). To demonstrate (ii) we can use the result from part (a) by replacing the constraint $|y_k| \le 1$ with $|u_k| \le 1$ and noting that $u_{i|k} = -\beta y_{i|k}$ for all $i \ge N$.

(c) Although the terminal equality constraint $x_{N|k} = 0$ would ensure recursive feasibility closed loop stability, it would severely restrict the operating region of the controller. In particular the second element of the state is uncontrollable so this terminal constraint would require the second element of the state to be equal to zero at all points in the operating region.

3 (a) If $u_k = Kx_k$ and $y_k = Cx_k$, with $C = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}$ and $K = \frac{1}{\sqrt{2}}C$, then

$$I - (A+BK)^T(A+BK) = \frac{1}{2}(C^TC + K^TK) = \frac{3}{4}C^TC = \frac{3}{8}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

Hence the solution of $P - (A+BK)^T P(A+BK) = \frac{1}{2}(C^TC + K^TK)$ is $P = I$, which implies $\sum_{k=0}^{\infty}\frac{1}{2}(y_k^2 + u_k^2) = x_0^T x_0$.

(b) From part (a), the cost function is equal to $\sum_{i=0}^{\infty}\frac{1}{2}(y_{i|k}^2 + u_{i|k}^2)$ and the control input is $u_{i|k} = \frac{1}{\sqrt{2}}y_{i|k}$ for all $i \ge N$. Let $J^*(x_k)$ be the minimum value of this cost over $u_{0|k}, \ldots, u_{N-1|k}$, at time k. Then, at time $k+1$, the predicted input sequence $u_{i|k+1} = u_{i+1|k}$, $i = 0, 1, \ldots$ gives

$$J(x_{k+1}) = \sum_{i=1}^{\infty}\frac{1}{2}(y_{i|k}^2 + u_{i|k}^2) = J^*(x_k) - \frac{1}{2}(y_k^2 + u_k^2).$$

and since the optimal cost at time $k+1$ satisfies $J^*(x_{k+1}) \le J(x_{k+1})$, we can conclude that $J^*(x_{k+1}) \le J^*(x_k) - \frac{1}{2}(y_k^2 + u_k^2)$. This implies closed loop stability because $J^*(x_k)$ is positive definite in x_k since (A, C) is observable.

(c) The closed loop system will be stable if the predicted trajectories satisfy $-1 \le y_{i|k} \le 1$ for all $i \ge 0$. The constraints give $-1 \le y_{i|k} \le 1$ for $i = 0, 1, \ldots, N-1$ and

$$-1 \le y_{N+i|k} = C(A+BK)^i x_{N|k} \le 1, \quad i = 0, 1.$$

Here $C = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}$, $C(A+BK) = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}$ and $(A+BK)^2 = -\frac{1}{2}I$. Therefore

$$\left.\begin{aligned} -1 \le Cx \le 1 \\ -1 \le C(A+BK)x \le 1 \end{aligned}\right\} \implies -1 \le C(A+BK)^2 x \le 1$$

Hence $-1 \le C(A+BK)^i x \le 1$ for all $i \ge 0$ which implies $-1 \le y_{N+i|k} \le 1$ for all $i \ge 0$ if $x = x_{N|k}$.

4 (a) The largest invariant set compatible with the constraints is given by

$$S_\nu = \{x : F\Phi^i x \le \mathbf{1},\ i = 0, \dots \nu\},\ F = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ \Phi = \begin{bmatrix} 0.42 & -0.025 \\ -0.16 & -0.35 \end{bmatrix}$$

where ν is such that $F\Phi^{\nu+1}x \le \mathbf{1}$ for all $x \in S_\nu$. Since S_ν is symmetric about $x = 0$ this condition can be checked by solving the linear program: $\mu = \max_{x \in S_\nu} \begin{bmatrix} 1 & -1 \end{bmatrix} \Phi^{\nu+1} x$, and determining whether $\mu \le 1$.

(b) To check $\nu = 1$:

$$\mu = \max_x \begin{bmatrix} 0.193 & -0.127 \end{bmatrix} x \text{ subject to } \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0.578 & 0.324 \\ -0.578 & -0.324 \end{bmatrix} x \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

gives $\mu = 0.224$, so $\mu \le 1$ as required.

5 (a) Solving $W - \Phi^T W \Phi = I + K^T K$ for W with $K = \begin{bmatrix} 0.244 & 1.751 \end{bmatrix}$ gives

$$W = \begin{bmatrix} 1.33 & 0.58 \\ 0.58 & 4.64 \end{bmatrix}$$

and hence $-(B^T W B + 1)^{-1} B^T W A = \begin{bmatrix} 0.244 & 1.751 \end{bmatrix}$, which confirms that K is the LQ-optimal feedback gain.

(b) By construction $\begin{bmatrix} F & 0 \end{bmatrix} \Psi^i z_k \le \mathbf{1}$ implies $F x_{i|k} \le \mathbf{1}$ for $i = 0, \dots, N+1$. But $F x_{N|k} \le \mathbf{1}$ and $F \Phi x_{N|k} \le \mathbf{1}$ implies that $x_{N|k}$ lies in the invariant set of Question 4(b) and hence $F \Phi^i x_{N|k} \le \mathbf{1}$ for all $i \ge 0$.

(c) The quadratic form of the cost, with $\rho = B^T W B + 1 = 6.56$, follows from Theorem 2.10.

(d) Since $u = Kx$ is the feedback law that minimizes the MPC cost index for the case of no constraints, and since the MPC cost is evaluated over an infinite horizon, there cannot be any reduction in the predicted cost when N is increased above the minimum value, say $\bar{N}$, for which the terminal constraints are inactive, i.e. $J_N^*(x_0) = J_{\bar{N}}^*(x_0)$ for all $N > \bar{N}$. This is likely to be the case for this initial condition with $\bar{N} = 9$ since the cost seems to have converged, with $J_9(x_0) = J_{10}(x_0)$.

If the terminal constraints are inactive, then the optimal predicted control sequence is optimal for an infinite mode 1 horizon and hence it must be equal to the closed loop control sequence generated by the receding horizon control law.

6 (a) Solving the SDP defining the invariant ellipsoid $\mathcal{E}_z = \{z : z^T P_z z \leq 1\}$ gives

$$P_z = \begin{bmatrix} 1.09 & -1.16 & -0.07 & 0.45 \\ -1.16 & 4.20 & -3.06 & 1.07 \\ -0.07 & -3.06 & 4.20 & -3.10 \\ 0.45 & 1.07 & -3.10 & 4.20 \end{bmatrix}.$$

The projection onto the x-subspace is $\mathcal{E}_x = \{x : x^T P_x x \leq 1\}$, where

$$P_x = \left(\begin{bmatrix} I & 0 \end{bmatrix} P_z^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1.01 & -0.96 \\ -0.96 & 1.24 \end{bmatrix}$$

and the maximum value of α is $\alpha = 1/(v^T P_x v)^{1/2} = 1.79$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Solving the linear program

$$\max_{\alpha, \mathbf{c}} \ \alpha \ \text{ subject to } \ \begin{bmatrix} F & 0 \end{bmatrix} \Psi^i \begin{bmatrix} \alpha v \\ \mathbf{c} \end{bmatrix} \leq \mathbf{1}, \ i = 0, \ldots, N+1$$

with $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gives $\alpha = 2.41$.

This value is necessarily greater than the value of α in (a) because the set $\mathcal{Z} = \{z : \begin{bmatrix} F & 0 \end{bmatrix} \Psi^i z \leq \mathbf{1}, \ i = 0, \ldots, N+1\}$ is the maximal invariant set for the dynamics $z_{k+1} = \Psi z_k$ and constraints $\begin{bmatrix} F & 0 \end{bmatrix} z_k \leq \mathbf{1}$, so it must contain $\mathcal{E}_z = \{z : z^T P_z z \leq 1\}$ as a subset, and therefore the projection of $\mathcal{E}_z$ onto the x-subspace must be a subset of the projection of $\mathcal{Z}$ onto the x-subspace.

(c) Solving the SDP for the invariant ellipsoidal set $\hat{\mathcal{E}}_z = \{z : z^T \hat{P}_z z \leq 1\}$ and the optimized prediction dynamics gives C_c and A_c as stated in the question and

$$\hat{P}_z = \begin{bmatrix} 2.43 & -1.31 & 1.36 & -1.31 \\ -1.31 & 3.12 & 1.21 & 3.12 \\ 1.36 & 1.21 & 2.45 & 1.21 \\ -1.31 & 3.12 & 1.21 & 8.88 \end{bmatrix}.$$

The projection onto the x-subspace is $\hat{\mathcal{E}}_x = \{x : x^T \hat{P}_x x \leq 1\}$, where

$$\hat{P}_x = \left(\begin{bmatrix} I & 0 \end{bmatrix} \hat{P}_z^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1.19 & -1.38 \\ -1.38 & 1.75 \end{bmatrix}$$

and hence the maximum value of α is $\alpha = 1/(v^T P_x v)^{1/2} = 2.32$.

(d) With $\hat{\Psi}$ defined on the basis of the optimized prediction dynamics, the maximal invariant set for the dynamics $z_{k+1} = \hat{\Psi} z_k$ and constraints $\begin{bmatrix} F & 0 \end{bmatrix} z_k \leq \mathbf{1}$ is $\{z : \begin{bmatrix} F & 0 \end{bmatrix} \hat{\Psi}^i z_k \leq \mathbf{1}, \ i = 0, \ldots, 5\}$. Solving the LP

$$\max_{\alpha,\mathbf{c}} \ \alpha \ \text{subject to} \ \begin{bmatrix} F & 0 \end{bmatrix} \hat{\Psi}^i \begin{bmatrix} \alpha v \\ \mathbf{c} \end{bmatrix} \leq \mathbf{1}, \ i = 0, \ldots, 5$$

with $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gives $\alpha = 3.82$.

(e) Computing the MPC cost for the optimized prediction dynamics by solving the Lyapunov equation (2.57) gives

$$J(x_k, \mathbf{c}_k) = \|x_k\|_W^2 + \mathbf{c}_k^T \begin{bmatrix} 106 & 32.8 \\ 32.8 & 10.2 \end{bmatrix} \mathbf{c}_k.$$

Solving $\min_{\mathbf{c}} J(x, \mathbf{c})$ subject to $\begin{bmatrix} F & 0 \end{bmatrix} \hat{\Psi}^i (x, \mathbf{c}) \leq \mathbf{1}$, $i = 0, \ldots, 5$ gives the optimal predicted cost for $x = (3.8, 3.8)$ as 1686. This is larger than (in fact more than double) the optimal predicted cost in Question 5(d) for $N = 9$, which is the minimum that can be obtained by any control sequence. The advantage of the optimized prediction dynamics is that the associated set of feasible initial conditions is almost identical to that of the MPC law in Question 5(d) despite using only 2 degrees of freedom rather than 9 degrees of freedom.

