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A FREQUENCY DOMAIN METHOD FOR ESTIMATING THE PARAMETERS OF A NON-LINEAR STRUCTURAL DYNAMIC MODEL THROUGH FEEDBACK

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A unifying perspective of non-linear structural dynamic systems as linear in the open loop with non-linear feedback in the closed loop has recently been revisited by the authors. The authors have previously used feedback to derive a new formulation of frequency response function matrices of non-linear systems, which are described as modulations of nominal linear systems. The modulation creates a pseudo-separation of the linear and non-linear dynamics of the system. The present article derives a new method for estimating parameters of non-linear parametric models that uses internal feedback to account for non-linearities. The main advantage of the new formulation of non-linear system identification is its simplicity. Moreover, the method estimates the linear frequency response matrix and non-linear system parameters at forced and unforced degrees of freedom of general multiple-degree-of-freedom non-linear systems simultaneously. This article demonstrates the implementation of this method on simulated data from single- and multiple-degree-of-freedom lumped parameter models.

1. INTRODUCTION

The goal of non-linear system identification is to estimate the parameters in a model that has been chosen to describe the dynamics of a system. There are three steps in the identification process: characterisation, model selection and parameter estimation. The first step, characterisation, can be the most challenging task in the identification process. Non-linear systems are said to be characterised after the presence, type and location of all the non-linearities throughout the system are determined. There are many methods that can be used to characterise the underlying structure of the non-linear system. A general survey of these methods is given in [1]. For real-world multiple-degree-of-freedom (mdof) systems with multiple non-linearities, the mdof-based techniques in [2, 3] have proven useful.

The second step in the identification process is model selection. Since the most descriptive model of a system can be selected only after the system has been fully characterised, the success of the second step depends greatly on the success of the first step of the identification process. This article does not discuss characterisation and model selection. Instead, the article focuses on the third step in the identification process, non-linear parameter estimation. The reader is referred to the aforementioned articles for research that discusses characterisation in detail.

The goals of parameter estimation are determined by the type of model that is chosen. Parametric models are lumped parameter models, which describe the detailed linear and

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non-linear interaction between a specific spatial configuration of discrete mass elements. In contrast, non-parametric models do not make any assumptions about the mechanisms for interaction between neighbouring mass elements. Although non-parametric models are more general than parametric models, the non-parametric models are less physical in nature; consequently, they are not as useful as parametric models for understanding and exploiting the behaviour of the system in different situations. This articles deals solely with parametric system identification. Several of the most popular and general methods of non-linear structural dynamic system identification are discussed in [4].

There are several types of parametric models in the time and frequency domains. Time-domain models include the non-linear autoregressive moving average with exogenous inputs (NARMAX) model [5] and the direct parameter model [6]. These time-domain models are conceptually simple to understand and offer significant advantages over frequency-domain methods for dealing with sources of correlated noise in the model [4]. The most modern frequency domain model is the 'reverse path' model [7–9]. The advantage of using the reverse path model is that it tracks changes in a parameter with frequency. The disadvantage of this model is that its implementation requires a thorough knowledge of conditioned spectral analysis, which adds complexity to an already complex non-linear parameter estimation problem. The reverse path approach also requires a feedforward solution procedure to obtain parameter estimates for non-linearities at unforced degrees of freedom (dofs).

The goal of the research in this article is to develop a less complex method for estimating non-linear parameters in a structural dynamic model that is easier to implement. More specifically, spatial data is used in addition to temporal input and output data to resolve the individual contributions that the linear and non-linear elements make to the system dynamics. The importance of spatial information for non-linear systems analysis is a consequence of the feedback nature of non-linear systems [10]. Although the notion of non-linearities as a feedback is not new, this article uses spatial information in a novel way to derive a simple parameter estimation method for non-linear structural dynamic models. The method estimates linear frequency response functions (FRFs) and non-linear parameters in a single step at both forced and unforced dofs.

Section 2 begins by establishing the non-linear structural dynamic model from which the new parameter estimation method is derived in Section 3. Since a spatial perspective is used to view the non-linear structural dynamics as internal feedback into the underlying linear system, Section 2 also discusses the notion of non-linear feedback and its relationship to the spatial characteristics of systems. Section 4 gives detailed sdof and mdof examples on the implementation of the new frequency-domain parameter estimation technique. Finally, Section 5 provides a summary of the advantages of this new technique over the reverse path technique in particular.

2. NON-LINEAR FEEDBACK—MODULATION RELATIONSHIPS

Although the choice between time- or frequency-domain models can lead to certain advantages and disadvantages in the parameter estimation process, each method of parameter estimation is carried out in a similar way. More specifically, the inputs, outputs and the configuration of the dofs that define the model are experimentally determined. These three pieces of information are then used to estimate the non-linear model parameters in a linear least-squares or total least-squares system of equations through averaging. The input and output measurements have always played an important role in the identification process; however, the dof information, which accounts for the spatial data, has not been used as effectively.

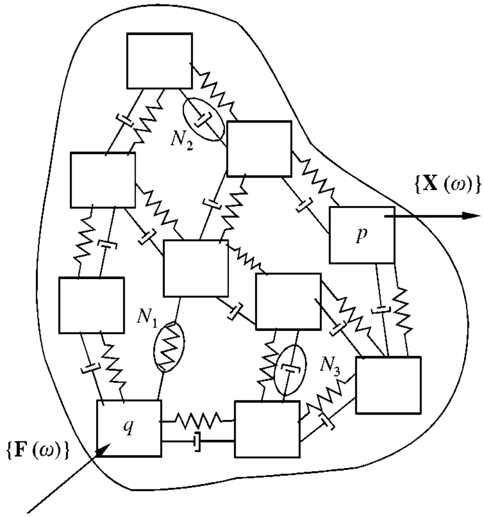

Figure 1. A general lumped parameter non-linear vibrating system with different types of non-linearities at different degrees of freedom.

A previous article by the authors has proposed that the spatial properties of non-linear models are determined by the non-linear feedback within systems [11]. The need to use spatial information or data for non-linear system identification has been motivated by the relatively recent impact that spatial data has had in the modal analysis of linear systems [12]. Spatial data has also proved useful for characterising non-linear vibrating systems using frequency response reciprocity checks to characterise system non-linearities [13].

In order to establish a framework for the derivation in Section 3, we begin by considering the following general frequency domain model for forced linear systems:

$$\{\mathbf{X}(\omega)\}_{N_0\times 1} = [\mathbf{H}_L(\omega)]_{N_0\times N_0}\{\mathbf{F}(\omega)\}_{N_0\times 1}. \tag{1}$$

The inputs and outputs in this equation are written as frequency transforms of the input and output time domain records, $\{\mathbf{x}(t)\}$ and $\{\mathbf{f}(t)\}$. Note that there are N_0 outputs and N_0 total inputs, but here are only N_i non-zero inputs. The time-domain records satisfy the following set of lumped parameter, time-invariant, ordinary differential equations:

$$[\mathbf{M}]_{N_0\times N_0}\{\ddot{\mathbf{x}}(t)\}_{N_0\times 1} + [\mathbf{C}]_{N_0\times N_0}\{\dot{\mathbf{x}}(t)\}_{N_0\times 1} + [\mathbf{K}]_{N_0\times N_0}\{\mathbf{x}(t)\}_{N_0\times 1} = \{\mathbf{f}(t)\}_{N_0\times 1}. \tag{2}$$

If the system is non-linear, then equations (1) and (2) are not accurate or complete models of the system. Furthermore, if $[\mathbf{H}_L(\omega)]$ is calculated from the frequency-domain model using a standard multiple-input, multiple-output (MIMO) FRF least-squares procedure with spectral averaging, the FRF estimates will contain bias errors. For the most common types of non-linearities, the non-linear characteristics are an explicit function of the outputs. Because of this commonality, it is more convenient to begin the derivation with the impedance equations of motion.