7 (a) In this case $\begin{bmatrix} 1 & 1 \end{bmatrix} (A + BK)x = 0$ for all $x \in \mathbb{R}^2$, and the eigenvalues of $A + BK$ lie inside the unit circle. It follows that there exists a stabilizing control law (namely $u = Kx$) such that, starting from the initial condition $x_0 = (\alpha, -\alpha)$ for arbitrarily large $|\alpha|$, the constraints $|\begin{bmatrix} 1 & 1 \end{bmatrix} (A + BK)^k x_0| \leq 1$ are satisfied for all k. Hence the maximal CPI set is unbounded (in fact it is equal to $\{x : |\begin{bmatrix} 1 & 1 \end{bmatrix} x| \leq 1\}$), and the maximal feasible initial condition set of an MPC law will increase monotonically with N.

(b) The transfer function from u_k to the constrained output $y_k = \begin{bmatrix} 1 & -1 \end{bmatrix} x_k$ is nonminimum-phase (its zero lies outside the unit circle at 1.33). Hence there is no stabilizing control law under which the constraints $|\begin{bmatrix} 1 & -1 \end{bmatrix} x_k| \leq \mathbf{1}$ are satisfied for all k when $\|x_0\|$ is arbitrarily large, in other words the maximal CPI set is bounded.

8 The predicted cost is

$$J_k = \mathbf{u}_k^T (\hat{R} + C_u^T \hat{Q} C_u) \mathbf{u}_k + 2\mathbf{u}_k C_u^T \hat{Q} C_x x_k + x_k^T C_x \hat{Q} C_x x_k,$$

and the optimal control sequence $\mathbf{u}_k^* = -(\hat{R} + C_u^T \hat{Q} C_u)^{-1} C_u^T \hat{Q} C_x x_k$ for the case of no constraints can be obtained by differentiation. The MPC law is given by the first element of this sequence, and is therefore a feedback law of the form $u_k = K_{(N,N_u)} x_k$.

For the given (A, B, C), computing the spectral radius of $A + BK_{(N,N_u)}$ for $N = 1, 2, \ldots$ and for $1 \leq N_u \leq N$ shows that $N = 9$ is the smallest output horizon for which stability can be achieved for any input horizon $N_u \leq N$. In fact for $N_u = 1$, stability is achieved only if $N \geq 10$.

The poles of the open loop system are at 0.693, 0.997, and thus both lie within the unit circle, whereas the system zero is at 1.16. This non-minimum phase zero implies that the predicted output sequence initially sets off in the wrong direction and this effect will be exacerbated at the next time step, when larger inputs will needed in order to return the output to the correct steady state. This indicates the tendency towards instability. For a sufficiently large output horizon (in this case for $N \geq 9$) the predicted cost can be shown to be monotonically non-increasing along closed loop system trajectories, indicating closed loop stability.

9 (a) The numerator and denominator polynomials of the system transfer function are given by

$$B(z^{-1}) = B_1 z^{-1} + B_0 = -0.6527z^{-1} + 0.5647$$
$$A(z^{-1}) = A_2 z^{-1} + A_1 z^{-1} + A_0 = 0.6908z^{-2} - 1.69z^{-1} + 1$$

Hence for the given $\tilde{X}(z^{-1})$ and $\tilde{Y}(z^{-1})$ polynomials we obtain $\tilde{Y}(z^{-1})A(z^{-1}) + z^{-1}\tilde{X}(z^{-1})B(z^{-1}) = 1$.

(b) For the single input single output case we have $\tilde{A}(z^{-1}) = A(z^{-1})$, $\tilde{B}(z^{-1}) = B(z^{-1})$, $\tilde{X}(z^{-1}) = X(z^{-1})$, $\tilde{Y}(z^{-1}) = Y(z^{-1})$ so that

$$C_{z^{-1}\tilde{X}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ X_0 & 0 & 0 & 0 \\ X_1 & X_0 & 0 & 0 \\ 0 & X_1 & X_0 & 0 \end{bmatrix} \qquad C_{\tilde{Y}} = \begin{bmatrix} Y_0 & 0 & 0 & 0 \\ Y_1 & Y_0 & 0 & 0 \\ 0 & Y_1 & Y_0 & 0 \\ 0 & 0 & Y_1 & Y_0 \end{bmatrix}$$
$$C_{\tilde{A}} = \begin{bmatrix} A_0 & 0 & 0 & 0 \\ A_1 & A_0 & 0 & 0 \\ A_2 & A_1 & A_0 & 0 \\ 0 & A_2 & A_1 & A_0 \end{bmatrix} \qquad C_{\tilde{B}} = \begin{bmatrix} B_0 & 0 & 0 & 0 \\ B_1 & B_0 & 0 & 0 \\ 0 & B_1 & B_0 & 0 \\ 0 & 0 & B_1 & B_0 \end{bmatrix}$$

where $X_0 = -32.2308$, $X_1 = 21.0529$, $Y_0 = 1$, $Y_1 = 19.89$. Inverting the 16×16 matrix consisting of these four blocks gives the matrix with blocks $C_A, C_Y, C_B, -C_{z^{-1}X}$, as required. This inverse can then be used to obtain the predicted output and control sequences from (2.72).

(c) From the predicted output and control sequences it is obvious that $y_{i+4|k} = 0$ and $u_{i+3|k} = 0$ for all $i \geq 1$. This implies that SGPC invokes (implicitly) a terminal equality constraint, and therefore the optimal predicted cost is monotonically non-increasing, from which it can be deduced that SGPC guarantees closed loop stability. The property that the predicted output and control sequences reach their steady state values of zero after $\nu + n_A$ prediction steps, where ν is the length of $\mathbf{c}_k$ and n_A the system order (i.e. here $\nu = n_A = 2$) is generally true given

Table A.1 Frequency responses of $S_a(\omega T)$ and $S_b(\omega T)$

ωT	0°	18°	36°	54°	72°	90°	108°	126°	144°	162°	180°
$\lvert S_a\rvert$	0.009	1.74	7.3	19.4	39.3	66.1	97.3	128.7	155.6	173.7	180.1
$\lvert S_b\rvert$	0.009	1.72	6.7	15.7	26.2	33.5	34.3	29.9	27.5	31.7	33.8

the structure of the convolution and Hankel matrices in the expression for the predicted output and control sequences. Hence, in the absence of constraints, SGPC ensures closed loop stability for any initial condition.

10 Denoting by S_a and S_b the transfer functions $K(z^{-1})/\big(1 + G(z^{-1})K(z^{-1})\big)$ corresponding to (a) $Q = 0$ and (b) $Q(z^{-1}) = -11.7z^{-1} + 43$, respectively, and denoting the sampling interval as T, we obtain the transfer function moduli given in Table A.1. These indicate that $|S_b(\omega T)| < |S_a(\omega T)|$ at all frequencies, and the ratio $|S_a(\omega T)|/|S_b(\omega T)|$ becomes larger at high frequencies ($|S_a(\omega T)|/|S_b(\omega T)| > 5$ for $\omega T > 140°$). Thus for $Q(z^{-1})$ as given in (b) the closed loop system will have enhanced robustness to additive uncertainty in the open loop system transfer function.

Solutions to Exercises for Chap. 3

1 (a) Two advantages of receding horizon control for this application:

- The receding horizon optimization is repeated at each time step, thus providing feedback (since the optimal predicted input sequence at k depends on the state x_k) and reducing the effect of the uncertainty in w_k.
- The optimization has to be performed over a finite number of free variables because of the presence of constraints. Using a receding horizon optimization reduces the degree of suboptimality with respect to the infinite horizon optimal control problem.

(b) With $u_{i|k} = \hat{w} - (x_{i|k} - x^0) + c_{i|k}$ we get $x_{i+1|k} - x^0 = \hat{w} - w_{k+i} + c_{i|k}$, so setting the disturbance equal to its nominal value, $w_{k+i} = \hat{w}$, gives $x_{i+1|k} - x^0 = s_{i+1|k} = c_{i|k}$ for all $i \geq 0$, and hence the nominal cost is $J(x_k, \mathbf{c}_k) = s_{0|k}^2 + \|\mathbf{c}_k\|^2$.

(c) Setting $s_{i|k} + e_{i|k} = x_{i|k} - x^0$ gives $e_{0|k} = 0$ and $e_{i+1|k} = \hat{w} - w_{k+i}$ for all $i \geq 0$. Hence $e_{i|k}$ lies in the interval $[\hat{w} - W, \hat{w}]$ for all $i \geq 0$. Using these bounds and $x_{i|k} = s_{i|k} + x^0 + e_{i|k}$, $u_{i|k} = \hat{w} - s_{i|k} - e_{i|k} + c_{i|k}$ we obtain

$$x_{i|k} \in [0, X] \iff \begin{cases} c_{i-1|k} + x^0 + \hat{w} \in [W, X], & 1 \leq i \leq N \\ W \leq X, & i > N \end{cases}$$

$$u_{i|k} \in [0, U] \iff \begin{cases} c_{0|k} - x_k + x^0 + \hat{w} \in [0, U - W], & i = 0 \\ c_{i|k} - c_{i-1|k} \in [0, U - W], & 1 \le i \le N - 1 \\ -c_{N-1|k} \in [0, U - W], & i = N \\ W \le U, & i > N \end{cases}$$

(d) By construction, $\mathbf{c}_{k+1} = (c^*_{1|k}, \ldots, c^*_{N-1|k}, 0)$ is feasible at time $k + 1$ if $\mathbf{c}^*_k$ is optimal at time k, so the problem is recursively feasible. Convergence of $c^*_{0|k}$ to zero as $k \to \infty$ then follows from the property that $\|c^*_{k+1}\|^2 \le \|c^*_k\|^2 - (c^*_{0|k})^2$, which implies that the l_2 norm of the sequence $\{c^*_{0|0}, c^*_{0|1}, \ldots\}$ is finite.

(e) The constraint $u_{i|k} \ge 0$ for $i = 1, \ldots, N$ requires $c_{i|k} \ge c_{i-1|k}$ and $c_{N-1} \le 0$. Hence $c_{0|k} \le 0$ so $u_{0|k} \ge 0$ requires $x_k \le x^0 + \hat{w}$. To relax this condition we need to use a less aggressive auxiliary control law, e.g. $u_{i|k} = \hat{w} - \alpha(x_{i|k} - x^0) + c_{i|k}$ for $0 < \alpha < 1$.