For a general non-linear system like the one shown in Fig. 1, the impedance model is given by

$$[\mathbf{B}_L(\omega)]_{N_0\times N_0}\{\mathbf{X}(\omega)\}_{N_0\times 1} + \sum_{i=1}^{N_n}\mu_i(\omega)\{B_{ni}\}_{N_0\times 1}X_{ni}(\omega) = \{\mathbf{F}(\omega)\}_{N_0\times 1}. \tag{3}$$

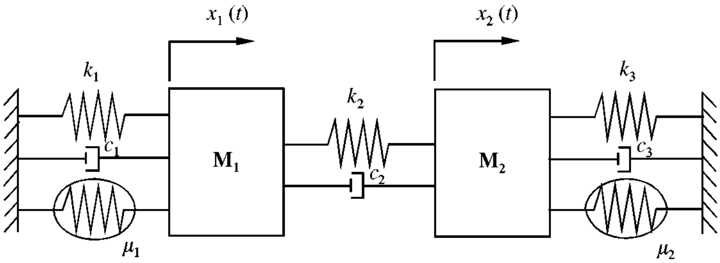

Figure 2. Two-degree-of-freedom system with two non-linear stiffnesses to illustrate the selection of the three defining quantities of each non-linearity; μ_i, $\{\mathbf{B}_{ni}(\omega)\}$, and $X_{ni}(\omega)$.

The linear impedance matrix, $[\mathbf{B}_L(\omega)]$, in this equation accounts for the contributions of the lumped spring-mass-damper elements that make up the nominal or underlying linearised system in equation (2). The summation term accounts for the effects of the lumped non-linearities. Each of the $\mu_i(\omega)$ determines the strength of the associated non-linear element, while each $\{\mathbf{B}_{ni}\}$ determines the location of the non-linear element. In most structural dynamic systems, the scalar spectrum $X_{ni}(\omega)$ is the Fourier spectrum of the non-linear function of the output time histories. Each of these spectra determines the type of non-linearity (e.g. hardening/softening stiffness, clearance, quadratic damping, stiction, etc.)

As an example of how to calculate the three defining non-linear quantities described above, consider the two dof system in Fig. 2. This system contains two non-linear elements: a non-linear hardening spring to ground at dof 1 and a non-linear hardening spring to ground at dof 2. The $\mu_i(\omega)$, $\{\mathbf{B}_{ni}\}$, and $X_{ni}(\omega)$ for this system are given by

$$X_{n1}(\omega) = \mathbf{F}[x_1^3(t)], \quad \mu_1(\omega)\{\mathbf{B}_{n1}\} = \mu_1(\omega)\begin{pmatrix}1\\0\end{pmatrix} \tag{4}$$

$$X_{n2}(\omega) = \mathbf{F}[x_3^3(t)], \quad \mu_2(\omega)\{\mathbf{B}_{n2}\} = \mu_2(\omega)\begin{pmatrix}0\\1\end{pmatrix} \tag{5}$$

where $\mathbf{F}[\cdot]$ denotes the Fourier transform operator. The two $\mu_i(\omega)$ for this system are both constants.

Notice that the non-linearities create unmeasured, internal feedback forces in the underlying linear model of the system. This can be more easily seen by moving the non-linear terms in equation (3) to the right-hand side as follows:

$$[\mathbf{B}_L(\omega)]\{X(\omega)\} = \{\mathbf{F}(\omega)\} + \{\mathbf{F}_n(\omega)\} = \{\mathbf{F}(\omega)\} - \sum_{i=1}^{N_n} \mu_i(\omega)\{\mathbf{B}_{ni}\}X_{ni}(\omega). \tag{6}$$

Equation (6) simply states that the linear system is acted upon by two sets of forces: one set contains the external forces, $\{\mathbf{F}(\omega)\}$, whereas the second set contains the internal feedback forces due to the non-linearities, $\{\mathbf{F}_n(\omega)\}$. In other words, equation (6) is a statement of the 'principle of superposition' for nominal linear systems.

This is an important point because it suggests that the spatial nature of non-linear systems is equivalent to the notion that a non-linearity acts as an internal feedback force. Figure 3 illustrates the concept of internal feedback due to non-linearities. This illustration can be thought of as a *superposition principle for non-linear systems*. Equation (6) is used in the next section to derive a new parameter estimation method for non-linear structural dynamic systems.

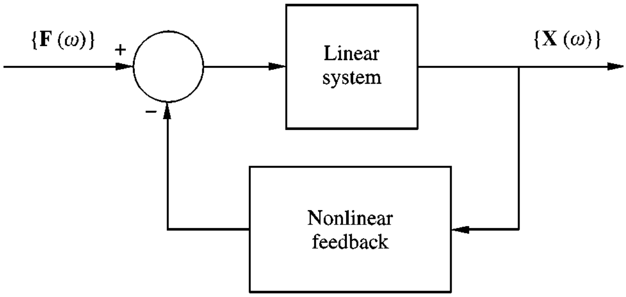

Figure 3. Feedback by non-linearities into a linear vibrating system illustrating the superposition of the external forces and the internal feedback forces.

3. PARAMETER ESTIMATION THROUGH FEEDBACK

The goal in this section is to estimate the underlying linear parameters and the (possibly frequency-dependent) parameters that determine the magnitudes of the internal feedback forces due to the non-linearities in equation (6). It will be shown that the feedback formulation affords a simple, MIMO least-squares parameter estimation method for non-linear systems that naturally decouples the linear and non-linear parameters.

Recall that in other frequency-domain parameter estimation methods like the reverse path method, the linear and non-linear parameters must be estimated in two stages because the linear and non-linear dynamics are coupled. In the first stage, conditioning is used to remove the effects of the non-linearities before estimating the linear FRFs. In the second stage, the non-linear parameters are estimated by utilising the linear FRFs that will have already been computed in the first stage. This method is well documented in the literature [4, 8]. Although the reverse path technique is useful, it is not easy to apply and does not lend itself to more general applications. In contrast, the compact FRF relationship in equation (6) is simple to understand and easy to use.

The new parameter estimation formulation can be derived from equation (6) because all of the inputs, outputs and non-linear functions $X_{ni}(\omega)$ are known for each ensemble without averaging. When this equation is pre-multiplied by the FRF matrix of the underlying linear system and then rewritten by separating the measured (known) quantities from the un-measured (unknown) quantities, the following set of linear equations at each frequency is obtained:

$$\{\mathbf{X}(\omega)\} = [[\mathbf{H}_L(\omega) \quad [\mathbf{H}_L(\omega)]\mu_1(\omega)\{\mathbf{B}_{n1}\} \; [\mathbf{H}_L(\omega)]\mu_2(\omega)\{\mathbf{B}_{n2}\} \cdots$$

$$[\mathbf{H}_L(\omega)]\mu_{N_n}(\omega)\{\mathbf{B}_{nN_n}\}] \begin{pmatrix} \{\mathbf{F}(\omega)\} \\ -\begin{pmatrix} X_{n1}(\omega) \\ X_{n2}(\omega) \\ \vdots \\ X_{N_n}(\omega) \end{pmatrix} \end{pmatrix}. \quad (7)$$

This formulation allows measured external inputs to act together with internal non-linear feedback forces on and within the underlying linear system to produce the measured outputs. Since the inputs and outputs are measured, and since the non-linear functions can be calculated *explicity* in terms of the measured inputs and outputs, the set of equations in equation (7) can be used to solve for the best unbiased least-squares estimates of the linear FRFs at forced dofs and the non-linear parameters $\mu_i(\omega)$ at forced and unforced dofs in a single step. In other words, the data associated with the spatial configuration of the non-linearities provide additional information that is needed to simultaneously identify the linear and non-linear parts of the system.