2 The structure of Ψ implies that, for any integer q,

$$\Psi^q = \begin{bmatrix} \Phi^q & \Gamma_q \\ 0 & M^q \end{bmatrix}$$

for some matrix Γ_q. Since $\Phi^n = 0$ and by construction $M^N = 0$, it follows that

$$\Psi^{n+N} = \Psi^n \Psi^N = \begin{bmatrix} \Gamma_n \\ M^n \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} \Phi^N & \Gamma_N \end{bmatrix} = 0.$$

(a) Since $\Phi^n = 0$ the minimal RPI set (3.23) is given by

$$\mathcal{X}^{\mathrm{mRPI}} = D\mathcal{W} \oplus \cdots \oplus \Phi^{n-1} D\mathcal{W}.$$

(b) Let $\bar{F} = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$ and define $h_0 = 0$ and h_i for $i \ge 1$ as in (3.13):

$$h_i = \sum_{j=0}^{i-1} \max_{w_j \in \mathcal{W}} F_1 \Phi^j D w_j,$$

Since $h_i = h_n$ for all $i > n$ and $\Psi^{n+N} = 0$, the MRPI set (3.16) is

$$\mathcal{Z}^{\mathrm{MRPI}} = \{z : \bar{F}\Psi^i z \le \mathbf{1} - h_i, \quad i = 0, \ldots, n + N\}.$$

(c) The MRPI set must contain the origin, while from (a) the maximum of $F_1 e$ over $e \in \mathcal{X}^{\mathrm{mRPI}}$ is h_n. Hence from (b) the MRPI set is non-empty if and only if

$$h_n \le \mathbf{1}.$$

3 For the given A, B and K:

$$\Phi = A + BK = 0.5 \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

and $\Phi^2 = 0$, which verifies that Φ is nilpotent. The constraint tightening parameters h_i that bound the effects of future disturbances on the constraints can be determined either by solving linear programs or by using the vertices of the disturbance set $\mathcal{W}$. In this example $\mathcal{W}$ has 4 vertices:

$$\mathcal{W} \doteq \text{Co}\left\{ \begin{bmatrix} 0 \\ \sigma \end{bmatrix}, \begin{bmatrix} 0 \\ -\sigma \end{bmatrix}, \begin{bmatrix} \sigma \\ 0 \end{bmatrix}, \begin{bmatrix} -\sigma \\ 0 \end{bmatrix} \right\} = \text{Co}\{w^{(j)},\ j = 1, 2, 3, 4\}$$

Hence $h_0 = 0$ and

$$\begin{aligned} h_1 &= \max_j F w^{(j)} = \sigma \mathbf{1}, \\ h_2 &= h_1 + \max_j F(A + BK) w^{(j)} = 1.5\sigma \mathbf{1}, \end{aligned}$$

with $h_i = h_2$ for all $i > 2$. For $N = 2$ the MRPI set is therefore given by

$$\begin{aligned} \mathcal{Z}^{\text{MRPI}}(\sigma) = \{z : & \bar{F} z \le \mathbf{1}, \\ & \bar{F} \Psi z \le (1 - \sigma)\mathbf{1}, \\ & \bar{F} \Psi^i z \le (1 - 1.5\sigma)\mathbf{1}, \quad i = 2, 3\} \end{aligned}$$

and $\mathcal{Z}^{\text{MRPI}}(\sigma)$ is non-empty if $1 - 1.5\sigma \ge 0$, i.e. $\sigma \le \frac{2}{3}$.

4 (a) Robust invariance of $\mathcal{Z}^{\text{MRPI}}(\sigma)$ implies that $(x_{k+1}, M\mathbf{c}_k) \in \mathcal{Z}^{\text{MRPI}}(\sigma)$ holds for all $w_k \in \sigma\mathcal{W}_0$ if $(x_k, \mathbf{c}_k^*) \in \mathcal{Z}^{\text{MRPI}}(\sigma)$. Hence if the MPC optimization is feasible at time k, then $\mathbf{c}_{k+1} = M\mathbf{c}_k^*$ is feasible at time $k+1$. By optimality therefore $\|\mathbf{c}_{k+1}^*\|^2 \le \|\mathbf{c}_k^*\|^2 - (c_{0|k}^*)^2$, which implies that $\sum_{k=0}^{\infty} (c_{0|k}^*)^2 \le \|\mathbf{c}_0^*\|^2$ and hence $c_{0|k}^* \to 0$ as $k \to \infty$.
Lemma 3.2 implies that the state x_k of the closed loop system satisfies the quadratic bound (3.35) since $A + BK$ is stable, and since the sequence $\{c_{0|0}^*, c_{0|1}^*, \ldots\}$ is square-summable, the argument of Theorem 3.2 implies that x_k converges to the minimal RPI set:

$$\begin{aligned} \mathcal{X}^{\text{mRPI}}(\sigma) &= \sigma\mathcal{W}_0 \oplus (A + BK)\mathcal{W}_0 \\ &= \sigma\text{Co}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\} \oplus \text{Co}\left\{ \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix} \right\} \\ &= \sigma\text{Co}\left\{ \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} -0.5 \\ -1.5 \end{bmatrix}, \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix} \right\} \end{aligned}$$

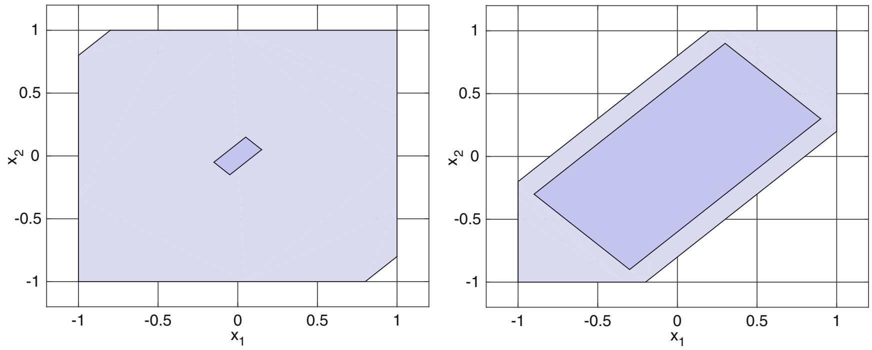

Fig. A.1 The set of feasible initial conditions for the MPC law of Question 4(a) (i.e. the projection $\mathcal{Z}^{\text{MRPI}}(\sigma)$ onto the x-subspace) and the minimal RPI set $\mathcal{X}^{\text{mRPI}}(\sigma)$ for $\sigma = 0.1$ (*left*) and $\sigma = 0.6$ (*right*)

The feasible initial condition sets and the minimal RPI sets for two values of σ are shown in Fig. A.1.

(b) Although the objective of the suggested MPC optimization problem is equal to the nominal predicted value of the cost, this is not a good suggestion since closed loop stability cannot, in general, be guaranteed with this combination of predicted cost and constraints. This is because the choice $\mathbf{c}_{k+1} = M\mathbf{c}_k^*$ does not ensure that the optimal predicted cost is monotonically non-increasing here due to the unknown disturbance that acts on the terms of the cost that depend on x_k. Note also that because K is not equal to the unconstrained optimal feedback gain, the predicted cost cannot be separated into terms that depend only on x_k and $\mathbf{c}_k$, as was done in the stability analysis of Sect. 3.3.

5 (a) Since $\Phi^r \mathcal{W} \subseteq \rho\mathcal{W}$ we have, for any $i \geq r$,

$$\max_{w\in\mathcal{W}} F\Phi^i w = \max_{w\in\Phi^r\mathcal{W}} F\Phi^{i-r} w \leq \max_{w\in\rho\mathcal{W}} F\Phi^{i-r} w = \rho \max_{w\in\mathcal{W}} F\Phi^{i-r} w,$$

which implies

$$\begin{aligned} h_\infty &= \sum_{j=0}^{\infty} \max_{w_j\in\mathcal{W}} F\Phi^j w_j \\ &\leq \sum_{j=0}^{r-1} \max_{w_j\in\mathcal{W}} F\Phi^j w_j + \rho \sum_{j=0}^{r-1} \max_{w_j\in\mathcal{W}} F\Phi^j w_j + \rho^2 \sum_{j=0}^{r-1} \max_{w_j\in\mathcal{W}} F\Phi^j w_j + \cdots \\ &= \frac{1}{1-\rho} \sum_{j=0}^{r-1} \max_{w_j\in\mathcal{W}} F\Phi^j w_j = \hat{h}_\infty. \end{aligned}$$

Also $h_\infty \geq h_r = \sum_{j=0}^{r-1} \max_{w_j \in \mathcal{W}} F\Phi^j w_j$ implies

$$\frac{\hat{h}_\infty - h_\infty}{h_\infty} \leq \frac{\hat{h}_\infty - h_r}{h_\infty} \leq \frac{\hat{h}_\infty - h_r}{h_r} = \frac{\rho}{1-\rho}.$$

(b) In order that the fractional error in the approximation of h_∞ is no greater than 0.01 we need $\rho \leq 0.01/1.01 = 0.0099$.
Two alternative methods of finding ρ such that $\Phi^r \mathcal{W} \subseteq \rho\mathcal{W}$ for given r: (i) Using the representation $\mathcal{W} = \{w : Vw \leq \mathbf{1}\}$, ρ is given by the maximum element of $\max_{w\in\mathcal{W}} V\Phi^r w$. (ii) Using the vertex representation, $\mathcal{W} = \mathrm{Co}\{w^{(1)}, \ldots, w^{(4)}\}$, the value of ρ is the maximum element of $\max_{j\in\{1,\ldots,4\}} V\Phi^r w^{(j)}$.
For the system of Question 4 with $K = \begin{bmatrix}0.479 & 0.108\end{bmatrix}$ we need $r = 7$ for $\rho \leq 0.0099$, and this gives $\rho = 0.0055$ and

$$\hat{h}_\infty = \begin{bmatrix}0.175 & 0.199 & 0.175 & 0.199\end{bmatrix}^T.$$

(c) An approximation to the minimal RPI set $\mathcal{X}^{\mathrm{mRPI}}$ is given by

$$\hat{\mathcal{X}}^{\mathrm{mRPI}} = \frac{1}{1-\rho} \bigoplus_{j=0}^{r-1} \Phi^j \mathcal{W}.$$

The discussion in (a) implies that $\hat{\mathcal{X}}^{\mathrm{mRPI}}$ contains $\mathcal{X}^{\mathrm{mRPI}}$ and, for any vector v, the support function $\max_e\{v^T e$ subject to $e \in \mathcal{X}^{\mathrm{mRPI}}\}$ is approximated by $\max_e\{v^T e$ subject to $e \in \hat{\mathcal{X}}^{\mathrm{mRPI}}\}$ with a fractional error no greater than $\rho/(1-\rho)$.

6 (a) For the given system parameters with $K = \begin{bmatrix}0.479 & 0.108\end{bmatrix}$ and $N = 1$,

$$\Psi = \begin{bmatrix} -0.521 & 0.308 & 1 \\ -0.489 & 0.596 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

The maximal RPI set for the dynamics $z_{i+1|k} = \Psi z_{i|k} + \bar{D} w_{k+i}$ and constraint $\bar{F} z_{i|k} \leq \mathbf{1}$ is given by

$$\mathcal{Z}^{\mathrm{MRPI}} = \{z : \bar{F}\Psi^i z \leq \mathbf{1} - h_i,\ i = 0, 1, 2\},$$

$$\{h_0,\ h_1,\ h_2\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.761 \\ 0.798 \\ 0.761 \\ 0.798 \end{bmatrix} \right\}.$$

since $\bar{F}\Psi^3 z \le \mathbf{1} - h_3$ holds for all $z \in \mathcal{Z}^{\text{MRPI}}$. Since this is the maximal RPI set for the prediction dynamics, its projection onto the x-subspace:

$$\mathcal{F}_1 = \{x \;:\; \exists \mathbf{c}_k \text{ such that } (x_k, \mathbf{c}_k) \in \mathcal{Z}^{\text{MRPI}}\}$$

must be equal to the maximal set of feasible model states x_k.

(b) Solving $W_z - \Psi^T W_z \Psi = \bar{Q}$ for W_z, where $\bar{Q} = \text{diag}\{I, 0\} + \begin{bmatrix} K & 1 \end{bmatrix}^T \begin{bmatrix} K & 1 \end{bmatrix}$:

$$W_z = \begin{bmatrix} 1.87 & -0.48 & 0.00 \\ -0.48 & 1.57 & 0.00 \\ 0.00 & 0.00 & 3.74 \end{bmatrix}.$$

With $x_0 = (0, 1)$ the optimal solution of the QP,

$$\underset{\mathbf{c}_0}{\text{minimize}} \quad \left\| \begin{bmatrix} x_0 \\ \mathbf{c}_0 \end{bmatrix} \right\|_{W_z}^2 \quad \text{subject to} \quad \bar{F}\Psi^i \begin{bmatrix} x_0 \\ \mathbf{c}_0 \end{bmatrix} \le \mathbf{1} - h_i, \; i = 0, 1, 2$$

is $\mathbf{c}_0^* = 0.192$ and the corresponding cost value is $z_0^T W_z z_0 = 1.707$.

(c) With the disturbance sequence as given in the question, the closed loop state and control sequences are

$$\{x_0, \; x_1, \; x_2, \; x_3, \ldots\} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \begin{bmatrix} -0.867 \\ -0.192 \end{bmatrix}, \begin{bmatrix} 0.393 \\ 0.810 \end{bmatrix} \right\}$$
$$\{u_0, \; u_1, \; u_2, \; u_3, \ldots\} = \{0.3, \; 0.533, \; -0.436, \; 0.276\}$$

and hence $\sum_{k=0}^{3} \big(\|x_k\|^2 + u_k^2 \big) = 4.489$.

7 (a) The solution $\check{W}_x$ of the Riccati equation (3.42) and the optimal gain K given in the question can be computed by using semidefinite programming to minimize $\text{tr}(\check{W}_x)$ subject to (3.43) and $\left[\begin{smallmatrix} \check{W}_x & I \\ I & S \end{smallmatrix}\right] \succeq 0$, with the value of γ^2 fixed at 3.3. Lemma 3.3 then implies that $\check{J}(x_k, \mathbf{c}_k) = \|x_k\|_{W_x}^2 + \|\mathbf{c}_k\|_{W_c}^2$ where

$$W_x = \check{W}_x = \begin{bmatrix} 2.336 & -0.904 \\ -0.904 & 2.103 \end{bmatrix},$$
$$W_c = B^T \big(\check{W}_x + \check{W}_x (\gamma^2 I - \check{W}_x)^{-1} \check{W}_x \big) B + 1 = 72.78.$$

(b) Repeating the procedure of Question 6(a) with the new value of K we get $\nu = 2$, so the online MPC optimization becomes the QP

$$\mathbf{c}_k^* = \arg \min_{\mathbf{c}_k} \; \|\mathbf{c}_k\|_{W_c}^2 \;\; \text{subject to} \;\; \bar{F}\Psi^i \begin{bmatrix} x_k \\ \mathbf{c}_k \end{bmatrix} \le \mathbf{1} - h_i, \; i = 0, 1, 2$$

with $\bar{F}$ as defined in Question 6 and

$$\Psi = \begin{bmatrix} -0.460 & 0.449 & 1 \\ -0.520 & 0.526 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}, \ \{h_0,\ h_1,\ h_2\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.730 \\ 0.763 \\ 0.730 \\ 0.763 \end{bmatrix} \right\}.$$

Solving this QP with $x_0 = (0, 1)$ is $\mathbf{c}_0^* = 0.051$ and $\check{J}^*(x_0) = 2.29$.

(c) From Lemma 3.4 we get

$$\Delta = \begin{bmatrix} 0.964 & 0.904 \\ 0.904 & 1.197 \end{bmatrix},$$

$$W_{\mu z} = \begin{bmatrix} -1.29 & 1.25 & 4.17 \\ 1.29 & -1.25 & -4.17 \\ -0.01 & -0.61 & 56.6 \\ 0.01 & 0.61 & -56.6 \end{bmatrix}, \quad W_{\mu\mu} = \begin{bmatrix} 4.19 & -4.19 & 2.78 & -2.78 \\ -4.19 & 4.19 & -2.78 & 2.78 \\ 2.78 & -2.78 & 47.2 & -47.2 \\ -2.78 & 2.78 & -47.2 & 47.2 \end{bmatrix},$$

and hence the predicted cost is given by

$$\check{J}(x_k, \mathbf{c}_k) = \min_{\mu_k \geq 0} \begin{bmatrix} x_k \\ \mathbf{c}_k \\ \mu_k \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} W_x & 0 \\ 0 & W_c \end{bmatrix} & -W_{\mu z}^T \\ -W_{\mu z} & W_{\mu\mu} \end{bmatrix} \begin{bmatrix} x_k \\ \mathbf{c}_k \\ \mu_k \end{bmatrix} + 2\mu^T \mathbf{1}.$$

With this cost and $x_0 = (0, 1)$ the solution of the online MPC optimization is $\mathbf{c}_0^* = 0.051$ and $\check{J}^*(x_0) = 2.22$.

(d) The cost in (c) is the maximum over $\{w_k, w_{k+1}, \ldots\}$ subject to the disturbance bounds $w_{k+i} \in \mathcal{W}$ for $i = 0, \ldots, N-1$. Therefore its minimum value over $\mathbf{c}_k$ is no greater than that of the cost in (b) for any given x_k. For this example $x_0 = (0, 1)$ lies on the boundary of the feasible set and this is why the optimal $\mathbf{c}_0$ is the same for both costs.

By taking into account the disturbance bounds over the first N predicted times steps, the cost of (c) provides a tighter worst case bound and hence is more representative of the worst case predicted performance than (b). On the other hand, the cost of (c) requires $5N$ optimization variables rather than N for (b). Also (b) ensures convergence to a limit set that is relatively easy to compute, namely the minimal RPI set for $e_{k+1} = \Phi e_k + w_k$, $w_k \in \mathcal{W}$, whereas this is not necessarily the case for (c). In this example however, the closed loop state sequences under the MPC laws for (b) and (c) are identical. This is illustrated in Fig. A.2, which compares the closed loop evolution of the state under the two MPC laws for a random disturbance sequence.

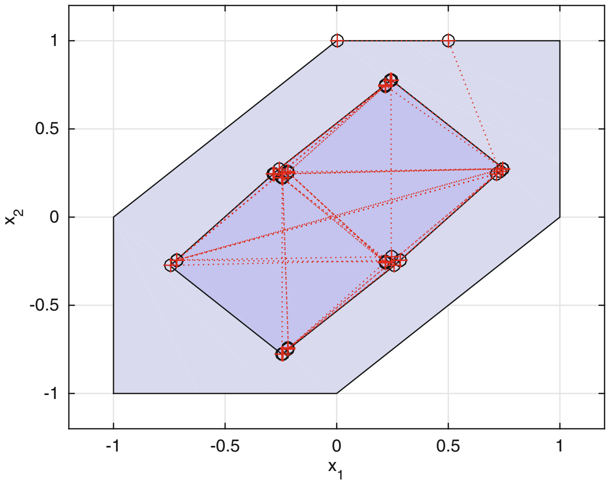

Fig. A.2 The evolution of the closed loop system state for the MPC laws in Question 7 part (b) (*marked* ∘) and part (c) (*marked* +) for a random sequence of disturbances $\{w_0, w_1, \ldots\}$ in which w_k is equal to a vertex of $\mathcal{W}$ for all k. Also shown are the set of feasible states and the minimal RPI set for $e_{k+1} = \Phi e_k + w_k$, $w_k \in \mathcal{W}$

8 (a) The minimum ρ such that $\Phi^2 \mathcal{W} \subseteq \rho \mathcal{W}$ is equal to the largest element of $\max_{w \in \mathcal{W}} V \Phi^2 w$. Using the disturbance set representation

$$\mathcal{W} = \{w : Vw \leq \mathbf{1}\}, \quad V = \begin{bmatrix} 2.5 & 2.5 \\ -2.5 & -2.5 \\ -2.5 & 2.5 \\ 2.5 & -2.5 \end{bmatrix}$$

we can compute ρ by solving a linear program:

$$\rho = \min_{t,w} t \ \text{ subject to } \ \begin{aligned} V\Phi^2 w &\leq t\mathbf{1} \\ Vw &\leq \mathbf{1} \end{aligned}$$

Hence $\rho = 0.228$ and $\mathcal{S} = 1.295(\mathcal{W} \oplus \Phi \mathcal{W})$. With this expression for $\mathcal{S}$ we can obtain $h_{\mathcal{S}} = \max_{e \in \mathcal{S}} Fe$ by solving a set of linear programs (one pair of linear programs for each element of $h_{\mathcal{S}}$):

$$h_{\mathcal{S}} = 1.295\left(\max_{w : Vw \leq \mathbf{1}} Fw + \max_{w : Vw \leq \mathbf{1}} F\Phi w\right) = \begin{bmatrix} 0.788 \\ 0.827 \\ 0.788 \\ 0.827 \end{bmatrix}.$$

(b) With the numerical values of Ψ and $\bar{F}$ given in the solution of Question 6(a), we get

$$\max_{z}\left\{\bar{F}\Psi^3 z \text{ subject to } \bar{F}\Psi^i z \leq \mathbf{1} - h_{\mathcal{S}},\ i = 0, 1, 2\right\} = \begin{bmatrix}0.030\\0.041\\0.030\\0.041\end{bmatrix} \leq \mathbf{1} - h_{\mathcal{S}}$$

which implies that $\{z : \bar{F}\Psi^i z \leq \mathbf{1} - h_{\mathcal{S}},\ i = 0, 1, 2\}$ is the maximal invariant set for the nominal prediction system $z_{k+1} = \Psi z_k$ and constraints $\bar{F}z_k \leq \mathbf{1} - h_{\mathcal{S}}$.