The parameter estimation method that is based on the compact formulation in equation (7) will be referred to as non-linear identification through feedback of the outputs (NIFO). A standard least-squares solution with spectral averaging can be used to estimate the parameters that govern the linear and non-linear dynamics; however, a more numerically stable orthogonal least-squares solution is recommended. Because the parameters are estimated in a single step, parameter estimation using the NIFO formulation is not subject to progressive (round-off) errors, which may affect parameter estimates that are obtained using the reverse path technique.

Note that equation (7) does not require the standard feedforward solution procedure to obtain parameter estimates for non-linearities that are attached to unforced dofs. In fact, equation (7) implicitly uses the reciprocity of the FRF matrix of the underlying linear system to eliminate the need for an often notationally cumbersome feedforward solution technique [9]. This latter solution technique is required in all existing time- and frequency-domain parameter estimation methods [4]. The ability to feedforward the parameter estimates at forced dofs in this manner is due to Newton's law of action and reaction.

It is important to emphasise that characterisation should be a pre-requisite for implementing (7). In particular, the correct dofs must be used to compute the non-linear functions ($X_{ni}(\omega)$), which account for the internal feedback forces. Otherwise, a kind of 'spatial truncation' can significantly affect the parameter estimates [14]. In other words, if the system is not well characterised, the parameter estimation problem is not well-posed because the NIFO formulation involves a correlated source of noise, which produces biased parameter estimates. It is best to avoid this possibility by performing *in situ* experimental characterisation procedures that detect, locate and classify non-linearities as in [2, 3].

In general, there will be more output dofs than there are non-linear elements and input dofs combined; consequently, there will usually be two parameter estimators corresponding to equation (7). One uses the 'H_1' estimator and the other uses the 'H_2' estimator. Quotes are used to indicate that the input auto- and cross-power spectra in the non-linear parameter estimation problem involve both the external forces and the internal feedback forces due to the non-linearities. The 'H_2' estimator should only be used when the number of outputs N_0 is greater than or equal to the number of non-zero inputs N_i. This same requirement holds when estimating FRFs for linear systems.

Although there are many similarities between the formulation in equation (7) and the standard formulation for estimating FRF matrices for linear systems, there is an important difference associated with the effects of noise on the estimates in the two different formulations. When FRFs of linear systems are estimated, H_1 always produces better estimates when there is measurement noise on the outputs and H_2 produces better estimates when the noise is on the input measurements.

In contrast, because internal feedback forces are derived from non-linear functions of the outputs, *non-linearities always convert uncorrelated* ('*white*') *random noise into correlated* ('*coloured*') *noise*. This means that output measurement noise produces bias errors in parameter estimates that result from the formulation in equation (7); consequently; the 'H_1' estimators do not necessarily give the best estimates of the linear FRFs and the non-linear parameters when there is noise on the outputs. This difference between the linear and non-linear parameters when there is noise on the outputs. This difference between the linear and non-linear H_1 and H_2 estimators in the presence of measurement noise is important for practical applications. In spite of these issues,the authors have demonstrated that the NIFO parameter estimation technique works well for real-world systems when a reasonable effort has been made to reduce noise on the data [15].

Another practical issue concerning the use of equation (7) is the possible correlation of the external forces, $\{\mathbf{F}(\omega)\}$, and internal non-linear forces, $X_{ni}(\omega)$. Although user-defined external forces can always be constructed to be uncorrelated, the internal non-linear forces will always be *partially* correlated because they are computed as non-linear functions of the measured or simulated responses. Because these internal forces are only partially correlated, the pseudo-inverse needed to estimate the parameters will be well defined throughout most of the frequency range of interest with the possible exception of regions near the system resonances. Furthermore, in the authors' experience, the input signatures and configurations can be selected to obtain internal non-linear forces that produce a well-conditioned pseudo-inverse calculation.

One final point to consider in equation (7) is the notion of 'underlying linearity'. In many complicated non-linear systems, the underlying linear system is not known beforehand and may not be uniquely determined in the course of the experimental system identification process. By working instead towards the notion of a 'nominal linear system', the goal of finding *an unique underlying linear system* is abandoned in favour of accurately describing the input–output linear and non-linear dynamics. This point will be addressed elsewhere by the authors [16].

4. PARAMETER ESTIMATION EXAMPLES

A sdof system and a two-dof system will be used in Sections 4.1 and 4.2 to illustrate the NIFO parameter estimation procedure and performance. The simulation parameters that were used to numerically integrate the equations of motion of these two systems are given in Table 1. Note that the time step must be small enough to accommodate the highest frequency of interest in the simulation. Since 15 Hz was the end of the frequency band of interest, the sampling rate was chosen to be greater than six times this frequency. If the simulation time step is increased, the integrator will lose accuracy and the non-linear parameters will increase in magnitude with increasing frequency. This frequency-dependent behaviour should not be misinterpreted as a true change in the non-linear parameters.

Also note that since the Hanning window attenuates data at the ends of each time record, a 51% overlap in contiguous windowed blocks of time data was used to decrease the random error in the estimates by utilising the data that is attenuated in previous averages. Lastly, the amplitudes of the band-limited Gaussian inputs are chosen throughout the simulations to make internal feedback forces due to the non-linearities 15% of the total root-mean-square value of the corresponding linear internal forces, i.e. the inputs are chosen so that non-linearities in these examples are weak to moderate in strength.

4.1. SINGLE DEGREE OF FREEDOM

A sdof system with a non-linear hardening stiffness to ground is shown in Fig. 4. Recall from equation (3) that the location and type of a single non-linear element are determined

TABLE 1

Simulation parameters

Δt	Blocksize	Averages	Nyquist frequency	Percent overlap
0.01 s	2500	40	50 Hz	0.51

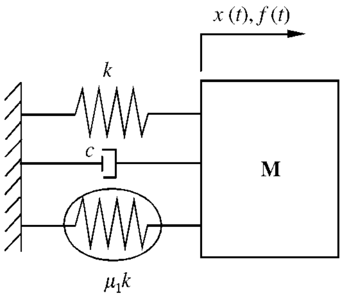

Figure 4. One-degree-of-freedom system with a non-linear hardening stiffness to ground, $k\mu_1 x^3(t)$, at degree of freedom 1.