(c) The predicted state is decomposed as $x_{i|k} = s_{i|k} + e_{i|k}$ where $e_{i|k} \in \mathcal{S}$ for all $i \geq 0$ and $s_{i|k}$ evolves according to the nominal dynamics (in which $w_{k+i} = 0$ for all $i \geq 0$ so that $s_{i|k} = \begin{bmatrix} I & 0\end{bmatrix} z_{i|k}$ where $z_{i+1|k} = \Psi z_{i|k}$). This decomposition allows the constraints $Fx_{i|k} \leq \mathbf{1}$ to be imposed robustly through the conditions $\bar{F}\Psi^i z_k \leq \mathbf{1} - h_{\mathcal{S}},\ i = 0, 1, 2$ with $z_k = (s_{0|k}, \mathbf{c}_k)$. Thus $s_{0|k}$ is the initial state of the nominal prediction system, the predicted trajectories of which are constrained so that $F(s_{i|k} + e_{i|k}) \leq \mathbf{1}$ for all $i \geq 0$.

To ensure that $e_{i|k} \in \mathcal{S}$ for all $i \geq 0$, we require that $s_{0|k}$ satisfies $e_{0|k} = x_k - s_{0|k} \in \mathcal{S}$. Using the hyperplane description of $\mathcal{W}$:

$$\mathcal{W} = \text{Co}\{w^{(j)},\ j = 1, \ldots, 4\} = \text{Co}\left\{\pm\begin{bmatrix}0.4\\0\end{bmatrix}, \pm\begin{bmatrix}0\\0.4\end{bmatrix}\right\}$$

it is possible to compute $\mathcal{S}$ as the convex hull of the 16 vertices that are formed from $w^{(i)} + \Phi w^{(j)}$ for all $i, j = 1, \ldots, 4$. This convex hull has 8 vertices:

$$\mathcal{S} = \text{Co}\left\{\pm\begin{bmatrix}0.788\\0.254\end{bmatrix}, \pm\begin{bmatrix}0.270\\0.772\end{bmatrix}, \pm\begin{bmatrix}0.159\\0.827\end{bmatrix}, \pm\begin{bmatrix}-0.359\\0.309\end{bmatrix}\right\}$$

In order to invoke the constraint $x_k - s_{0|k} \in \mathcal{S}$ in a manner that avoids introducing additional optimization variables, we need to compute the hyperplane representation:

$$\mathcal{S} = \{e : V_{\mathcal{S}}e \leq \mathbf{1}\}, \quad V_{\mathcal{S}} = \begin{bmatrix}-0.960 & -0.960\\-0.551 & -1.103\\1.498 & -1.498\\1.681 & -1.284\\0.960 & 0.960\\0.551 & 1.103\\-1.498 & 1.498\\-1.681 & 1.284\end{bmatrix}.$$

For $\mathcal{S} \subset \mathbb{R}^2$ with a small number of vertices this can done simply by plotting the vertices to determine which lie on each facet of $\mathcal{S}$, then computing the corresponding hyperplanes. For more complicated sets in higher dimensions, dedicated software (for example lrslib [1], cddlib [2]) can be used to convert between vertex and hyperplane representations.

(d) The objective function of the MPC optimization is $\|(s_{0|k}, \mathbf{c}_k)\|^2_{W_z}$, where $W_z = \operatorname{diag}\{W_x, W_c\}$ is as given in the solution to Question 6(c). The MPC optimization is then

$$\underset{s_{0|k},\mathbf{c}_k}{\text{minimize}} \quad \left\| \begin{bmatrix} s_{0|k} \\ \mathbf{c}_k \end{bmatrix} \right\|^2_{W_z} \quad \text{subject to} \quad \begin{aligned} \bar{F}\Psi^i \begin{bmatrix} s_{0|k} \\ \mathbf{c}_k \end{bmatrix} &\le \mathbf{1} - h_{\mathcal{S}}, \; i = 0, 1, 2 \\ V_{\mathcal{S}}(x_k - s_{0|k}) &\le \mathbf{1} \end{aligned}$$

For $x_0 = (0, 1)$ the optimal solution is given by $s^*_{0|0} = (-0.159, 0.173)$, $\mathbf{c}^*_0 = 0.0163$ and $\|s^*_{0|0}\|^2_{W_x} + \|\mathbf{c}^*_0\|^2_{W_c} = 0.122$.

(e) Rigid tubes provide conservative bounds on the effects of disturbances on predicted trajectories, thus reducing the size of the feasible set of states relative to e.g. the robust MPC strategy of Question 6. This effect can be countered (but not eliminated, in general) by choosing K_e to improve the disturbance rejection properties of the system $e_{k+1} = (A + BK_e)e_k + w_k$, thus reducing the size of $\mathcal{S}$ and allowing tighter bounds on the effects of disturbances on constrained variables.

9 (a) From $\Phi\mathcal{S} \oplus \mathcal{W} \subseteq \mathcal{S}$, where $\mathcal{S} = \{s : V_{\mathcal{S}} s \le \mathbf{1}\}$, we have

$$V_{\mathcal{S}}\Phi e + V_{\mathcal{S}} w \le V_{\mathcal{S}} e \le \mathbf{1}, \quad \forall e \in \mathcal{S}, \; \forall w \in \mathcal{W}.$$

This condition is equivalent to $\bar{e} + \bar{w} \le \mathbf{1}$.

(b) Let

$$\mathcal{Z}^{(\nu)} = \left\{(z, \boldsymbol{\alpha}) \;:\; \bar{F}\Psi^i z \le \mathbf{1} - \alpha_i h_{\mathcal{S}}, \; \alpha_i \bar{e} + \bar{w} \le \alpha_{i+1}, \; i = 0, \ldots, \nu\right\}$$

where $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{N-1})$ and $\alpha_i = 1$ for all $i \ge N$. If $\nu \ge N - 1$ satisfies $\bar{F}\Psi^{\nu+1} z \le h_{\mathcal{S}}$ for all $(z, \boldsymbol{\alpha}) \in \mathcal{Z}^{(\nu)}$, then from Theorem 2.3 it follows that $\mathcal{Z}^{(\nu)}$ is an invariant set (in fact it is the MPI set) for the dynamics $z^+ = \Psi z$ and $\boldsymbol{\alpha}^+ = (\alpha_1, \ldots, \alpha_{N-1}, 1)$, and the constraints $\bar{F}z \le \mathbf{1} - \alpha_0 h_{\mathcal{S}}$.

From the robust invariance property of $\mathcal{S}$ and the definition of $\mathcal{Z}^{(\nu)}$, if $x_k - s_{0|k} \in \alpha_{0|k}\mathcal{S}$ and $(z_k, \boldsymbol{\alpha}_k) \in \mathcal{Z}^{(\nu)}$ where $z_k = (s_{0|k}, \mathbf{c}_k)$, then by construction $x_{k+1} - s_{1|k} \in \alpha_{1|k}\mathcal{S}$ for all $w_k \in \mathcal{W}$. Furthermore the invariance of $\mathcal{Z}^{(\nu)}$ implies $\big(\Psi z_k, (\alpha_{1|k}, \ldots, \alpha_{N-1|k}, 1)\big) \in \mathcal{Z}^{(\nu)}$. Hence

$$\begin{bmatrix} s_{0|k+1} \\ \mathbf{c}_{k+1} \end{bmatrix} = \Psi \begin{bmatrix} s_{0|k} \\ \mathbf{c}_k \end{bmatrix}$$
$$\boldsymbol{\alpha}_k + 1 = (\alpha_{1|k}, \ldots, \alpha_{N-1|k}, 1)$$

satisfy $x_{k+1} - s_{0|k+1} \in \alpha_{0|k+1}\mathcal{S}$ and $(z_{k+1}, \boldsymbol{\alpha}_k + 1) \in \mathcal{Z}^{(\nu)}$ at time $k + 1$.

(c) From part (b), feasibility at $k = 0$ implies feasibility at all times $k > 0$. Therefore the definition of $h_{\mathcal{S}}$ and the constraints $x_k - s_{0|k} \in \alpha_{0|k}\mathcal{S}$ and $\bar{F}z_k = Fs_{0|k} \leq \mathbf{1} - \alpha_{0|k}h_{\mathcal{S}}$ imply that $Fx_k \leq \mathbf{1}$ for all $k \geq 0$.
From the feasible solution in part (b) we obtain the cost bound:

$$\begin{aligned} J(s^*_{0|k+1}, \mathbf{c}^*_{k+1}, {}^*_{k+1}) &\leq \begin{bmatrix} s^*_{0|k} \\ \mathbf{c}^*_k \end{bmatrix}^T \Psi^T W_z \Psi \begin{bmatrix} s^*_{0|k} \\ \mathbf{c}^*_k \end{bmatrix} + \sum_{i=1}^{N-1} q_\alpha(\alpha^*_{i|k} - 1)^2 \\ &= \begin{bmatrix} s^*_{0|k} \\ \mathbf{c}^*_k \end{bmatrix}^T \left(W_z - \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} K^T \\ E^T \end{bmatrix} R \begin{bmatrix} K & E \end{bmatrix} \right) \begin{bmatrix} s^*_{0|k} \\ \mathbf{c}^*_k \end{bmatrix} \\ &\quad + \sum_{i=1}^{N-1} q_\alpha(\alpha^*_{i|k} - 1)^2 \\ &= J(s^*_{0|k}, \mathbf{c}^*_k, {}^*_k) - (\|s_{0|k}\|^2_Q + \|v_{0|k}\|^2_R) - q_\alpha(\alpha^*_{0|k} - 1)^2. \end{aligned}$$

Using, for example, the argument of the proof of Theorem 3.6, it then follows that $x_k \to \mathcal{S}$ asymptotically as $k \to \infty$ (in fact the minimum distance from x_k to any point in $\mathcal{S}$ decays exponentially with k).

10 (a) The solution of Question 8 gives $\rho = 0.228$ and $\mathcal{S} = 1.295(\mathcal{W} \oplus \Phi\mathcal{W})$, thus allowing $V_{\mathcal{S}}$ to be determined so that $\mathcal{S} = \{s \,:\, V_{\mathcal{S}}s \leq \mathbf{1}\}$. Hence by solving a set of linear programs (one LP for each element of $\bar{w}$ and $\bar{e}$) we obtain

$$\bar{e} = \max_{e: V_{\mathcal{S}}e \leq \mathbf{1}} V_{\mathcal{S}}\Phi e = \begin{bmatrix} 0.616 & 0.552 & 0.365 & 0.259 & 0.616 & 0.552 & 0.365 & 0.259 \end{bmatrix}^T$$

$$\bar{w} = \max_{w: Vw \leq \mathbf{1}} V_{\mathcal{S}}w = \begin{bmatrix} 0.384 & 0.441 & 0.599 & 0.673 & 0.384 & 0.441 & 0.599 & 0.673 \end{bmatrix}^T$$

For $N = 1$ and the numerical values of Ψ, $\bar{F}$ given in the solution of Question 6(a) we obtain

$$\max_{z,} \left\{ \bar{F}\Psi^3 z \;\text{ subject to } (z,) \in \mathcal{Z}^{(2)} \right\} = \begin{bmatrix} 0.030 \\ 0.041 \\ 0.030 \\ 0.041 \end{bmatrix} \leq \mathbf{1} - h_{\mathcal{S}}$$

and hence $\nu = 2$ satisfies the conditions for invariance of $\mathcal{Z}^{(\nu)}$.

(b) Solving the QP that defines the MPC optimization at $k = 0$ with $x_0 = (0, 1)$ we get

$$s^*_{0|0} = (-0.159, 0.173), \quad \mathbf{c}^*_0 = 0.0163, \quad {}^*_0 = 1.0,$$

and hence $\mathbf{c}^*_0$ is equal to the optimal solution for rigid tube MPC for the same initial condition. The explanation for this is that the HTMPC online optimization places a penalty on $|1 - \alpha_{i|k}|$ and hence its optimal solution will be equal to that of rigid tube MPC if it is feasible.

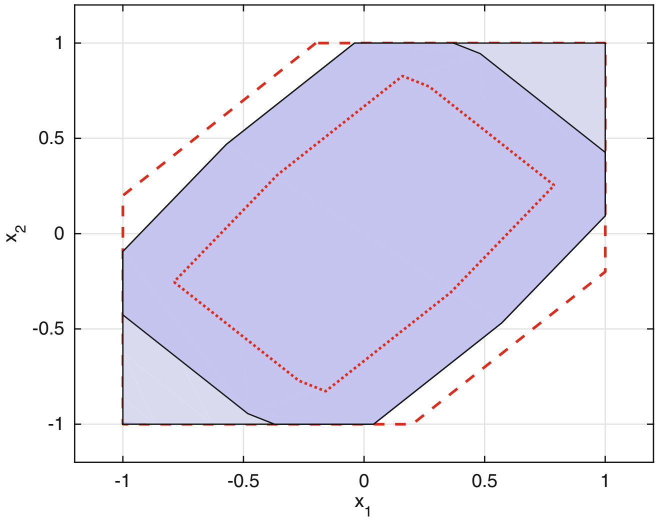

Fig. A.3 The feasible sets for the rigid and homothetic tube MPC laws in Questions 8 and 9, *shaded dark* and *light blue* respectively. Also shown are the set of feasible states for the MPC law of Question 6 (*outer*, *red dashed line*) and the minimal RPI set for $e_{k+1} = \Phi e_k + w_k$, $w_k \in \mathcal{W}$ (*inner*, *red dotted line*)

(c) Comments on the size of the feasible sets for these algorithms:

- The feasible initial condition set of the rigid tube MPC strategy is necessarily a subset of that for HTMPC, since if $\alpha_{i|k} = 1$ for all i, then the constraints of HTMPC are identical to those of rigid tube MPC. Since it is able to scale the uncertainty set $\mathcal{S}$, HTMPC may also be feasible for initial conditions for which the rigid tube MPC is infeasible.
- The feasible initial condition set of HTMPC is itself a subset of the robust MPC strategy of Question 6 (namely Algorithm 3.1), since this approach employs the tightest available bounds on the unknown future disturbances appearing in the constraints on predicted variables, whereas the corresponding bounds in HTMPC are computed using outer (possibly conservative) bounds based on $\alpha_{i|k}\mathcal{S}$.

  This nested property can be seen in Fig. A.3, which shows the feasible sets for the numerical examples in Questions 6, 8 and 9.
  Comments on performance:

- As mentioned in the solution to (b), the performance of the HTMPC and rigid tube MPC algorithms of Questions 8 and 9 are identical for every initial condition such that rigid tube MPC is feasible.
- The predicted cost for HTMPC must be greater than or equal to that of the robust MPC strategy of Question 6 since the overbounding of the constraints in HTMPC implies that any feasible $\mathbf{c}_k$ for HTMPC is also feasible for the strategy of Question 6. A possible advantage of the algorithms of Questions 7 and 8 over Question 6

is that they ensure exponential convergence to an outer approximation of the mRPI set, while the control law Question 6 ensures convergence of the state to the mRPI set, but not necessarily exponential convergence.

Solutions to Exercises for Chap. 5

1 (a) Each element of $M(x) = M_0 + x_1 M_1 + \cdots + x_n M_n$ is an affine function of x. For any $y, z \in \mathbb{R}^n$ and any scalar λ we therefore have

$$M\big(\lambda y + (1-\lambda)z\big) = \lambda M(y) + (1-\lambda)M(z).$$

Suppose that $M(y) \succ 0$ and $M(z) \succ 0$, so that $v^T M(y)v > 0$ and $v^T M(z)v > 0$ for all vectors $v \neq 0$. Then for all $0 \leq \lambda \leq 1$ we have

$$v^T M\big(\lambda y + (1-\lambda)z\big)v = \lambda v^T M(y)v + (1-\lambda)v^T M(z)v > 0$$

for all $v \neq 0$. This implies that, if $x = \lambda y + (1-\lambda)z$, then $M(x) \succ 0$ for all $0 \leq \lambda \leq 1$ and thus demonstrates that $M(x) \succ 0$ is a convex condition on x.

(b) The matrix $\begin{bmatrix} P & Q \\ Q^T & P \end{bmatrix}$ is positive definite if

$$\begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} > 0 \qquad (\star)$$

holds for all $(v, w) \neq 0$. Consider the cases of $v = 0$ and $w \neq 0$ separately:

(I) If $v = 0$, then the condition $(\star)$ is equivalent to $w^T R w > 0$ for all $w \neq 0$, i.e. $R \succ 0$.

(II) For non-zero v, consider the minimum of the LHS of $(\star)$ over all w. This is achieved with $w = -R^{-1}Qv$ so that

$$\min_w (v^T P v + 2v^T Q w + w^T R w) = v^T (P - Q^T R^{-1} Q)v.$$

Hence $(\star)$ holds for all $v \neq 0$ if and only if $P - Q^T R^{-1} Q \succ 0$.

From (I) and (II) we conclude that $(\star)$ is equivalent to the Schur complements $R \succ 0$ and $P - Q^T R^{-1} Q \succ 0$.

(c) Pre- and post-multiplying $P - A^T P A \succ 0$ by $S = P^{-1} > 0$ gives the equivalent inequality $S - SA^T S^{-1} AS \succ 0$. Using Schur complements (noting that $S \succ 0$), this is equivalent to

$$\begin{bmatrix} S & SA^T \\ AS & S \end{bmatrix} \succ 0$$

Using Schur complements again gives the equivalent conditions:

$$S - ASS^{-1}SA^T \succ 0, \quad S \succ 0$$

i.e. $S - ASA^T \succ 0$ whenever $P - A^T PA \succ 0$ if $S = P^{-1} \succ 0$.

2 (a) For $x \in \mathcal{X}$ we require $V_i x \leq 1$ for $i = 1, \ldots, n_V$. Inserting P and $S = P^{-1}$ into these conditions gives $V_i S^{1/2} P^{1/2} x \leq 1$, so the Cauchy-Schwarz inequality implies $V_i S^{1/2} P^{1/2} x \leq (V_i S V_i^T)^{1/2} (x^T P x)^{1/2}$, which immediately shows that the conditions

$$V_i P^{-1} V_i^T \leq 1, \quad i = 1, \ldots, n_V$$

are sufficient to ensure that $V_i x \leq 1$, $i = 1, \ldots, n_V$ for all $x \in \mathcal{E}$. However these conditions are also necessary for $\mathcal{E} \subseteq \mathcal{X}$ because the maximum of $V_i x$ subject to $x^T P x \leq 1$ is equal to the quantity on the LHS since

$$\max_{\substack{x \\ x^T P x \leq 1}} V_i x = \max_{\substack{\xi \\ \|\xi\| \leq 1}} V_i P^{-1/2} \xi = V_i P^{-1} V_i^T.$$

(b) The result of part (a) implies $(F_i + G_i K)x \leq 1$ for all x such that $x^T S^{-1} x \leq 1$ if and only if $1 - (F_i + G_i K) S (F_i + G_i K)^T \geq 0$. Using Schur complements, the last condition is equivalent to $S \succ 0$ and

$$\begin{bmatrix} 1 & F_i S + G_i Y \\ (F_i S + G_i Y)^T & S \end{bmatrix} \succeq 0$$

for $i = 1, \ldots, n_C$.

3 (a) The conditions given in the question ensure that:

(i) $x_k \in \mathcal{E}$, where $\mathcal{E} = \{x : x^T P x \leq 1\}$,
(ii) $-1 \leq u_{i|k} \leq 1$ for all $i \geq 0$ if $x_k \in \mathcal{E}$ and $u_{i|k} = K_k x_{i|k}$,
(iii) $\check{J}(x_k, K_k) \leq \gamma_k x_k^T P x_k \leq \gamma_k$ if $x_k \in \mathcal{E}$ and $u_{i|k} = K_k x_{i|k}$.

(b) When expressed in terms of the variables $S = P^{-1}$ and $Y = K_k P^{-1}$, the optimization becomes the semidefinite programming problem given in (5.18). Solving this using the model parameters given in the question and $x_0 = (4, -1)$ results in

$$\gamma_0 = 152.4, \quad S = \begin{bmatrix} 16.24 & -3.932 \\ -3.932 & 1.019 \end{bmatrix}, \quad Y = \begin{bmatrix} -1.170 & 0.034 \end{bmatrix}$$

and therefore $K_0 = \begin{bmatrix} -0.962 & -3.678 \end{bmatrix}$.