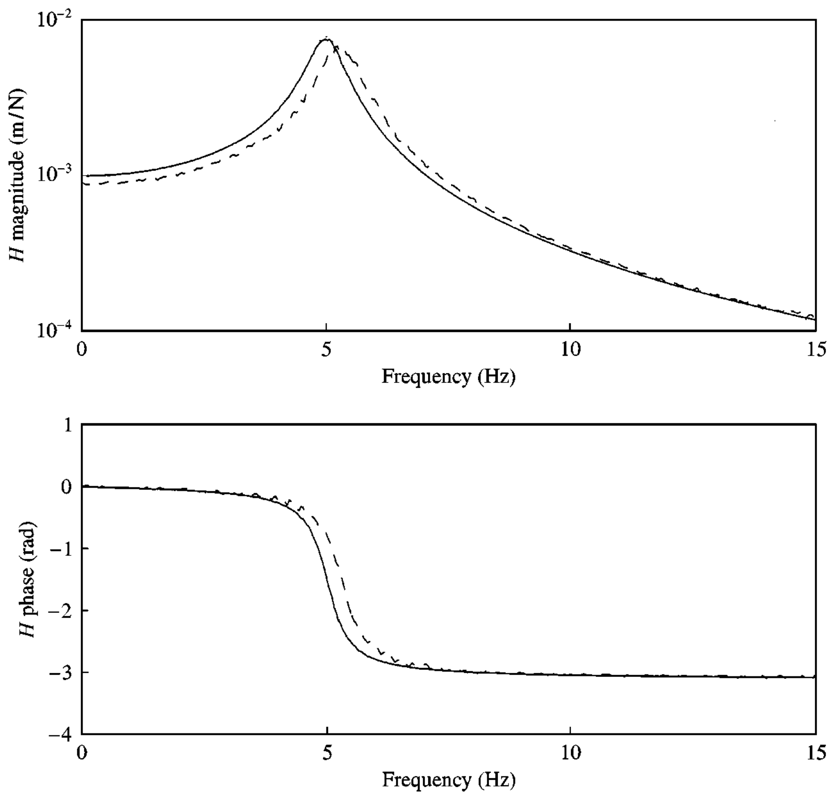

Figure 5. Magnitude and phase of the frequency response functions of the one-degree-of freedom system with a non-linear hardening stiffness to ground, $k\mu_1 x^3(t)$: ($\cdots$), true FRF of linear system; (– – –), FRF of non-linear system; (—), estimated FRF of linear system.

by $\{\mathbf{B}_{n1}\}$ and $X_{n1}(\omega)$, respectively. The strength of the non-linear stiffness in Fig. 4 relative to the linear stiffness, k, is determined by μ_1. The two quantities that determine the location and type of the non-linear stiffness are given below:

$$\{\mathbf{B}_{n1}\} = 1 \tag{8}$$

$$X_{n1}(\omega) = \mathbf{F}[x^3(t)] \tag{9}$$

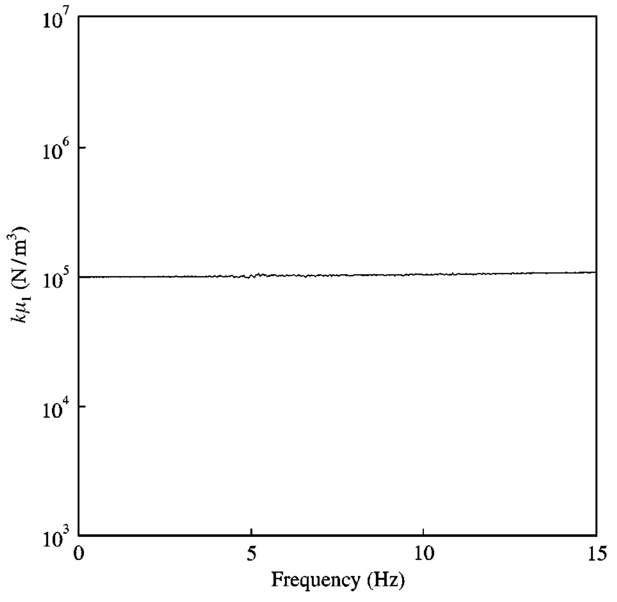

Figure 6. Non-linear parameter estimate, $\hat{\mu}_1(\omega)$, for the one-degree-of-freedom system with a non-linear hardening stiffness to ground, $k\mu_1 x^3(t)$.

TABLE 2

Single-degree-of-freedom non-linear system parameters

m (kg)	c (N m/s)	k (N/m)	μ_1 (1/m^2)
1	4	1000	100

When these quantities are inserted into equation (7), the following expression is obtained for each average at each frequency:

$$X(\omega) = [H_L(\omega) \quad k\mu_1 H_L(\omega)] \begin{pmatrix} F(\omega) \\ -X_{n1}(\omega) \end{pmatrix} \tag{10}$$

The linear parameter in this equation is $H_L(\omega)$ and the scaled non-linear parameter is $k\mu_1$ units of [force/(length)^3]. In this example μ_1 is constant; however, non-linear parameters that vary with frequency can also be estimated. Both of these parameters are estimated simultaneously by writing this equation N_{avg} times, where N_{avg} is the number of spectral averages, and then solving the resulting set of equations using orthogonal least squares.

The results of this procedure are shown in Figs 5 and 6. The upper and lower plots in Fig. 5 show the magnitude and phase, respectively, of three FRFs. The true FRF of the linear system, $H_L(\omega)$, is drawn with a dotted line; the FRF of the non-linear system is drawn with a dashed line; and the estimate of the linear FRF, $\hat{H}_L(\omega)$, is drawn with a solid line. Note that the estimated FRF matches the true linear system FRF to within 0.5% of the total squared error in both magnitude and phase. The estimate of the non-linear stiffness parameter, $\mu_1(\omega)$, is shown scaled by the linear stiffness parameter in Fig. 6. This scaling is applied in keeping with the notion of the non-linearity as an internal feedback force,

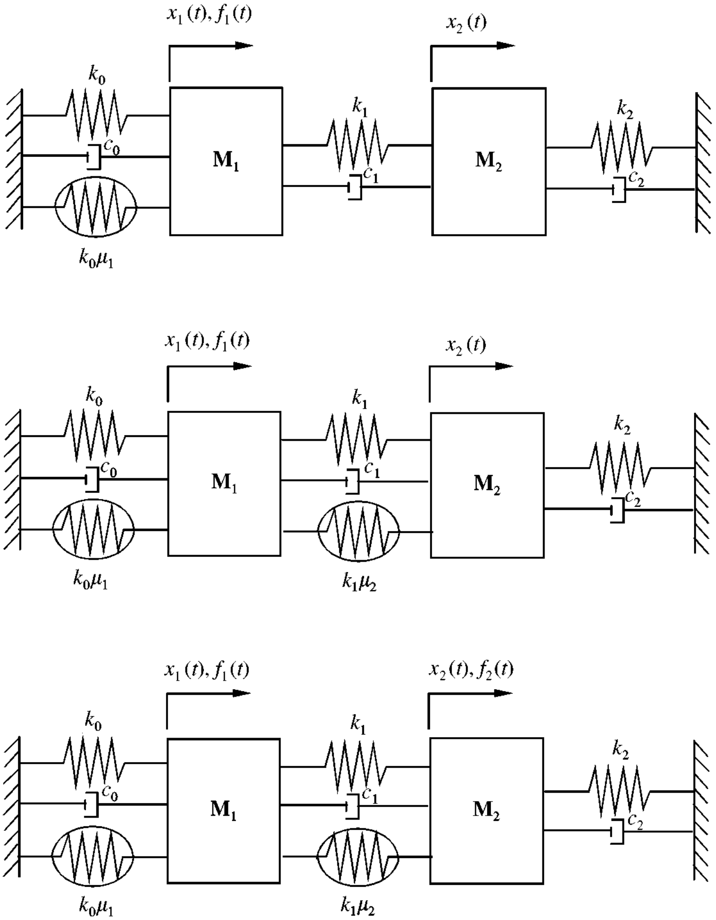

Figure 7. Three two-degree-of-freedom non-linear systems: (top) one non-linear hardening stiffness to ground at dof 1, $k_0\mu_1 x_1^3(t)$, and a single input; (middle) a nonlinear hardening stiffness to ground at dof 1 and a hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t) - x_2(t))^3$, and a single input; (bottom) system with two non-linearities and two inputs.

$k\mu_1 x^3(t)$. The spectral mean of this estimate is within 0.5% of the total root-mean-squared error of the true parameter, which is given in Table 2. Note that even if the non-linear parameters are real constants, their estimates will always be found as functions of frequency.

4.2. MULTIPLE DEGREE OF FREEDOM

Practically, any method of parameter estimation can be used to estimate the non-linear parameters of a sdof structural dynamic system; however, the power of NIFO is its ability to estimate the parameters of mdof systems with multiple inputs and outputs efficiently and easily. Three two-dof systems with slightly different configurations of non-linearities and external inputs will now be used to demonstrate the use of NIFO for mdof non-linear parameter estimation.

The first two-dof system is shown at the top of Fig. 7. There is a single non-linear stiffness to ground at dof 1, $k_0\mu_1 x_1^3(t)$, and a single force applied at dof 1. The results of this estimation procedure are shown in Figs 8–10. Since the force is only applied at dof 1, only

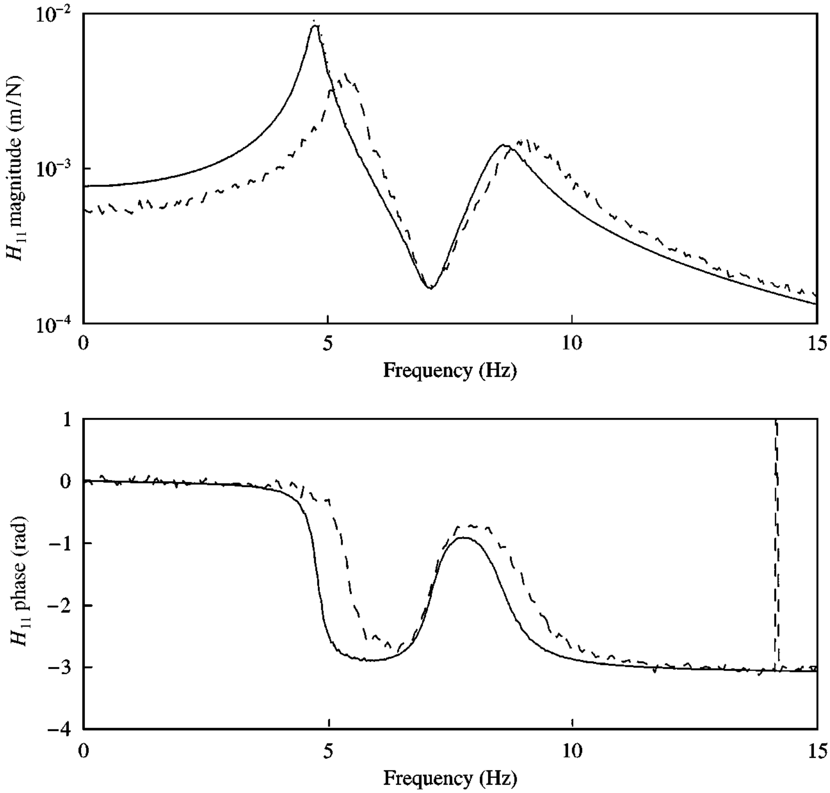

Figure 8. Magnitude and phase of the driving point frequency response function of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_1 x_1^3(t)$, at dof 1 and a single input: (···), true FRF of linear system; (– – –), FRF of non-linear system; (—), estimated FRF of linear system.

the first column of the linear FRF matrix is estimated. Figures 8 and 9 show that the magnitude and phase of the estimated FRFs of the linear system are practically identical to the true FRFs of the linear system. Similarly, a comparison of Fig. 10 and the true parameter value in Table 3 indicates that the non-linear parameter estimate is within 0.5% of the total root-mean-squared error of the true value.

The two-dof system with two non-linear stiffnesses and a single input at dof 1 is shown in the middle illustration in Fig. 7. The results of the parameter estimation procedure for this system are shown in Figs 11–13. The estimates of the FRFs of the linear system are once again in excellent agreement with the true FRFs (refer to Figs. 11 and 12). Likewise, the spectral means of the parameter estimates in Fig. 13 are within 1% total root-mean-squared error of the true parameters from Table 3. The solid line in this figure denotes the estimate $\hat{\mu}_1(\omega)$ and the dashed line denotes the estimate $\hat{\mu}_2(\omega)$. The error in $\hat{\mu}_2(\omega)$ near the first mode of vibration is seen in other frequency-domain parameter estimation methods as well [9]. This error is due to a high sensitivity to leakage errors near the peaks in the FRFs and to correlation of the internal non-linear forces. The leakage errors can be reduced by increasing the frequency resolution.

When 8% of the root-mean-square Gaussian zero-mean noise is added to each simulated response channel, the parameter estimation results for the system in the middle of Fig. 7 are as shown in Fig. 14. The top two plots show the magnitude and phase estimates for the

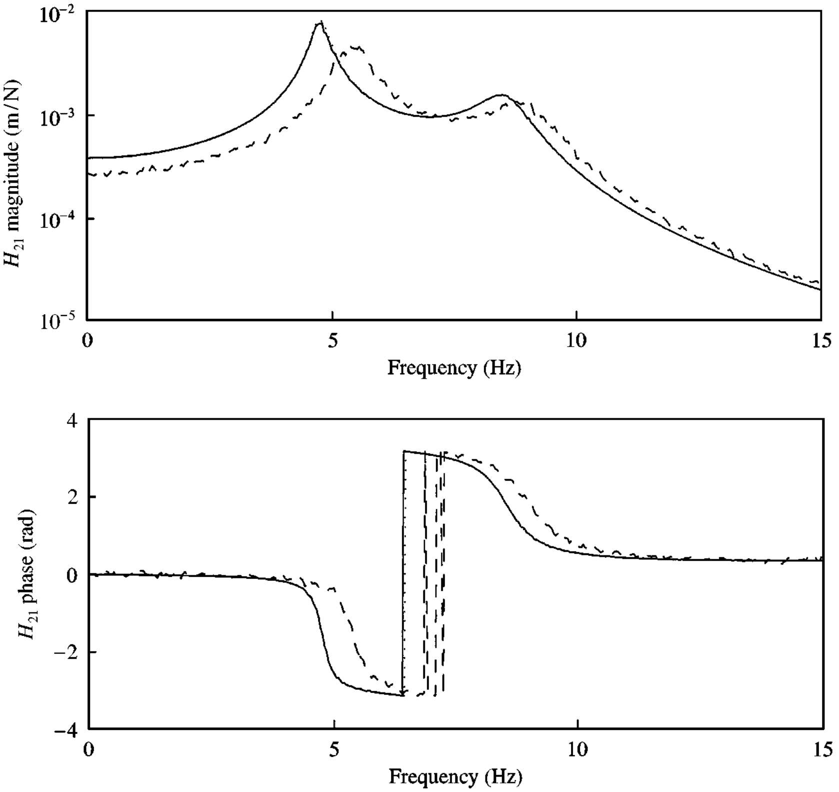

Figure 9. Magnitude and phase of the frequency response function between input dof 1 and output dof 2 of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_1 x_1^3(t)$, at dof 1 and a single input: (···), true FRF of linear system; (– – –), FRF of non-linear system; (—), estimated FRF of linear system.