(c) The constraints on Θ ensure that $\|x_{i+1|k}\|_\Theta^2 \leq \|x_{i|k}\|_\Theta^2 - \|x_{i|k}\|_Q^2 - \|u_{i|k}\|_R^2$ holds along all predicted trajectories of the model under the predicted control law $u_{i|k} = K_k x_{i|k}$. Therefore the worst case predicted cost has the upper bound

$\check{J}(x_k, K_k) \leq \|x_k\|^2_\Theta$. The cost bound in (c) will in general be smaller than the bound in (a) because $\Theta = P\gamma$ is feasible for the optimization in (c) whenever P and γ satisfy the constraints of the optimization in (a), and because (a) includes additional constraints.

(d) The suggested optimization is likely to result in an MPC law with improved performance since the implied online MPC optimization minimizes a tighter upper bound on the worst case predicted cost. However the constraints involve products of optimization variables and hence are nonconvex, and furthermore there is no convexifying transformation of variables that can be employed in this case. Therefore it will in general be difficult to compute efficiently the global optimum for the suggested optimization, and the computational is likely to grow rapidly with problem size.

4 (a) Any point u that belongs to the projection onto the u-subspace of the set $\mathcal{E} = \{x = (u, v) : x^T P x \leq 1\}$ satisfies, by definition,

$$\min_v \left(u^T P_{uu} u + 2u^T P_{uv} v + v^T P_{vv} v\right) \leq 1,$$

where P_{uu}, P_{uv}, P_{vv} are the blocks of P:

$$P = \begin{bmatrix} P_{uu} & P_{uv} \\ P_{uv}^T & P_{vv} \end{bmatrix}.$$

Since $P \succ 0$ implies that $P_{vv} \succ 0$, the u-subspace projection of $\mathcal{E}$ is therefore given by

$$\text{proj}_u(\mathcal{E}) = \left\{u : u^T (P_{uu} - P_{uv} P_{vv}^{-1} P_{uv^T}) u \leq 1\right\}$$

Thus P_u is equal to the Schur complement $P_{uu} - P_{uv} P_{vv}^{-1} P_{uv^T}$. This can equivalently be expressed in terms of the blocks of P^{-1} as

$$P_u^{-1} = \begin{bmatrix} I_m & 0 \end{bmatrix} P^{-1} \begin{bmatrix} I_m \\ 0 \end{bmatrix}.$$

(b) The x-subspace projection of $\mathcal{E}_z$ is maximized by solving the SDP:

$$\begin{aligned} \underset{S}{\text{maximize}} \quad & \log\det(S_{xx}) \\ \text{subject to} \quad & \begin{bmatrix} S & \Psi^{(j)} S \\ S\Psi^{(j)T} & S \end{bmatrix} \succeq 0, \quad j = 1, 2 \\ & \begin{bmatrix} 1 & \begin{bmatrix} K & E \end{bmatrix} S \\ S \begin{bmatrix} K^T \\ E^T \end{bmatrix} & S \end{bmatrix} \succeq 0 \end{aligned}$$

The optimal solution gives $P_x = S_{xx}^{-1} = \begin{bmatrix} 0.839 & 3.211 \\ 3.211 & 13.23 \end{bmatrix}$ and $\det(P_x) = 0.783$.

(c) The matrix W appearing in the expression $J(x_k, \mathbf{c}_k) = z_k^T W z_k$ for the nominal cost could be determined by solving the Lyapunov equation $W - \Psi^{(0)\,T} W \Psi^{(0)} = \text{diag}\{Q, 0\} + \begin{bmatrix} K & E \end{bmatrix}^T R \begin{bmatrix} K & E \end{bmatrix}$. However the question states that K is the unconstrained optimal feedback gain for the nominal cost, and by Theorem 2.10 we must therefore have

$$W = \begin{bmatrix} W_x & 0 \\ 0 & W_c \end{bmatrix}, \quad W_c = \text{diag}\{B^{(0)\,T} W_x B^{(0)} + R, \ldots, B^{(0)\,T} W_x B^{(0)} + R\}$$

where W_x is the solution of the Riccati equation that is provided in the question and $B^{(0)\,T} W_x B^{(0)} + R = 4.891$.

The minimization of the nominal predicted cost is equivalent to

$$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \ \|\mathbf{c}_k\|^2 \ \text{ subject to } \ (x_k, \mathbf{c}_k) \in \mathcal{E}_z,$$

which is a convex quadratic programming problem with a single quadratic constraint. In applications requiring very fast online computation this can be solved using an efficient Newton-Raphson iteration as discussed in Sect. 2.8. If computational load is not important, it can alternatively (and more conveniently) be rewritten as a second order cone programming problem and solved using a generic SOCP solver. The solution for $x_0 = (4, -1)$ gives

$$J^*(x_0) = \|x_0\|_{W_x}^2 + 4.891\|\mathbf{c}_0\|^2 = 39.9.$$

5 (a) Whenever the constraint $(x_k, \mathbf{c}_k) \in \mathcal{E}_z$ is inactive in the optimization in Question 4(c) (i.e. whenever $\mathbf{c}_k^* \neq 0$), the line search defined in the question results in $z_k = (x_k, \alpha_k^* \mathbf{c}_k^*) \notin \mathcal{E}_z$. Therefore the constraints of the line search are needed in order to ensure that:

(i) The input (or more generally mixed input/state) constraints are satisfied at the current sampling instant
(ii) The optimization in Question 4(c) is feasible at time $k + 1$

The second of these conditions is imposed in the line search through a robust constraint on the one step-ahead prediction $z_{1|k}$ in order to ensure that the predicted cost decreases along closed loop system trajectories. This provides a way to guarantee closed loop stability.

(b) Minimizing the value of $\sigma \geq 0$ subject to the LMI of the question gives the optimal values:

$$\sigma^2 = 9.849, \quad \Theta = \begin{bmatrix} 2.856 & -0.009 \\ -0.009 & 15.385 \end{bmatrix}.$$

From Lemma 5.3 it follows that the state x_k under $u_k = Kx_k + c_{0|k}$ satisfies the bound

$$\sum_{k=0}^{\infty} \|x_k\|^2 \le \|x_0\|_{\Theta}^2 + 9.85 \sum_{k=0}^{\infty} c_{0|k}^2.$$

(c) The constraints of the line search ensure that the optimization is recursively feasible, since $z_k = (x_k, \mathbf{c}_k^*)$ satisfies $\Psi^{(j)} z_k \in \mathcal{E}_z$, $j = 1, 2$, and hence by convexity we have $z_{k+1} = \Psi_k z_k \in \mathcal{E}_z$ whenever $z_k \in \mathcal{E}_z$.
The feasibility of $z_{k+1} = \Psi_k z_k$ and the definition of W_c implies that the solution of the online optimization, $\alpha_k^* \mathbf{c}_k^*$ satisfies the bound

$$\|\alpha_{k+1}^* \mathbf{c}_{k+1}\|^2 - \|\alpha_k^* \mathbf{c}_k\|^2 \le -\alpha_k^* c_{0|k}^2$$

and hence

$$\sum_{k=0}^{\infty} \alpha_k^* c_{0|k}^2 \le \|\alpha_0^* \mathbf{c}_0\|^2.$$

From the answer to part (b) the state of the closed loop system therefore satisfies the quadratic bound

$$\sum_{k=0}^{\infty} \|x_k\|^2 \le \|x_0\|_{\Theta}^2 + 9.85 \|\alpha_0^2 \mathbf{c}_0\|^2,$$

implying asymptotic convergence: $x_k \to 0$ as $k \to \infty$. Since the origin of the closed loop system state space is necessarily Lyapunov stable (because $u_k = Kx_k$ is feasible at all points in some region that contains $x = 0$), it follows that the closed loop system is asymptotically stable. The region of attraction is the feasible set for the optimization in Question 4(c), namely the projection of $\mathcal{E}_z$ onto the x-subspace.

6 (a) Performing the optimization

$$\underset{\Xi^{(1)}, \Xi^{(2)}, \Gamma, X, Y}{\text{maximize}} \quad \log\det(Y) \ \text{ subject to (5.47) and (5.44b)}$$

and using the inverse transformation (5.45), we get the values of $A_c^{(1)}$, $A_c^{(2)}$ and C_c given in the question and

$$P_z = S^{-1} = \begin{bmatrix} 19.11 & -4.68 & -19.11 & 0.00 \\ -4.68 & 1.24 & 4.68 & -0.10 \\ -19.11 & 4.68 & 19.11 & 0.00 \\ 0.00 & -0.10 & 0.00 & 0.10 \end{bmatrix}^{-1}.$$

(b) Minimizing $\mathrm{tr}(W_c)$ subject to the LMI in the question gives

$$W_c = \begin{bmatrix} 0.561 & -0.175 \\ -0.175 & 3.715 \end{bmatrix}$$

Hence the minimum value of $J(x_k, \mathbf{c}_k)$ over $\mathbf{c}_k$ subject to $(x_k, \mathbf{c}_k) \in \mathcal{E}_z$ is $J^*(x_0) = 44.66$.

(c) The LMI satisfied by W_c implies that the optimal $\mathbf{c}_k^*$ sequence for $k = 0, 1, \ldots$ satisfies

$$\|\mathbf{c}_{k+1}^*\|_{W_c}^2 \le \|\mathbf{c}_k^*\|_{W_c}^2 - 4.891(C_c\mathbf{c}_k^*)^2$$

and hence

$$\sum_{k=0}^{\infty}(C_c\mathbf{c}_k^*)^2 \le \frac{1}{4.891}\|\mathbf{c}_0^*\|_{W_c}^2.$$

Therefore, from the quadratic bound on the l^2-norm of the closed loop state sequence in Question 4(c) we get

$$\sum_{k=0}^{\infty}\|x_k\|^2 \le \|x_0\|_{\Theta}^2 + 9.85\sum_{k=0}^{\infty}(C_c\mathbf{c}_k)^2 \le \|x_0\|_{\Theta}^2 + 2.01\|\mathbf{c}_0^*\|_{W_c}^2$$

which implies $x_k \to 0$ as $k \to \infty$, and hence by the argument that was used in Question 5(c), the origin of the closed loop system is asymptotically stable with region of attraction equal to the projection of $\mathcal{E}_z$ onto the x-subspace.

(d) For $x_0 = (4, -1)$, the maximum scaling σ such that σx_0 is feasible in each case is given in the following table.