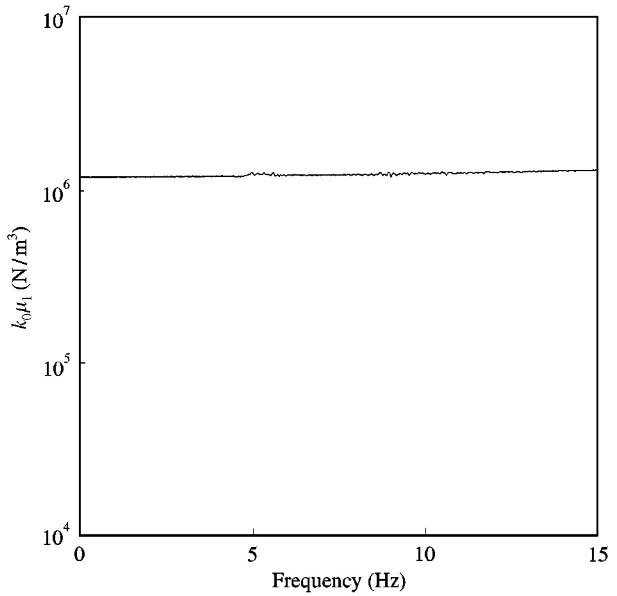

Figure 10. Non-linear parameter estimate, $\hat{\mu}_1(\omega)$, for the two-degree-of-freedom system with a non-linear hardening stiffness to ground at dof 1, $k_0\mu_1 x_1^3(t)$, and a single input.

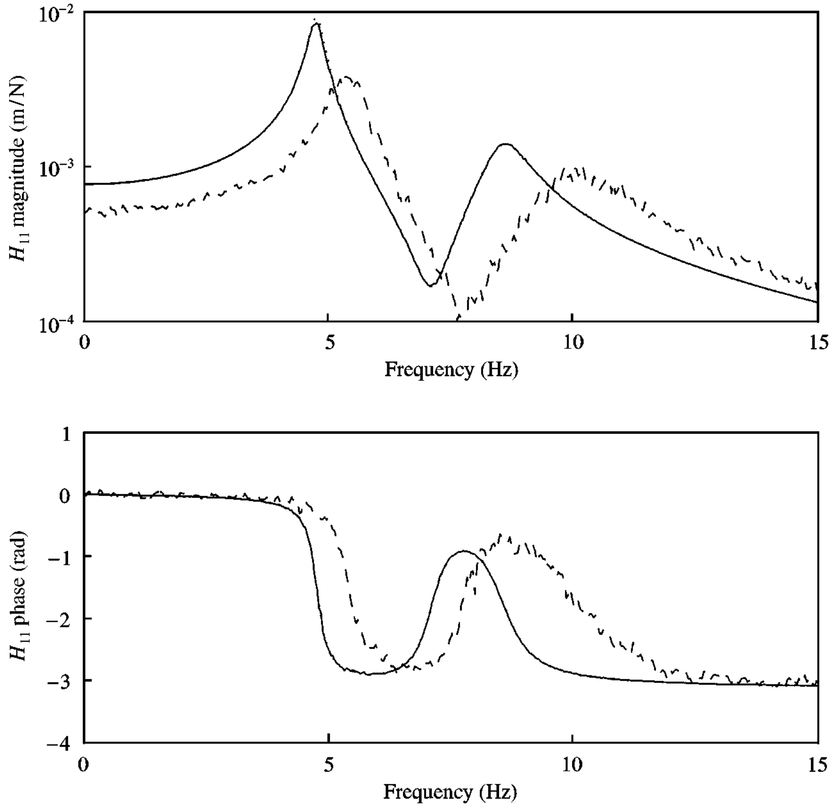

Figure 11. Magnitude and phase of the driving point frequency response function of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_1 x_1^3(t)$, at dof 1, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t) - x_2(t))^3$, and a single input: (···), true FRF of linear system; (– – –), FRF of non-linear system; (—), estimated FRF of linear system.

driving point dof and the bottom two plots show the estimates of the non-linear parameters. Note that the magnitude estimate is in error throughout the frequency range of interest (0–10 Hz); however, the phase estimate is accurate except in the frequency range between the two modes of vibration (6–9 Hz). This example illustrates the serious effects that noise on the output measurements can have on non-linear parameter estimates. The authors are currently developing a best least-squares unbiased frequency-domain parameter estimation method that reduces the noise-induced errors in the parameter estimates. The basis for this method is that the parameter estimation residuals are correlated with frequency when there are bias errors due to measurement noise.

The two dof system with the same two non-linear stiffness as in the previous example but with two inputs instead of one is shown in the bottom illustration of Fig. 7. The results of the NIFO parameter estimation procedure for this system are shown in Figs 15–19. Since there are two inputs, both columns of the FRF matrix of the linear system can be estimated with NIFO. All of the FRF estimates are in excellent agreement with the true FRFs of the linear system. Likewise, the parameter estimates in Fig. 19 closely match the true parameter values from Table 3. Note in particular that the overall results for the multiple-input case are better than for the single-input case. This is true in general because the input energy is more evenly distributed to the non-linear elements and the system as a whole when there is more than one input.

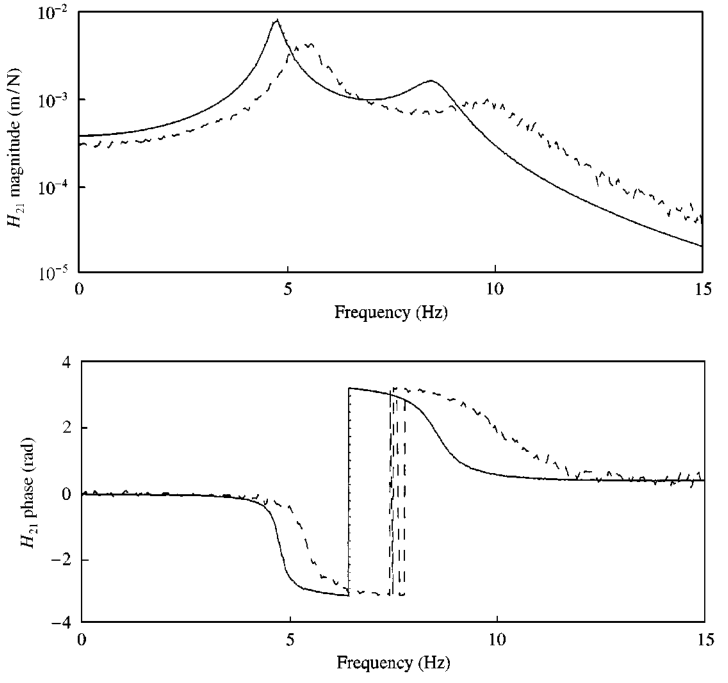

Figure 12. Magnitude and phase of the frequency response function between input dof 1 and output dof 2 of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_1x_1^3(t)$, at dof 1, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t) - x_2(t))^3$, and a single input: (⋯), true FRF of linear system; (– – –), FRF of non-linear system; (—), estimated FRF of linear system.

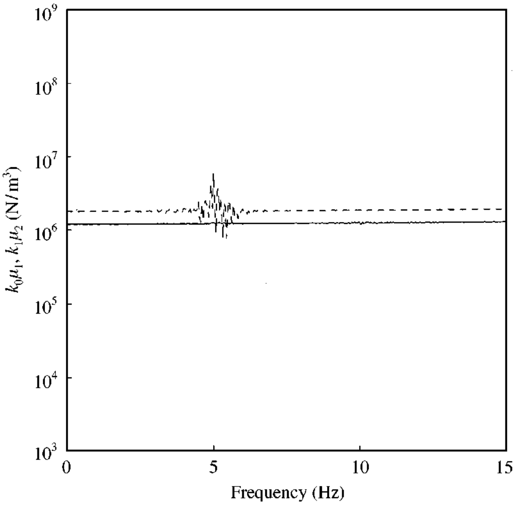

Figure 13. Non-linear parameter estimates, $\hat{\mu}_1(\omega)$ and $\hat{\mu}_2(\omega)$, for the two-degree-of-freedom system with a non-linear hardening stiffness to ground at dof 1, $k_0\mu_1x_1^3(t)$, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t) - x_2(t))^3$, and a single input.

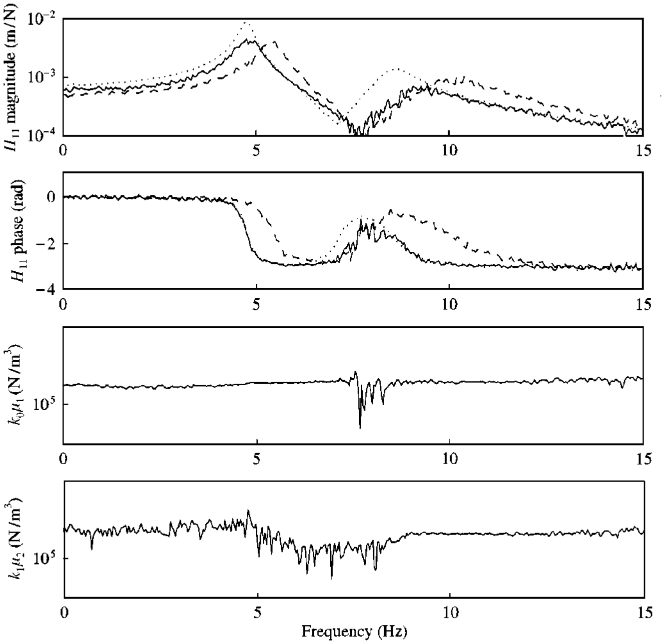

Figure 14. Magnitude and phase of the driving point frequency response function of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_2 x_1^3(t)$, at dof 1, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t)-x_2(t))^3$, and a single input with 8% noise on the output data showing effects of noise in non-linear parameter estimation: ($\cdots$), true FRF of linear system; (– – –), FRF of non-linear system; (—), estimated FRF of linear system.

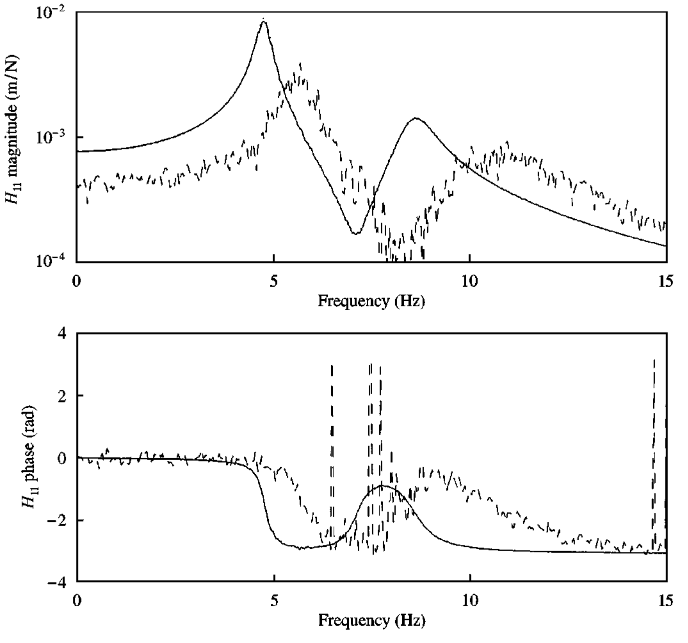

Figure 15. Magnitude and phase of the driving point dof 1 frequency response function of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_1 x_1^3(t)$, at dof 1, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t)-x_2(t))^3$, and two inputs: ($\cdots$), true FRF of linear system; (– – –), FRF of non-linear system; (—), estimated FRF of linear system.

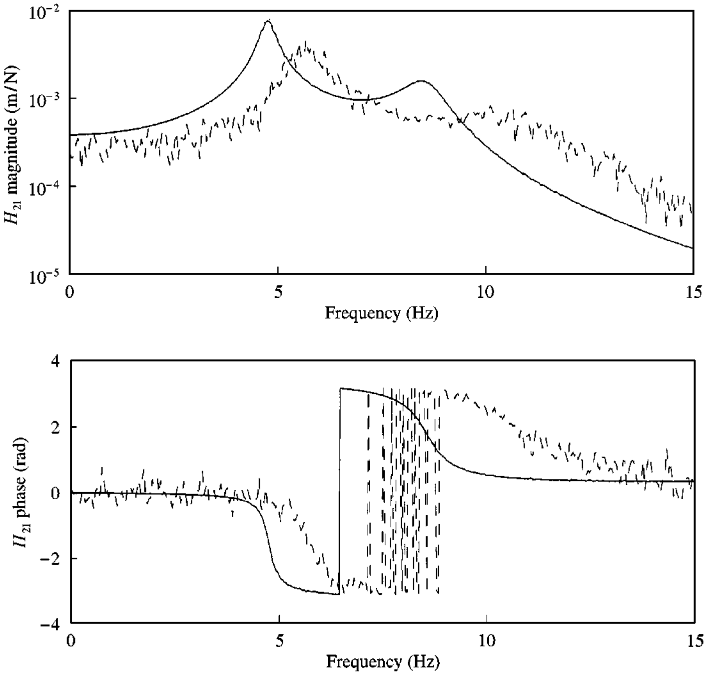

Figure 16. Magnitude and phase of the frequency response function between input dof 1 and output dof 2 of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_1 x_1^3(t)$, at dof 1, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t) - x_2(t))^3$, and two inputs: (···), true FRF of linear system; (– – –), FRF of non-linear system; (—), estimated FRF of linear system.

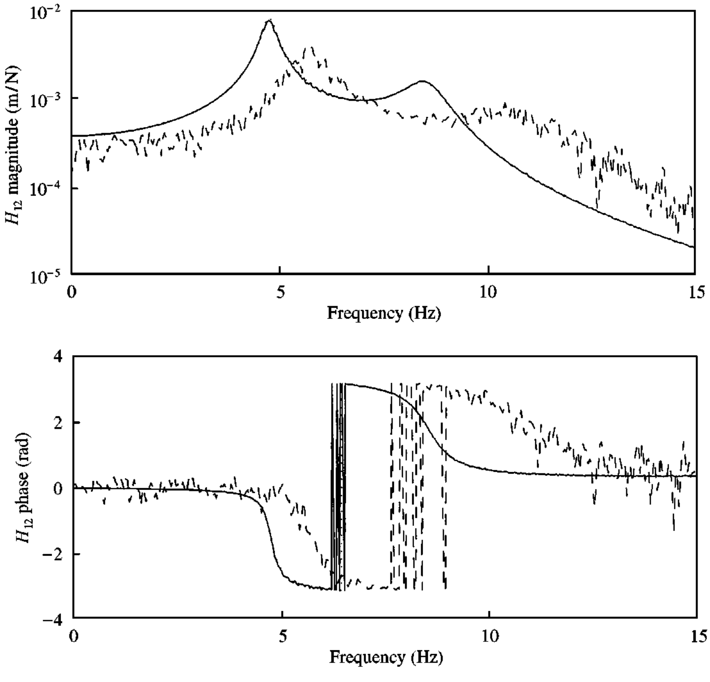

Figure 17. Magnitude and phase of the frequency response function between input dof 2 and output dof 1 of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_1 x_1^3(t)$, at dof 1, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t) - x_2(t))^3$, and two inputs: (···), true FRF of linear system; (– – –), FRF of non-linear system; (—); estimated FRF of linear system.