Optimization:	Question 3(b) (Algorithm 5.1)	Question 4(c) (Algorithm 5.3)	Question 6(b) (Algorithm 5.4)
σ	1.17	1.02	1.09

This is consistent with the expectation that Algorithm 5.1 has the largest feasible set of these three algorithms, since it computes (where possible) a robustly invariant ellipsoidal set online that contains the current state, whereas the robustly invariant ellipsoidal sets in the other two algorithms are determined offline, and hence without reference to the current state, so as to maximize the volume of their x-subspace projections. Similarly the feasible set of Algorithm 5.4 is expected to be at least as large as that of Algorithm 5.2 since it coincides with the maximal volume robustly invariant ellipsoidal set under any linear feedback law. These observations are confirmed by the feasible sets plotted in Fig. A.4.

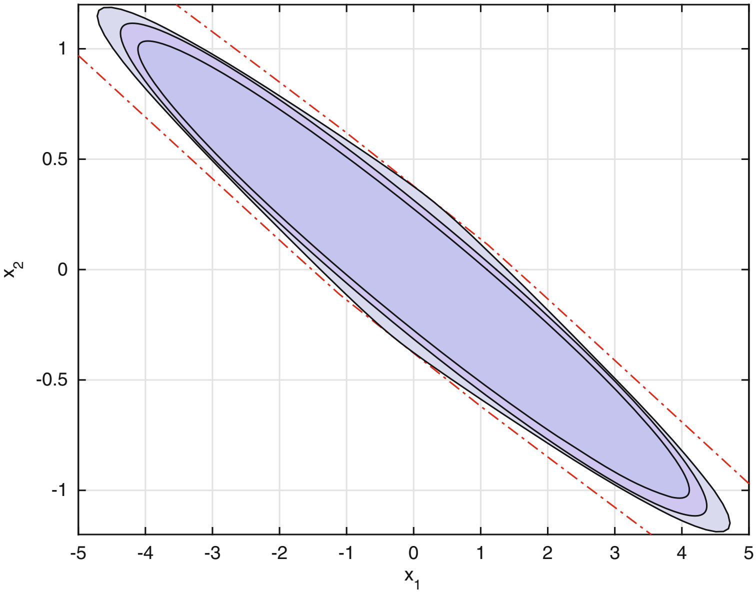

Fig. A.4 The feasible sets for Algorithm 5.2 (*inner shaded set*), Algorithm 5.4 and Algorithm 5.1 (*outer shaded set*) in Questions 6(b), 4(c) and 3(b). The *dashed line* shows the boundary of the maximal controllable set for this system and constraints

7 The volume of the low-complexity polytope $\Pi(V, \alpha) = \{x \,:\, |Vx| \le \alpha\}$ in $\mathbb{R}^{n_x}$ can be evaluated by considering it to be a linear transformation of the hypercube $\{x : |x| \le \mathbf{1}\}$. This gives

$$\text{volume}\big(\Pi(V, \alpha)\big) = C_{n_x}\, |\det(W)| \prod_{i=1}^{n_x} \alpha_i$$

where $W = V^{-1}$, $\alpha = (\alpha_1, \ldots, \alpha_{n_x})$, and C_{n_x} is a constant. Although the maximization of the determinant of a symmetric positive definite matrix P can be expressed in terms of the maximization of a concave function of its elements, e.g. $\log\big(\det(P)\big)$, which can therefore form the objective of a convex optimization, the matrix W is here neither symmetric nor positive definite in general. However for fixed V, maximizing the product of the elements of then non-negative vector α is equivalent to maximizing the determinant of the symmetric positive definite matrix $P = \text{diag}\{\alpha_1, \ldots, \alpha_{n_x}\}$, which can be expressed as a concave function of α.

8 (a) For the matrix V given in the question, the matrix $\bar{\Phi}$ whose (i, j)th element is equal to the larger of the (i, j)th element of $V\Phi^{(1)}W$ and the (i, j)th element of $V\Phi^{(2)}W$, is given by

$$\bar{\Phi} = \begin{bmatrix} 0.719 & 0.229 \\ 0.031 & 0.583 \end{bmatrix}.$$

The maximum eigenvalue of $\bar{\Phi}$ is equal to 0.760, and since this is less than unity, it follows that with α equal to the corresponding eigenvector we necessarily obtain

$$V\Phi^{(j)}W\alpha \leq \alpha, \quad j = 1, 2$$

which is the condition for robust invariance of the set

$$\Pi(V, \alpha) = \{x \,:\, |Vx| \leq \alpha\}$$

under the dynamics $x_{k+1} \in \text{Co}\{\Phi^{(1)}x_k, \Phi^{(2)}x_k\}$ (see Lemma 5.4 for the proof of this result). Here we also require that $-1 \leq Kx \leq 1$ holds for all $x \in \Pi(V, \alpha)$, and this can be ensured by scaling α.
Checking this result numerically, we have

$$\alpha = \begin{bmatrix} 0.985 \\ 0.175 \end{bmatrix}, \quad |V\Phi^{(1)}W|\alpha = \begin{bmatrix} 0.748 \\ 0.098 \end{bmatrix}, \quad |V\Phi^{(2)}W|\alpha = \begin{bmatrix} 0.615 \\ 0.133 \end{bmatrix}$$

which confirms that $|V\Phi^{(j)}W|\alpha \leq \alpha$ for $j = 1, 2$.

(b) The volume of $\Pi(V, \alpha)$ is maximized by the optimization

$$\begin{aligned} \underset{\alpha=(\alpha_1,\alpha_2)}{\text{maximize}} \quad \log(\alpha_1\alpha_2) \quad \text{subject to} \quad & \alpha > 0 \\ & |V\Phi^{(j)}W|\alpha \leq \alpha, \quad j = 1, 2 \\ & |KW|\alpha \leq 1 \end{aligned}$$

which is convex and can be solved using, for example, any method for solving determinant maximization problems subject to linear constraints. For the problem data in the question, the optimal solution is

$$\alpha = \begin{bmatrix} 1.247 \\ 0.181 \end{bmatrix}$$

for which the conditions for robust invariance are satisfied since

$$|V\Phi^{(1)}W|\alpha = \begin{bmatrix} 0.938 \\ 0.108 \end{bmatrix}, \quad |V\Phi^{(2)}W|\alpha = \begin{bmatrix} 0.770 \\ 0.144 \end{bmatrix}, \quad |KW|\alpha = 1.$$

9 (a) Although this question uses the general complexity polytopic tube framework of Sect. 5.5, the set $\{x \,:\, Vx \leq \mathbf{1}\}$ is a low-complexity polytope. Hence the linear programs (5.94) and (5.95) that define the matrices $H^{(1)}$, $H^{(2)}$ and H_c with minimum row-sums have closed form solutions (described on p. 214). Using these solutions (or alternatively by solving an LP to determine each row) we get

$$H_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$H^{(1)} = \begin{bmatrix} 0 & 0.041 & 0.678 & 0 \\ 0.041 & 0 & 0 & 0.678 \\ 0.391 & 0 & 0.379 & 0 \\ 0 & 0.391 & 0 & 0.379 \end{bmatrix}, \quad H^{(2)} = \begin{bmatrix} 0 & 0 & 0.999 & 0 \\ 0 & 0 & 0 & 0.999 \\ 0.348 & 0 & 0.002 & 0 \\ 0 & 0.347 & 0 & 0.002 \end{bmatrix}$$

(b) Assume that $\mathbf{c}_k = (c_{0|k}, \ldots, c_{N-1|k})$ and $\boldsymbol{\alpha}_k = (\alpha_{0|k}, \ldots, \alpha_{N|k})$ satisfy the constraints given in the question at time k. If $u_k = Kx_k + c_{0|k}$, then a feasible solution at time $k+1$ is given by

$$\begin{aligned} \mathbf{c}_{k+1} &= (c_{1|k}, c_{2|k}, \ldots, c_{N-1|k}, 0) \\ \boldsymbol{\alpha}_{k+1} &= (\alpha_{1|k}, \ldots, \alpha_{N|k}, \alpha_{N|k}) \end{aligned}$$

Therefore the constraint set is recursively feasible (Fig. A.5).

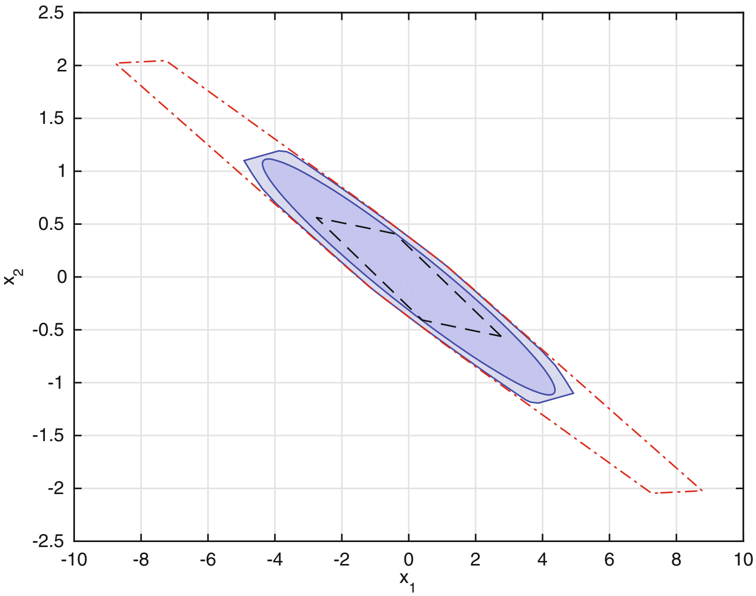

Fig. A.5 The feasible sets for the constraints in Question 9(b) (*outer shaded set*) and for Algorithm 5.4 (*inner shaded ellipsoidal set*). The boundary of the maximal controllable set (*dash-dotted line*) and the boundary of the maximal robustly invariant set under $u = Kx$ (*dotted line*) are also shown

(c) For a horizon of $N = 8$, the maximum scaling σ such that σx_0 is feasible, where $x_0 = (4, -1)$, is $\sigma = 1.134$.

(d) Solving the MPC optimization with $x_0 = (4, -1)$ gives the optimal solution for $\mathbf{c}_0$ as

$$\mathbf{c}_0^* = (0.482, 0.306, 0.251, 0.195, 0.128, 0.070, -0.020, -0.062)$$

and $s_{0|k}^* = (4, -1)$. Hence the optimal value of the nominal predicted cost is $\|s_{0|k}^*\|_{W_x}^2 + \|\mathbf{c}_0^*\|_{W_c}^2 = 37.88$.

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