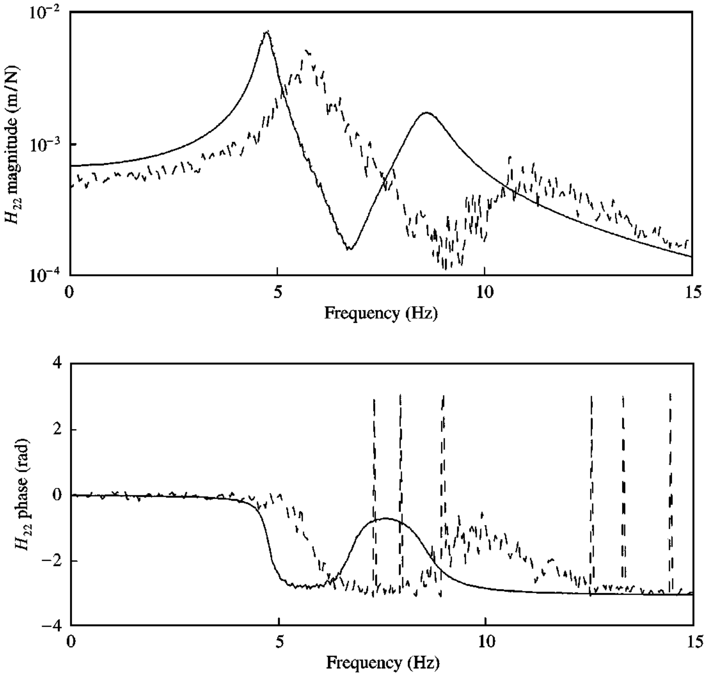

Figure 18. Magnitude and phase of the driving point dof 2 frequency response function of the two-degree-of-freedom system with a non-linear hardening stiffness to ground, $k_0\mu_1 x_1^3(t)$, at dof 1, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t) - x_2(t))^3$, and two inputs: (···); true FRF of linear system (– – –); FRF of non-linear system (—); estimated FRF of linear system.

TABLE 3

Two-degree-of-freedom non-linear system parameters

Mass (kg)	Linear damping (N m/s)	Linear stiffness (N/m)	Non-linear parameters (1/m^2)
$m_1 = m_2 = 1$	$c_1 = c_2 = c_3 = 2$	$k_0 = 800$ $k_1 = k_2 = 1000$	$\mu_1 = 1500$ $\mu_2 = 1800$

The final issue that was addressed in this research is how well NIFO discriminates between many possible non-linear terms that are included in equation (7). For instance, suppose two cubic stiffnesses are included in the parameter estimation process when only one of the two stiffnesses is truly affecting the dynamics of the system. In this case, NIFO correctly determines that the non-contributing element is extraneous and can be discarded. This capability is important because it allows NIFO to work from a general parametric model with with many arbitrary non-linearities to a specific model of the system, which contains only the important non-linear terms. Future research on experimental applications of NIFO will address these issues.

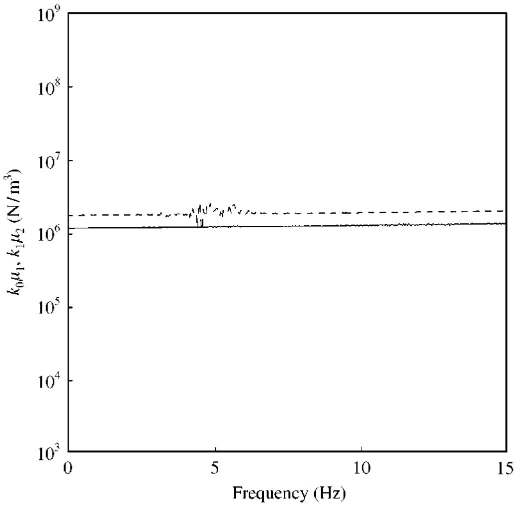

Figure 19. Non-linear parameter estimates, $\hat{\mu}_1(\omega)$ and $\hat{\mu}_2(\omega)$, for the two-degree-of-freedom system with a non-linear hardening stiffness to ground at dof 1, $k_0\mu_1 x_1^3(t)$, a non-linear hardening stiffness between dofs 1 and 2, $k_1\mu_2(x_1(t) - x_2(t))^3$, and two inputs.

5. CONCLUSIONS

The perspective of non-linearities as internal feedback forces has been used to derive a new frequency-domain parameter estimation method for non-linear structural dynamic models. The authors have proposed that non-linear feedback is directly linked to the spatial nature of non-linear systems. A new frequency-domain formulation for modelling and estimating parameters of non-linear structural dynamic systems was derived. The non-linear identification through feedback of the outputs (NIFO) formulation is useful because it estimates the linear FRF matrix and the non-linear parameters throughout the system simultaneously. It offers several advantages over the reverse path method including an ability to simultaneously estimate linear and non-linear parameters; a simple interpretation and compact implementation; better conditioning and computational efficiency; an ability to estimate non-linear parameters at unforced as well as forced dofs; and offers a clearer view of the effects of measurement noise on the linear and non-linear parameter estimates.

In addition, the feedback concept helps to unify the general theory of non-linear structural dynamic system characterisation and identification. NIFO has been shown to produce excellent results for both sdof and mdof structural dynamic models. Research has also shown that NIFO can detect extraneous non-linear terms in the parameter estimation process. Further research is being pursued to compensate for the effects of measurement noise on the outputs. Also, a time-domain equivalent to the frequency-domain NIFO technique is currently being developed by the authors.

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APPENDIX A. NOMENCLATURE

dof(s)	degrees(s) of freedom
sdof	single-degree of freedom
mdof	multiple-degrees of freedom
MIMO	multiple-input, multiple-output testing configuration
N_0	number of output (response) degrees of freedom
N_i	number of input (forced) degrees of freedom at which the input is non-zero; there are N_0 total input degrees of freedom
NARMAX	non-linear autoregressive moving average with exogenous inputs
FRF(s)	frequency response function(s)
$\{\mathbf{x}(t)\}_{N_0\times 1}$	measured output time history vector of length N_0
$\{\mathbf{f}(t)\}_{N_0\times 1}$	measured input time history vector of length N_0
$\{\mathbf{X}(\omega)\}_{N_0\times 1}$	linear Fourier spectrum of the output vector
$\{\mathbf{F}(\omega)\}_{N_0\times 1}$	linear Fourier spectrum of the input vector with N_i non-zero components
$\{\mathbf{F}_n(\omega)\}$	linear Fourier spectrum of the non-linear internal force vector
$[\mathbf{H}_L(\omega)]_{N_0\times N_0}$	frequency response function matrix of a linear or linearised system
$H_{pq}(\omega)$	frequency response function between input degree of freedom q and output dof p
$[\mathbf{B}_L(\omega)]_{N_0\times N_0}$	impedance matrix of a linear or linearised system
$\mu_i(\omega)$	scalar non-linear parameter for non-linear element i

$X_{ni}(\omega)$	Fourier transform of scalar non-linear function of the outputs for non-linear element i
$\{\mathbf{B}_{ni}\}_{N_0 \times 1}$	vector of impedance with non-linear coefficient factored out to yield entries of 1 and -1 only; associated with non-linear element i
$(\,\hat{}\,)$	hat denotes an estimate of a frequency response function of the linear system or an estimate of a non-linear parameter